Stability in the energy space for chains of solitons of the Landau-Lifshitz equation

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Abstract

We prove the orbital stability of sums of solitons for the one-dimensional Landau-Lifshitz equation with an easy-plane anisotropy, under the assumptions that the (non-zero) speeds of the solitons are different, and that their initial positions are sufficiently separated and ordered according to their speeds.

1 Introduction

We consider the one-dimensional Landau-Lifshitz equation

$$\partial_t m + m \times (\partial_{xx} m + \lambda m_3 e_3) = 0, \tag{LL}$$

for a map $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$. Originally introduced by Landau and Lifshitz in [21], this equation describes the dynamics of magnetization in a one-dimensional ferromagnet, typically in samples of $CsNiF_3$ and TMNC (see e.g. [19, 26, 17]). The vector $e_3 = (0, 0, 1)$ and the parameter $\lambda \in \mathbb{R}$ take into account the anisotropy of the material. When $\lambda > 0$, the ferromagnet owns an easy-axis anisotropy along the axis spanned by e_3. When $\lambda < 0$, it owns an easy-plane anisotropy along the plane orthogonal to e_3. In the isotropic case $\lambda = 0$, the Landau-Lifshitz equation reduces to the one-dimensional Schrödinger map equation, which has been intensively studied in the mathematical literature (see e.g. [15, 18], and the references therein).

In this paper, we focus on the Landau-Lifshitz equation with an easy-plane anisotropy ($\lambda < 0$). Suitably scaling the map m, we assume from now on that $\lambda = -1$. The Landau-Lifshitz equation is integrable by means of the inverse scattering method (see e.g. [11]). Using this method, one can check the existence of multi-solitons for (LL) (see e.g. [4, Section 10] for their explicit formula). Multi-solitons are exact solutions to (LL) that can be regarded as a nonlinear superposition of single solitons. Our main goal in this paper is to investigate the stability along the Landau-Lifshitz flow of arbitrary perturbations of a (well-prepared) superposition of solitons (see Theorem 2).

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1.1 The hydrodynamical formulation

The Landau-Lifshitz equation is Hamiltonian. Its Hamiltonian, the so-called Landau-Lifshitz energy, is given by

$$E(m) := \frac{1}{2}\int_{\mathbb{R}} \big(|\partial_x m|^2 + m_3^2\big).$$

In the sequel, we restrict our attention to the solutions m to (LL) with finite Landau-Lifshitz energy, i.e. in the energy space

$$\mathcal{E}(\mathbb{R}) := \big\{v : \mathbb{R} \to \mathbb{S}^2, \text{ s.t. } v' \in L^2(\mathbb{R}) \text{ and } v_3 \in L^2(\mathbb{R})\big\}.$$

In particular, since the component m_3 belongs to $H^1(\mathbb{R})$, the map $\check{m} := m_1 + im_2$ satisfies the condition

$$|\check{m}(x)| = (1 - m_3^2(x))^{\frac{1}{2}} \to 1,$$

as $x \to \pm\infty$, due to the Sobolev embedding theorem. As a consequence, the Landau-Lifshitz equation shares many properties with nonlinear Schrödinger equations with non-zero conditions at infinity, e.g. with the Gross-Pitaevskii equation (see e.g. [1]).

One of the common features lies in the existence of an hydrodynamical framework for the Landau-Lifshitz equation. In terms of the maps $\check{m}$ and m_3, this equation may be recast as

$$\begin{cases} i\partial_t \check{m} - m_3 \partial_{xx}\check{m} + \check{m}\partial_{xx} m_3 - \check{m} m_3 = 0, \\ \partial_t m_3 + \partial_x \big\langle i\check{m}, \partial_x \check{m}\big\rangle_{\mathbb{C}} = 0. \end{cases}$$

When the map $\check{m}$ does not vanish, one can write it as $\check{m} = (1 - m_3^2)^{1/2} \exp i\varphi$. The equations for the hydrodynamical variables $v := m_3$ and $w := \partial_x\varphi$ then write as

$$\begin{cases} \partial_t v = \partial_x\big((v^2 - 1)w\big), \\ \partial_t w = \partial_x\Big(\dfrac{\partial_{xx} v}{1 - v^2} + v\dfrac{(\partial_x v)^2}{(1 - v^2)^2} + v\big(w^2 - 1\big)\Big). \end{cases} \tag{HLL}$$

This system is very similar to the hydrodynamical Gross-Pitaevskii equation (see e.g. [3]).[1] In the sequel, most of the analysis is performed in the hydrodynamical framework. This simplifies both the construction and the study of chains of solitons.

Before defining more precisely these special solutions, notice that the Landau-Lifshitz energy is expressed as

$$E(\mathfrak{v}) := \int_{\mathbb{R}} e(\mathfrak{v}) := \frac{1}{2}\int_{\mathbb{R}} \Big(\frac{(v')^2}{1 - v^2} + \big(1 - v^2\big)w^2 + v^2\Big), \tag{1}$$

in terms of the hydrodynamical pair $\mathfrak{v} := (v, w)$. Another conserved quantity is the momentum P, which is simply defined as

$$P(\mathfrak{v}) := \int_{\mathbb{R}} vw. \tag{2}$$

The momentum P, as well as the Landau-Lifshitz energy E, play an important role in the construction and the qualitative analysis of the solitons.

[1] The hydrodynamical terminology originates in the fact that the hydrodynamical Gross-Pitaevskii equation is very similar to the Euler equation for an irrotational fluid (see e.g. [1]).

1.2 The solitons

Solitons are special solutions to (LL) of the form

$$m(x,t) = u(x-ct),$$

for a given speed $c \in \mathbb{R}$. The profile u is solution to the ordinary differential equation

$$u'' + |u'|^2 u + u_3^2 u - u_3 e_3 + c u \times u' = 0. \tag{3}$$

This equation can be solved explicitly. When $|c| \geq 1$, all the solutions with finite Landau-Lifshitz energy are constant, namely the constant vectors u with $u_3 = 0$. In contrast, when $|c| < 1$, the non-constant solutions u_c to (3) are given by the formulae

$$[u_c]_1(x) = \frac{c}{\cosh\big((1-c^2)^{\frac{1}{2}} x\big)}, \quad [u_c]_2(x) = \tanh\big((1-c^2)^{\frac{1}{2}} x\big), \quad [u_c]_3(x) = \frac{(1-c^2)^{\frac{1}{2}}}{\cosh\big((1-c^2)^{\frac{1}{2}} x\big)}, \tag{4}$$

up to the invariances of the equation, i.e. translations, rotations around the axis x_3 and orthogonal symmetries with respect to any line in the plane $x_3 = 0$. A non-constant soliton with speed c may be written as

$$u_{c,a,\theta,s}(x) = \Big(\cos(\theta)[u_c]_1 - s\sin(\theta)[u_c]_2, \sin(\theta)[u_c]_1 + s\cos(\theta)[u_c]_2, s[u_c]_3\Big)(x-a),$$

with $a \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $s \in \{\pm 1\}$. We refer to [8] for more details.

In the isotropic case $\lambda = 0$, there is no travelling-wave solution to (LL) with non-zero speed and finite energy. However, breather-like solutions were found to exist in [20], and their numerical stability was investigated in [31]. In the easy-axis case $\lambda = 1$, there are travelling-wave solutions (see e.g. [5]). However their third coordinate $m_3(x)$ converges to ± 1 as $|x| \to +\infty$. This prevents from invoking the hydrodynamical formulation, and from using the strategy developed below in order to prove their orbital stability.

In contrast, in the easy-plane case, we can define properly the solitons in the hydrodynamical framework when $c \neq 0$, since the function $\check{u}_c$ does not vanish. More precisely, we can identify the soliton u_c with the pair $\mathfrak{v}_c := (v_c, w_c)$ given by

$$v_c(x) = \frac{(1-c^2)^{\frac{1}{2}}}{\cosh\big((1-c^2)^{\frac{1}{2}} x\big)}, \quad \text{and} \quad w_c(x) = \frac{c\, v_c(x)}{1 - v_c(x)^2} = \frac{c(1-c^2)^{\frac{1}{2}} \cosh\big((1-c^2)^{\frac{1}{2}} x\big)}{\sinh\big((1-c^2)^{\frac{1}{2}} x\big)^2 + c^2}. \tag{5}$$

In this framework, the only remaining invariances of (3) are translations, as well as the opposite map $(v,w) \mapsto (-v,-w)$. A non-constant soliton with speed c may be written as

$$\mathfrak{v}_{c,a,s}(x) := s\, \mathfrak{v}_c(x-a) := \big(s\, v_c(x-a), s\, w_c(x-a)\big),$$

with $a \in \mathbb{R}$ and $s \in \{\pm 1\}$.

Our goal in this paper is to establish the orbital stability of a single soliton u_c along the Landau-Lifshitz flow. We will also consider the case of a sum of solitons. In the original framework, defining this quantity properly is not so easy. As a matter of fact, we face the difficulty that a sum of unit vectors in $\mathbb{R}^3$ does not necessarily remain in the unit sphere $\mathbb{S}^2$.

In the hydrodynamical framework, this difficulty does not arise any longer. We can define a sum of solitons $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ as

$$S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} := (V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}, W_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}) := \sum_{j=1}^N \mathfrak{v}_{c_j,a_j,s_j},$$

with $N \in \mathbb{N}^*$, $\mathfrak{c} = (c_1, \ldots, c_N)$, $\mathfrak{a} = (a_1, \ldots, a_N) \in \mathbb{R}^N$, and $\mathfrak{s} = (s_1, \ldots, s_N) \in \{\pm 1\}^N$. However, we have to restrict the analysis to speeds $c_j \neq 0$, since the function $\check{u}_0$ vanishes at the origin.

Coming back to the original framework, we can define properly a corresponding sum of solitons $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$, when the third component of $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ does not reach the value ± 1. Due to the exponential decay of the functions v_c and w_c, this assumption is satisfied at least when the positions a_j are sufficiently separated. In this case, the sum $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ is given, up to a phase factor, by the expression

$$R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} := \Big((1 - V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}^2)^{\frac{1}{2}} \cos(\Phi_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}), (1 - V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}^2)^{\frac{1}{2}} \sin(\Phi_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}), V_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} \Big),$$

where we have set

$$\Phi_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}(x) := \int_0^x W_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}(y)\, dy,$$

for any $x \in \mathbb{R}$. This definition presents the advantage to provide a quantity with values in the sphere $\mathbb{S}^2$. On the other hand, it is only defined under restrictive assumptions on the speeds c_j and positions a_j. Moreover, it does not take into account the geometric invariance with respect to rotations around the axis x_3 in (3).

In the sequel, our main results are proved in the hydrodynamical framework. We establish that, if the initial positions a_j^0 are well-separated and the initial speeds c_j^0 are ordered according to the initial positions a_j^0, the solution corresponding to a chain of solitons at initial time, that is a perturbation of a sum of solitons $S_{\mathfrak{c}^0,\mathfrak{a}^0,\mathfrak{s}^0}$, is uniquely defined, and that it remains a chain of solitons for any positive time. We then rephrase this statement in the original framework.

1.3 Statement of the main results

Before detailing this stability result, we have to consider the Cauchy problem for both the original and hydrodynamical Landau-Lifshitz equations.

Concerning the original one, the existence of global solutions in the energy space $\mathcal{E}(\mathbb{R})$ can be achieved following arguments similar to the ones introduced by Zhou and Guo [32], or Sulem, Sulem and Bardos [30], for the Schrödinger map equation (see also [15] for more details). The uniqueness of solutions in the energy space $\mathcal{E}(\mathbb{R})$ is more involved. We refer to [18] for a discussion about this subject in the case of the Schrödinger map equation. In the sequel, we prove the uniqueness of a continuous flow for the Landau-Lifshitz equation, in particular, in the neighbourhood of the sums of solitons $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ (see Corollary 1 below). This property of the flow is required when dealing with orbital stability.

In order to establish this property rigorously, we first address the Cauchy problem in the hydrodynamical framework. In view of the expression of the energy in (1), the natural space for solving it is given by

$$\mathcal{NV}(\mathbb{R}) := \Big\{ \mathfrak{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{\mathbb{R}} |v| < 1 \Big\},$$

and we can endow it with the metric structure corresponding to the norm

$$\|\mathfrak{v}\|_{H^1 \times L^2} := \Big(\|v\|_{H^1}^2 + \|w\|_{L^2}^2 \Big)^{\frac{1}{2}}.$$

The non-vanishing condition on the maximum of $|v|$ is necessary to define properly the function w, which, in the original setting of a solution m to (LL), corresponds to the derivative of the phase φ of the map $\check{m}$. Due to the Sobolev embedding theorem, this non-vanishing condition is also enough to define properly, and then establish the continuity of the energy E and the momentum P on $\mathcal{NV}(\mathbb{R})$.

Concerning the Cauchy problem for (HLL), we show the following local well-posedness result.

Theorem 1. *Let* $\mathfrak{v}^0 = (v^0, w^0) \in \mathcal{NV}(\mathbb{R})$. *There exist a positive number* $T_{\max}$ *and a map* $\mathfrak{v} = (v, w) \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$, *which satisfy the following statements.*
(i) *The map* $\mathfrak{v}$ *is the unique solution to* (HLL), *with initial datum* $\mathfrak{v}^0$, *such that there exist smooth solutions* $\mathfrak{v}_n \in \mathcal{C}^\infty(\mathbb{R} \times [0, T])$ *to* (HLL), *which satisfy*

$$\mathfrak{v}_n \to \mathfrak{v} \quad \text{in } \mathcal{C}^0([0, T], \mathcal{NV}(\mathbb{R})), \tag{6}$$

as $n \to +\infty$, *for any* $T \in (0, T_{\max})$.
(ii) *The maximal time* $T_{\max}$ *is characterized by the condition*

$$\lim_{t \to T_{\max}} \max_{x \in \mathbb{R}} |v(x, t)| = 1, \quad \text{if } T_{\max} < +\infty.$$

(iii) *The energy* E *and the momentum* P *are constant along the flow.*
(iv) *When*

$$\mathfrak{v}_n^0 \to \mathfrak{v}^0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}), \tag{7}$$

as $n \to +\infty$, *the maximal time of existence* T_n *of the solutions* $\mathfrak{v}_n$ *to* (HLL), *with initial datum* $\mathfrak{v}_n^0$, *satisfies*

$$T_{\max} \leq \liminf_{n \to +\infty} T_n, \tag{8}$$

and

$$\mathfrak{v}_n \to \mathfrak{v} \quad \text{in } \mathcal{C}^0([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R})), \tag{9}$$

as $n \to +\infty$, *for any* $T \in (0, T_{\max})$.

In other words, Theorem 1 provides the existence and uniqueness of a continuous flow for (HLL) in the energy space $\mathcal{NV}(\mathbb{R})$. As mentioned previously, this is enough to consider the stability of the sums of solitons in the energy space. We will prove it for the solutions corresponding to this unique continuous flow (see Theorem 2 below).

Remark 1. On the other hand, this does not prevent from the existence of other solutions which could not be approached by smooth solutions according to (6). In particular, we do not claim that there exists a unique local solution to (HLL) in the energy space for a given initial datum.

Remark 2. To our knowledge, the question of the global existence (in the hydrodynamical framework) of the local solution $\mathfrak{v}$ is open. In the sequel, we by-pass this difficulty using the stability of a well-prepared sum of solitons $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$. Since the solitons in such a sum have exponential decay by (5), and are sufficiently well-separated, the sum $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ belongs to $\mathcal{NV}(\mathbb{R})$. Invoking the Sobolev embedding theorem, this remains true for a small perturbation in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. As a consequence, the global existence for a well-prepared chain of solitons follows from its stability by applying a continuation argument (see Subsection 1.5 below).

In the energy space $\mathcal{E}(\mathbb{R})$, which we endow with the metric structure corresponding to the distance

$$d_{\mathcal{E}}(f, g) := \Big(\big| \check{f}(0) - \check{g}(0) \big|^2 + \big\| f' - g' \big\|_{L^2}^2 + \big\| f_3 - g_3 \big\|_{L^2}^2 \Big)^{\frac{1}{2}},$$

we obtain the following statement for the original Landau-Lifshitz equation.

Corollary 1. *Let* $m^0 \in \mathcal{E}(\mathbb{R})$, *with* $\max_{\mathbb{R}} |m_3^0| < 1$. *Consider the corresponding hydrodynamical pair* $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$, *and denote by* $T_{\max} > 0$ *the maximal time of existence of the solution* $\mathfrak{v} \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$ *to* (HLL) *with initial datum* $\mathfrak{v}^0$, *which is provided by Theorem 1. There exists a map* $m \in \mathcal{C}^0([0, T_{\max}), \mathcal{E}(\mathbb{R}))$, *which satisfies the following statements.*
(i) *The hydrodynamical pair corresponding to* $m(x, t)$ *is well-defined for any* $(x, t) \in \mathbb{R} \times [0, T_{\max})$,

and is equal to $\mathfrak{v}(x,t)$.
(ii) The map m *is the unique solution to* (LL), *with initial datum* m^0, *such that there exist smooth solutions* $m_n \in \mathcal{C}^\infty(\mathbb{R} \times [0,T])$ *to* (LL), *which satisfy*

$$m_n \to m \quad \text{in } \mathcal{C}^0([0,T], \mathcal{E}(\mathbb{R})), \tag{10}$$

as $n \to +\infty$, *for any* $T \in (0, T_{\max})$.
(iii) The energy E *is constant along the flow.*
(iv) If

$$m_n^0 \to m^0 \quad \text{in } \mathcal{E}(\mathbb{R}),$$

as $n \to +\infty$, *then the solution* m_n *to* (LL) *with initial datum* m_n^0 *satisfies*

$$m_n \to m \quad \text{in } \mathcal{C}^0([0,T], \mathcal{E}(\mathbb{R})),$$

as $n \to +\infty$, *for any* $T \in (0, T_{\max})$.

Corollary 1 is nothing more than the translation of Theorem 1 in the original framework of the Landau-Lifshitz equation. It provides the existence of a unique continuous flow for (LL) in the neighbourhood of solutions m, such that the third component m_3 does not reach the value ± 1. The flow is only locally defined due to this restriction. This restriction is not a difficulty when dealing with the orbital stability of sums of solitons $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$, but a priori prevents from addressing the orbital stability of more general sums of solitons.

We refer to Appendix A for the proofs of Theorem 1 and Corollary 1. Concerning the Cauchy problem for the hydrodynamical Landau-Lifshitz equation, the main difficulty is to establish the continuity with respect to the initial datum in the energy space $\mathcal{NV}(\mathbb{R})$. In this direction, we rely on the strategy developed by Chang, Shatah and Uhlenbeck in [7] for the Schrödinger map equation (see also [16, 28]). We introduce the map

$$\Psi := \frac{1}{2}\Big(\frac{\partial_x v}{(1-v^2)^{\frac{1}{2}}} + i(1-v^2)^{\frac{1}{2}} w\Big) \exp i\theta, \tag{11}$$

where we have set

$$\theta(x,t) := -\int_{-\infty}^{x} v(y,t) w(y,t)\, dy. \tag{12}$$

The map Ψ solves the nonlinear Schrödinger equation

$$i\partial_t \Psi + \partial_{xx}\Psi + 2|\Psi|^2\Psi + \frac{1}{2}v^2\Psi - \operatorname{Re}\Big(\Psi\big(1 - 2F(v,\overline{\Psi})\big)\Big)\big(1 - 2F(v,\Psi)\big) = 0, \tag{13}$$

with

$$F(v,\Psi)(x,t) := \int_{-\infty}^{x} v(y,t)\Psi(y,t)\, dy, \tag{14}$$

while the function v satisfies the two equations

$$\begin{cases} \partial_t v = 2\partial_x \operatorname{Im}\Big(\Psi\big(2F(v,\overline{\Psi}) - 1\big)\Big), \\ \partial_x v = 2\operatorname{Re}\Big(\Psi\big(1 - 2F(v,\overline{\Psi})\big)\Big). \end{cases} \tag{15}$$

In this setting, deriving the continuous dependence in $\mathcal{NV}(\mathbb{R})$ of $\mathfrak{v}$ with respect to its initial datum reduces to establish it for v and Ψ in $L^2(\mathbb{R})$. This can be done combining a standard energy method for v, and classical Strichartz estimates for Ψ (see Appendix A for more details).

Concerning the stability of chains of solitons, our main result is

Theorem 2. *Let $\mathfrak{s}^* \in \{\pm 1\}^N$ and $\mathfrak{c}^* = (c_1^*, \ldots, c_N^*) \in (-1, 1)^N$ such that*

$$c_1^* < \ldots < 0 < \ldots < c_N^*. \tag{16}$$

There exist positive numbers α^*, L^* and A^*, depending only on $\mathfrak{c}^*$ such that, if $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$ satisfies the condition

$$\alpha^0 := \big\| \mathfrak{v}^0 - S_{\mathfrak{c}^*, \mathfrak{a}^0, \mathfrak{s}^*} \big\|_{H^1 \times L^2} \leq \alpha^*, \tag{17}$$

for points $\mathfrak{a}^0 = (a_1^0, \ldots, a_N^0) \in \mathbb{R}^N$ such that

$$L^0 := \min \big\{ a_{j+1}^0 - a_j^0, 1 \leq j \leq N - 1 \big\} \geq L^*,$$

then the solution $\mathfrak{v}$ to (HLL) *with initial datum $\mathfrak{v}^0$ is globally well-defined on $\mathbb{R}_+$, and there exists a function $\boldsymbol{a} = (a_1, \ldots, a_N) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$ such that*

$$\sum_{j=1}^N \big| a_j'(t) - c_j^* \big| \leq A^* \Big(\alpha^0 + \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L^0}{65} \Big) \Big), \tag{18}$$

and

$$\big\| \mathfrak{v}(\cdot, t) - S_{\mathfrak{c}^*, \boldsymbol{a}(t), \mathfrak{s}^*} \big\|_{H^1 \times L^2} \leq A^* \Big(\alpha^0 + \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L^0}{65} \Big) \Big), \tag{19}$$

for any $t \in \mathbb{R}_+$.

Theorem 2 provides the orbital stability of well-prepared chains of solitons with different, non-zero speeds for positive time. The chains are well-prepared in the sense that their positions at initial time are well-separated and ordered according to their speeds (see condition (16)). As a consequence, the solitons are more and more separated along the Landau-Lifshitz flow (see estimate (18)). Their interactions become weaker and weaker. The stability of the chain then results from the orbital stability of each single soliton in the chain.

As a matter of fact, the orbital stability of a single soliton appears as a special case of Theorem 2 when N is taken equal to 1.

Corollary 2. *Let $s^* \in \{\pm 1\}$ and $c^* \in (-1, 0) \cup (0, 1)$. There exist two positive numbers α^* and A^*, depending only on c^* such that, if $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$ satisfies the condition*

$$\alpha^0 := \big\| \mathfrak{v}^0 - \mathfrak{v}_{c^*, a^0, s^*} \big\|_{H^1 \times L^2} \leq \alpha^*,$$

for a point $a^0 \in \mathbb{R}$, then the solution $\mathfrak{v}$ to (HLL) *with initial datum $\mathfrak{v}^0$ is globally well-defined on $\mathbb{R}$, and there exists a function $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that*

$$\big| a'(t) - c^* \big| \leq A^* \alpha^0, \tag{20}$$

and

$$\big\| \mathfrak{v}(\cdot, t) - \mathfrak{v}_{c^*, a(t), s^*} \big\|_{H^1 \times L^2} \leq A^* \alpha^0, \tag{21}$$

for any $t \in \mathbb{R}$.

In this case, stability occurs for both positive and negative times due to the time reversibility of the Landau-Lifshitz equation. Time reversibility also provides the orbital stability of reversely well-prepared chains of solitons for negative time. The analysis of stability for both negative and positive time is more involved. It requires a deep understanding of the possible interactions between the solitons in the chain (see [22, 23] for such an analysis in the context of the Korteweg-de Vries equation). This issue is of particular interest because of the existence of multi-solitons (see e.g. [4, Section 10] for the explicit formula).

Special chains of solitons are indeed provided by the exact multi-solitons. However, there is a difficulty to define them properly in the hydrodynamical framework. As a matter of fact, multi-solitons can reach the values ± 1 at some times. On the other hand, an arbitrary multi-soliton becomes well-prepared for large time in the sense that the individual solitons are ordered according to their speeds and well-separated (see e.g. [4, Section 10]).

If we now consider a perturbation of a multi-soliton at initial time, there is no evidence that it remains a perturbation of a multi-soliton for large time. As a matter of fact, this property would follow from the continuity with respect to the initial datum of the (LL) flow, which remains, to our knowledge, an open question. Indeed, Corollary 1 only provides this continuity in the neighbourhood of solutions m, whose third component m_3 does not reach the value ± 1. As a consequence, Theorem 2 only shows the orbital stability of the multi-solitons, which do not reach the values ± 1 for any positive time. Well-prepared multi-solitons at initial time are examples of such multi-solitons. We can rephrase the orbital stability of the sums of solitons in the original formulation of the Landau-Lifshitz equation according to

Corollary 3. *Let $\mathfrak{s}^* \in \{\pm 1\}^N$ and $\mathfrak{c}^* = (c_1^*, \ldots, c_N^*) \in (-1, 1)^N$, with $c_1^* < \ldots < 0 < \ldots < c_N^*$. Given any positive number ϵ^*, there exist positive numbers ρ^* and L^* such that, if $m^0 \in \mathcal{E}(\mathbb{R})$ satisfies the condition*

$$d_{\mathcal{E}}\big(m^0, R_{\mathfrak{c}^*, \mathfrak{a}^0, \mathfrak{s}^*}\big) \leq \rho^*, \tag{22}$$

for points $\mathfrak{a}^0 = (a_1^0, \ldots, a_N^0) \in \mathbb{R}^N$ such that

$$\min\big\{a_{j+1}^0 - a_j^0, 1 \leq j \leq N-1\big\} \geq L^*, \tag{23}$$

then the solution m to (LL) *with initial datum m^0 is globally well-defined on $\mathbb{R}_+$. Moreover, there exists a function $\boldsymbol{a} = (a_1, \ldots, a_N) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$ such that, setting*

$$I_1 := \Big(-\infty, \frac{a_1 + a_2}{2}\Big], \quad I_j := \Big[\frac{a_{j-1} + a_j}{2}, \frac{a_j + a_{j+1}}{2}\Big], \quad \text{and} \quad I_N := \Big[\frac{a_{N-1} + a_N}{2}, +\infty\Big),$$

for $2 \leq j \leq N-1$, we have the estimates

$$\sum_{j=1}^N \big|a_j'(t) - c_j^*\big| \leq \epsilon^*, \tag{24}$$

and

$$\sum_{j=1}^N \inf_{\theta_j \in \mathbb{R}} \Big\{\big|\check{m}(a_j(t), t) - \check{u}_{c_j^*, a_j(t), \theta_j, s_j^*}(a_j(t))\big| + \big\|\partial_x m - u'_{c_j^*, a_j(t), \theta_j, s_j^*}\big\|_{L^2(I_j)} + \big\|m_3 - \big[u_{c_j^*, a_j(t), \theta_j, s_j^*}\big]_3\big\|_{L^2(I_j)}\Big\} \leq \epsilon^*, \tag{25}$$

for any $t \in \mathbb{R}_+$.

Corollary 3 only guarantees that a solution corresponding to a perturbation of a (well-prepared) sum of solitons $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ at initial time splits into localized perturbations of N solitons for any time. In particular, the solution does not necessarily remain a perturbation of a sum of solitons $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ for any time.

This difficulty is related to the main obstacle when constructing a function m corresponding to an hydrodynamical pair $\mathfrak{v}$, which is a possible phase shift of the map $\check{m}$. In the construction of the sum $R_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$, this phase shift is globally controlled. In contrast, the estimates into (19) do not seem to prevent a possible phase shift θ_j around each soliton in the hydrodynamical sum. This explains the difference between the controls in assumption (22) and in conclusion (25).

Observe also that we have no information on the dependence of the error ϵ^* on the numbers ρ^* and L^* in contrast with estimates (18) and (19) in Theorem 2. This is due to the property that the dependence of a function m with respect to the corresponding hydrodynamical pair $\mathfrak{v}$ is not a priori locally Lipschitz.

When $N = 1$, Corollary 3 states nothing more than the orbital stability of the solitons $u_{c,a,\theta,s}$, with $c \neq 0$. Taking into account the time reversibility of the Landau-Lifshitz equation, we can indeed show

Corollary 4. *Let $s^* \in \{\pm 1\}$ and $c^* \in (-1, 0) \cup (0, 1)$. Given any positive number ϵ^*, there exists a positive number ρ^* such that, if $m^0 \in \mathcal{E}(\mathbb{R})$ satisfies the condition*

$$d_{\mathcal{E}}\big(m^0, u_{c^*,a^0,\theta^0,s^*}\big) \leq \rho^*,$$

for numbers $(a^0, \theta^0) \in \mathbb{R}^2$, then the solution m to (LL) *with initial datum m^0 is globally well-defined on $\mathbb{R}_+$. Moreover, there exists a function $a \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ such that we have the estimates*

$$\big|a'(t) - c^*\big| \leq \epsilon^*,$$

and

$$\inf_{\theta \in \mathbb{R}} \big|\check{m}(a(t), t) - \check{u}_{c^*,a(t),\theta,s^*}(a(t))\big| + \big\|\partial_x m - u'_{c^*,a(t),\theta,s^*}\big\|_{L^2} + \big\|m_3 - \big[u_{c^*,a(t),\theta,s^*}\big]_3\big\|_{L^2} \leq \epsilon^*,$$

for any $t \in \mathbb{R}$.

To our knowledge, the orbital stability of the soliton u_0 remains an open question. In the context of the Gross-Pitaevskii equation, the orbital stability of the vanishing soliton was proved in [2, 13]. Part of the analysis in this further context certainly extends to the soliton u_0 for the Landau-Lifshitz equation.

In the rest of this introduction, we restrict our attention to the analysis of the stability of single solitons and sums of solitons in the hydrodynamical framework. In particular, we present below the main elements in the proof of Theorem 2. Before detailing this proof, we would like to underline that the arguments developed in the sequel do not make use of the inverse scattering transform. Instead, they rely on the Hamiltonian structure of the Landau-Lifshitz equation, in particular, on the conservation laws for the energy and momentum. As a consequence, our arguments can presumably be extended to non-integrable equations similar to the hydrodynamical Landau-Lifshitz equation.

1.4 Main elements in the proof of Theorem 2

Our strategy is reminiscent of the one developed to tackle the stability of well-prepared chains of solitons for the generalized Korteweg-de Vries equations [24], the nonlinear Schrödinger equations [25], or the Gross-Pitaevskii equation [3].

A key ingredient in the proof is the minimizing nature of the soliton $\mathfrak{v}_c$, which can be constructed as the solution of the minimization problem

$$E(\mathfrak{v}_c) = \min\big\{E(\mathfrak{v}), \mathfrak{v} \in \mathcal{NV}(\mathbb{R}) \text{ s.t. } P(\mathfrak{v}) = P(\mathfrak{v}_c)\big\}. \tag{26}$$

This characterization results from the compactness of the minimizing sequences for (26) on the one hand, and the classification of solitons in (5) on the other hand. The compactness of minimizing sequences can be proved following the arguments developed for a similar problem in the context of the Gross-Pitaevskii equation (see [1, Theorem 3]).

The Euler-Lagrange equation for (26) reduces to the identity

$$E'(\mathfrak{v}_c) = cP'(\mathfrak{v}_c). \tag{27}$$

The speed c appears as the Lagrange multiplier of the minimization problem. The minimizing energy is equal to

$$E(\mathfrak{v}_c) = 2(1 - c^2)^{\frac{1}{2}},$$

while the momentum of the soliton $\mathfrak{v}_c$ is given by

$$P(\mathfrak{v}_c) = \arctan\Big(\frac{(1 - c^2)^{\frac{1}{2}}}{c}\Big), \tag{28}$$

when $c \neq 0$. An important consequence of formula (28) is the inequality

$$\frac{d}{dc}\Big(P(\mathfrak{v}_c)\Big) = -\frac{1}{(1 - c^2)^{\frac{1}{2}}} < 0, \tag{29}$$

which is related to the Grillakis-Shatah-Strauss condition (see e.g. [14]) for the orbital stability of a soliton. As a matter of fact, we can use inequality (29) to establish the coercivity of the quadratic form

$$Q_c := E''(\mathfrak{v}_c) - cP''(\mathfrak{v}_c),$$

under suitable orthogonality conditions. More precisely, we show

Proposition 1. *Let $c \in (-1, 0) \cup (0, 1)$. There exists a positive number Λ_c, depending only on c, such that*

$$Q_c(\varepsilon) \geq \Lambda_c \|\varepsilon\|^2_{H^1 \times L^2}, \tag{30}$$

for any pair $\varepsilon \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying the two orthogonality conditions

$$\langle \partial_x \mathfrak{v}_c, \varepsilon \rangle_{L^2 \times L^2} = \langle P'(\mathfrak{v}_c), \varepsilon \rangle_{L^2 \times L^2} = 0. \tag{31}$$

Moreover, the map $c \mapsto \Lambda_c$ is uniformly bounded from below on any compact subset of $(-1, 1)\backslash\{0\}$.

The first orthogonality condition in (31) originates in the invariance with respect to translations of (HLL). Due to this invariance, the pair $\partial_x \mathfrak{v}_c$ lies in the kernel of Q_c. The quadratic form Q_c also owns a unique negative direction, which is related to the constraint in (26). This direction is controlled by the second orthogonality condition in (31).

As a consequence of Proposition 1, the functional

$$F_c(\mathfrak{v}) := E(\mathfrak{v}) - cP(\mathfrak{v}),$$

controls any perturbation $\varepsilon = \mathfrak{v} - \mathfrak{v}_c$ satisfying the two orthogonality conditions in (31). More precisely, we derive from (27) and (30) that

$$F_c(\mathfrak{v}_c + \varepsilon) - F_c(\mathfrak{v}_c) \geq \Lambda_c \|\varepsilon\|^2_{H^1 \times L^2} + \mathcal{O}\big(\|\varepsilon\|^3_{H^1 \times L^2}\big), \tag{32}$$

when $\|\varepsilon\|_{H^1 \times L^2} \to 0$. When $\mathfrak{v}$ is a solution to (HLL), its energy $E(\mathfrak{v})$ and its momentum $P(\mathfrak{v})$ are conserved along the flow. The left-hand side of (32) remains small for all time if it was small at initial time. As a consequence of (32), the perturbation ε remains small for all time, which implies the stability of $\mathfrak{v}_c$.

The strategy for proving Theorem 2 consists in extending this argument to a sum of solitons. This requires to derive a coercivity inequality in the spirit of (32) for the perturbation of a sum of solitons $\mathfrak{v}_{c_j,a_j,s_j}$. In a configuration where the solitons $\mathfrak{v}_{c_j,a_j,s_j}$ are sufficiently separated, a perturbation $\boldsymbol{\varepsilon}$, which is localized around the position a_k, essentially interacts with the soliton $\mathfrak{v}_{c_k,a_k,s_k}$ due to the exponential decay of the solitons. In order to extend (32), it is necessary to impose that $\boldsymbol{\varepsilon}$ satisfies at least the orthogonality conditions in (31) for the soliton $\mathfrak{v}_{c_k,a_k,s_k}$. In particular, we cannot hope to extend (32) to a general perturbation $\boldsymbol{\varepsilon}$ without imposing the orthogonality conditions in (31) for all the solitons in the sum.

It turns out that this set of orthogonal conditions is sufficient to derive a coercivity inequality like (32) when the solitons in the sum are well-separated (see Proposition 3 below). Before addressing this question, we have to handle with the usual tool to impose orthogonality conditions, that is modulation parameters. Here again, we take advantage of the exponential decay of the solitons to check that modulating their speeds and positions is enough to get the necessary orthogonality conditions, at least when the solitons are well-separated.

More precisely, we now fix a set of speeds $\mathfrak{c}^* = (c_1^*, \dots, c_N^*) \in (-1,1)^N$, with $c_j^* \neq 0$, and of orientations $\mathfrak{s}^* = (s_1^*, \dots, s_n^*) \in \{\pm 1\}^N$ as in the statement of Theorem 2. Given a positive number L, we introduce the set of well-separated positions

$$\mathrm{Pos}(L) := \big\{ \mathfrak{a} = (a_1, \dots, a_N) \in \mathbb{R}^N, \text{ s.t. } a_{j+1} > a_j + L \text{ for } 1 \leq j \leq N-1 \big\},$$

and we set

$$\mathcal{V}(\alpha, L) := \Big\{ \mathfrak{v} = (v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \inf_{\mathfrak{a} \in \mathrm{Pos}(L)} \big\| \mathfrak{v} - S_{\mathfrak{c}^*,\mathfrak{a},\mathfrak{s}^*} \big\|_{H^1 \times L^2} < \alpha \Big\},$$

for any $\alpha > 0$. We also define

$$\mu_{\mathfrak{c}} := \min_{1 \leq j \leq N} |c_j|, \quad \text{and} \quad \nu_{\mathfrak{c}} := \min_{1 \leq j \leq N} \big(1 - c_j^2 \big)^{\frac{1}{2}},$$

for any $\mathfrak{c} \in (-1,1)^N$. At least for α small enough and L sufficiently large, we show the existence of modulated speeds $\boldsymbol{c}(\mathfrak{v}) = (c_1(\mathfrak{v}), \dots, c_N(\mathfrak{v}))$ and positions $\boldsymbol{a}(\mathfrak{v}) = (a_1(\mathfrak{v}), \dots, a_N(\mathfrak{v}))$ such that any pair $\mathfrak{v} \in \mathcal{V}(\alpha, L)$ may be decomposed as $\mathfrak{v} = S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*} + \boldsymbol{\varepsilon}$, with $\boldsymbol{\varepsilon}$ satisfying suitable orthogonality conditions.

Proposition 2. *There exist positive numbers α_1^* and L_1^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that we have the following properties.*
(i) Any pair $\mathfrak{v} = (v,w) \in \mathcal{V}(\alpha_1^*, L_1^*)$ belongs to $\mathcal{NV}(\mathbb{R})$, with

$$1 - v^2 \geq \frac{1}{8} \mu_{\mathfrak{c}^*}^2. \tag{33}$$

(ii) There exist two maps $\boldsymbol{c} \in \mathcal{C}^1(\mathcal{V}(\alpha_1^*, L_1^*), (-1,1)^N)$ and $\boldsymbol{a} \in \mathcal{C}^1(\mathcal{V}(\alpha_1^*, L_1^*), \mathbb{R}^N)$ such that

$$\boldsymbol{\varepsilon} = \mathfrak{v} - S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*},$$

satisfies the orthogonality conditions

$$\big\langle \partial_x \mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}, \boldsymbol{\varepsilon} \big\rangle_{L^2 \times L^2} = \big\langle P'(\mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}), \boldsymbol{\varepsilon} \big\rangle_{L^2 \times L^2} = 0, \tag{34}$$

for any $1 \leq j \leq N$.
(iii) There exists a positive number A^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that, if

$$\big\| \mathfrak{v} - S_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*} \big\|_{H^1 \times L^2} < \alpha,$$

for $\mathfrak{a}^* \in \operatorname{Pos}(L)$*, with* $L > L_1^*$ *and* $\alpha < \alpha_1^*$*, then we have*

$$\|\boldsymbol{\varepsilon}\|_{H^1\times L^2} + \sum_{j=1}^N \big|c_j(\mathfrak{v}) - c_j^*\big| + \sum_{j=1}^N \big|a_j(\mathfrak{v}) - a_j^*\big| \leq A^*\alpha, \tag{35}$$

as well as

$$\boldsymbol{a}(\mathfrak{v}) \in \operatorname{Pos}(L-1), \quad \mu_{\boldsymbol{c}(\mathfrak{v})} \geq \frac{1}{2}\mu_{\mathfrak{c}^*} \quad \text{and} \quad \nu_{\boldsymbol{c}(\mathfrak{v})} \geq \frac{1}{2}\nu_{\mathfrak{c}^*}. \tag{36}$$

The next ingredient in the proof is to check the persistence of a coercivity inequality like (32) for the perturbation $\boldsymbol{\varepsilon}$ in Proposition 2. Once again, we rely on the property that the solitons $\mathfrak{v}_j := \mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}$ are well-separated and have exponential decay.

We indeed localize the perturbation $\boldsymbol{\varepsilon}$ around the position $a_j(\mathfrak{v})$ of each soliton $\mathfrak{v}_j$ by introducing cut-off functions, and we then control each localized perturbation using the coercivity of the quadratic form $Q_j = E''(\mathfrak{v}_j) - c_j(\mathfrak{v})P''(\mathfrak{v}_j)$ in (30). Such a control is allowed by the orthogonality conditions that we have imposed in (34). Collecting all the localized controls, we obtain a global bound on $\boldsymbol{\varepsilon}$, which is enough for our purpose.

More precisely, we consider a pair $\mathfrak{v} = (v,w) \in \mathcal{V}(\alpha_1^*, L_1^*)$, and we set

$$\boldsymbol{\varepsilon} = \mathfrak{v} - S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*},$$

as in Proposition 2, with $\boldsymbol{c}(\mathfrak{v}) = (c_1(\mathfrak{v}),\dots,c_N(\mathfrak{v}))$ and $\boldsymbol{a}(\mathfrak{v}) = (a_1(\mathfrak{v}),\dots,a_N(\mathfrak{v}))$. We next introduce the functions

$$\phi_j(x) := \begin{cases} 1 & \text{if } j = 1, \\ \frac{1}{2}\Big(1 + \tanh\Big(\frac{\nu_{\mathfrak{c}^*}}{16}\Big(x - \frac{a_{j-1}(\mathfrak{v}) + a_j(\mathfrak{v})}{2}\Big)\Big)\Big) & \text{if } 2 \leq j \leq N, \\ 0 & \text{if } j = N+1. \end{cases} \tag{37}$$

By construction, the maps $\phi_j - \phi_{j+1}$ are localized in a neighbourhood of the soliton $\mathfrak{v}_j$. Moreover, they form a partition of unity since they satisfy the identity

$$\sum_{j=1}^N (\phi_j - \phi_{j+1}) = 1. \tag{38}$$

Setting

$$\mathcal{F}(\mathfrak{v}) := E(\mathfrak{v}) - \sum_{j=1}^N c_j^* P_j(\mathfrak{v}), \tag{39}$$

where

$$P_j(\mathfrak{v}) := \int_{\mathbb{R}} (\phi_j - \phi_{j+1}) vw, \tag{40}$$

and following the strategy described above, we prove that the functional $\mathcal{F}$ controls the perturbation $\boldsymbol{\varepsilon}$ up to small error terms.

Proposition 3. *There exist positive numbers* $\alpha_2^* \leq \alpha_1^*$*,* $L_2^* \geq L_1^*$ *and* Λ^**, depending only on* $\mathfrak{c}^*$ *and* $\mathfrak{s}^*$*, such that* $\mathfrak{v} = S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*} + \boldsymbol{\varepsilon} \in \mathcal{V}(\alpha_2^*, L)$*, with* $L \geq L_2^*$*, satisfies the two inequalities*

$$\mathcal{F}(\mathfrak{v}) \geq \sum_{j=1}^N F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \Lambda^* \big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2 + \mathcal{O}\Big(\sum_{j=1}^N |c_j(\mathfrak{v}) - c_j^*|^2\Big) + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*}L}{16}\Big)\Big), \tag{41}$$

and

$$\mathcal{F}(\mathfrak{v}) \leq \sum_{j=1}^N F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \mathcal{O}\Big(\big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2\Big) + \mathcal{O}\Big(\sum_{j=1}^N |c_j(\mathfrak{v}) - c_j^*|^2\Big) + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*}L}{16}\Big)\Big). \tag{42}$$

Remark 3. Here as in the sequel, we have found convenient to use the notation $\mathcal{O}$ in order to simplify the presentation. By definition, we are allowed to substitute a quantity X by the notation $\mathcal{O}(Y)$ if and only if there exists a positive number A^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that

$$|X| \leq A^* Y.$$

In order to establish the stability of a sum of solitons with respect to the Landau-Lifshitz flow, we now consider an initial datum $\mathfrak{v}^0 \in \mathcal{V}(\alpha/2, 2L)$, with $\alpha \leq \alpha_2^*$ and $L \geq L_2^*$. Invoking the continuity of the flow with respect to the initial datum (see Theorem 1), we can assume the existence of a positive number T such that

$$\mathfrak{v}(\cdot, t) \in \mathcal{V}(\alpha, L) \subset \mathcal{V}(\alpha_2^*, L_2^*),$$

for any $t \in [0, T]$. As a consequence, we can specialize the statements in Propositions 2 and 3 to the pair $\mathfrak{v}(\cdot, t)$. We define

$$\mathfrak{c}(t) := \boldsymbol{c}(\mathfrak{v}(\cdot, t)) := \big(c_1(t), \ldots, c_N(t)\big), \quad \text{and} \quad \mathfrak{a}(t) := \boldsymbol{a}(\mathfrak{v}(\cdot, t)) := \big(a_1(t), \ldots, a_N(t)\big),$$

as well as

$$\boldsymbol{\varepsilon}(\cdot, t) := \big(\varepsilon_1(\cdot, t), \varepsilon_2(\cdot, t)\big) = \mathfrak{v}(\cdot, t) - S_{\mathfrak{c}(t), \mathfrak{a}(t), \mathfrak{s}^*}, \tag{43}$$

for any $t \in [0, T]$. In view of Proposition 2, we have

$$\|\boldsymbol{\varepsilon}(\cdot, t)\|_{H^1 \times L^2} + \sum_{j=1}^N \big|c_j(t) - c_j^*\big| + \sum_{j=1}^N \big|a_j(t) - a_j^*\big| \leq A^* \alpha, \tag{44}$$

and

$$\boldsymbol{a}(t) \in \operatorname{Pos}(L-1), \quad \mu_{\boldsymbol{c}(t)} \geq \frac{1}{2} \mu_{\mathfrak{c}^*}, \quad \text{and} \quad \nu_{\boldsymbol{c}(t)} \geq \frac{1}{2} \nu_{\mathfrak{c}^*}. \tag{45}$$

Similarly, Proposition 3 provides

$$\begin{aligned} \mathcal{F}(t) := \mathcal{F}(\mathfrak{v}(\cdot, t)) \geq \sum_{j=1}^N F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \Lambda^* \big\|\boldsymbol{\varepsilon}(\cdot, t)\big\|_{H^1 \times L^2}^2 + \mathcal{O}\Big(\sum_{j=1}^N |c_j(t) - c_j^*|^2\Big) \\ + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{16}\Big)\Big). \end{aligned}$$

Coming back to the strategy developed for the orbital stability of a single soliton (see the discussion after inequality (32)), we observe two major differences between the coercivity estimates (32) and (41). The first one lies in the two extra terms in the right-hand side of (41). There is no difficulty to control the second term, namely $\mathcal{O}(L \exp(-\nu_{\mathfrak{c}^*} L/16))$, since it becomes small when L is large enough. In contrast, we have to deal with the differences $|c_j(t) - c_j^*|^2$. In order to bound them, we rely on the equation satisfied by the perturbation $\boldsymbol{\varepsilon}$. Introducing identity (43) into (HLL) and using (27), we are led to the equations

$$\partial_t \varepsilon_1 = \sum_{j=1}^N \Big(\big(a_j'(t) - c_j(t)\big) \partial_x v_j - c_j'(t) \partial_c v_j\Big) + \partial_x \Big(\big((V + \varepsilon_1)^2 - 1\big)(W + \varepsilon_2) - \sum_{j=1}^N \big(v_j^2 - 1\big) w_j\Big), \tag{46}$$

and

$$\begin{aligned} \partial_t \varepsilon_2 = & \sum_{j=1}^N \Big(\big(a_j'(t) - c_j(t)\big) \partial_x w_j - c_j'(t) \partial_c w_j\Big) + \partial_{xx} \Big(\frac{\partial_x V + \partial_x \varepsilon_1}{1 - (V + \varepsilon_1)^2} - \sum_{j=1}^N \frac{\partial_x v_j}{1 - v_j^2}\Big) \\ & + \partial_x \Big(\big(V + \varepsilon_1\big)\big((W + \varepsilon_2)^2 - 1\big) - \frac{(V + \varepsilon_1)(\partial_x V + \partial_x \varepsilon_1)^2}{(1 - (V + \varepsilon_1)^2)^2} - \sum_{j=1}^N \Big(v_j \big(w_j^2 - 1\big) - \frac{v_j (\partial_x v_j)^2}{(1 - v_j^2)^2}\Big)\Big). \end{aligned} \tag{47}$$

Here, we have set $v_j(\cdot,t) := v_{c_j(t),a_j(t),s_j^*}(\cdot)$ and $w_j(\cdot,t) := w_{c_j(t),a_j(t),s_j^*}(\cdot)$ for any $1 \leq j \leq N$, as well as

$$V(\cdot,t) = V_{\boldsymbol{c}(t),\boldsymbol{a}(t),\mathfrak{s}^*}(\cdot) = \sum_{j=1}^N v_j(\cdot,t), \quad \text{and} \quad W(\cdot,t) = W_{\boldsymbol{c}(t),\boldsymbol{a}(t),\mathfrak{s}^*}(\cdot) = \sum_{j=1}^N w_j(\cdot,t),$$

in order to simplify the notation. We next differentiate with respect to time the orthogonality conditions in (34) to derive bounds on the time derivatives $a_j'(t)$ and $c_j'(t)$ of the modulation parameters. This provides

Proposition 4. *There exist positive numbers $\alpha_3^* \leq \alpha_2^*$ and $L_3^* \geq L_2^*$, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that, if $\alpha \leq \alpha_3^*$ and $L \geq L_3^*$, then the modulation functions $\boldsymbol{a}$ and $\boldsymbol{c}$ are of class $\mathcal{C}^1$ on $[0,T]$, and satisfy*

$$\sum_{j=1}^N \Big(\big| a_j'(t) - c_j(t) \big| + \big| c_j'(t) \big| \Big) = \mathcal{O}\Big(\big\| \boldsymbol{\varepsilon}(\cdot,t) \big\|_{H^1 \times L^2} \Big) + \mathcal{O}\Big(L \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{2} \Big) \Big), \tag{48}$$

for any $t \in [0,T]$.

Combining Proposition 4 with the bounds in (35), we conclude that the evolution of the modulation parameters is essentially governed by the initial speeds of the solitons in the sum $S_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*}$. In particular, when the speeds are well-ordered, that is when

$$c_1^* < \ldots < c_N^*, \tag{49}$$

the solitons in the sum $S_{\mathfrak{c}(t),\mathfrak{a}(t),\mathfrak{s}^*}$ remain well-separated for any $t \in [0,T]$. More precisely, setting

$$\delta_{\mathfrak{c}^*} = \frac{1}{2} \min \big\{ c_{j+1}^* - c_j^*, 1 \leq j \leq N-1 \big\},$$

we can derive from (35), (48) and (49), for a possible further choice of the numbers α_3^* and L_3^*, the estimates

$$a_{j+1}(t) - a_j(t) > a_{j+1}(0) - a_j(0) + \delta_{\mathfrak{c}^*} t \geq L - 1 + \delta_{\mathfrak{c}^*} t, \tag{50}$$

and

$$a_j'(t)^2 \leq 1 - \frac{\nu_{\mathfrak{c}^*}^2}{4}, \tag{51}$$

for any $t \in [0,T]$, when $\alpha \leq \alpha_3^*$ and $L \geq L_3^*$. In view of these bounds and the exponential decay of the solitons, the interactions between the solitons remain exponentially small for any $t \in [0,T]$.

A second difference between (32) and (41) lies in the fact that the left-hand side of (41) is not conserved along the (HLL) flow due to the presence of the cut-off function $\phi_j - \phi_{j+1}$ in the definition of P_j. As a consequence, we also have to control the evolution with respect to time of these quantities. We derive this control from the conservation law for the momentum, which may be written as

$$\partial_t (vw) = -\frac{1}{2} \partial_x \Big(v^2 + w^2 (1 - 3v^2) + \frac{3 - v^2}{(1 - v^2)^2} (\partial_x v)^2 \Big) - \frac{1}{2} \partial_{xxx} \ln \big(1 - v^2 \big). \tag{52}$$

As a consequence of this equation, we obtain a monotonicity formula for a localized version of the momentum. More precisely, we set

$$R_j(t) = \int_{\mathbb{R}} \phi_j(\cdot,t) v(\cdot,t) w(\cdot,t), \tag{53}$$

for any $1 \leq j \leq N$. Using (52), we establish

Proposition 5. *There exist positive numbers $\alpha_4^* \leq \alpha_3^*$, $L_4^* \geq L_3^*$ and A_4^*, depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$, such that, if $\alpha \leq \alpha_4^*$ and $L \geq L_4^*$, then the map R_j is of class $\mathcal{C}^1$ on $[0,T]$, and it satisfies*

$$R_j'(t) \geq -A_4^* \exp\Big(- \frac{\nu_{\mathfrak{c}^*}(L + \delta_{\mathfrak{c}^*} t)}{32}\Big), \tag{54}$$

for any $1 \leq j \leq N$ and any $t \in [0,T]$. In particular, the map $\mathcal{F}$ is of class $\mathcal{C}^1$ on $[0,T]$ and it satisfies

$$\mathcal{F}'(t) \leq \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*}(L + \delta_{\mathfrak{c}^*} t)}{32}\Big)\Big), \tag{55}$$

for any $t \in [0,T]$.

Estimate (55) is enough to overcome the fact that the function $\mathcal{F}$ is not any longer conserved along time. We now have all the elements to complete the proof of Theorem 2 applying the strategy developed for the orbital stability of a single soliton.

1.5 End of the proof of Theorem 2

In order to control the growth with respect to time of $\varepsilon(\cdot,t)$, we first take advantage of the monotonicity formulae in Proposition 5. They provide a control on the evolution between time 0 and time t of the momentum $R_j(t)$ at the right of the position $(a_{j-1}(t) + a_j(t))/2$. More precisely, the integration of (54) on $[0,t]$ leads to the inequality

$$R_j(0) - R_j(t) \leq \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big). \tag{56}$$

Since $P_j = R_j - R_{j+1}$ by (40) and (53), we deduce from (39), (49) and the conservation of E and P along the flow that

$$\begin{aligned} \big(c_j^* - c_{j-1}^*\big)\big(R_j(t) - R_j(0)\big) &= \mathcal{F}(0) - \mathcal{F}(t) + \sum_{2 \leq k \neq j \leq N} \big(c_k^* - c_{k-1}^*\big)\big(R_k(0) - R_k(t)\big) \\ &\leq \mathcal{F}(0) - \mathcal{F}(t) + \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big). \end{aligned} \tag{57}$$

In view of Proposition 3, we also have

$$\begin{aligned} \mathcal{F}(0) - \mathcal{F}(t) \leq &\mathcal{O}\big(\|\varepsilon(\cdot,0)\|_{H^1 \times L^2}^2\big) + \mathcal{O}\Big(\sum_{j=1}^N \big|c_j(0) - c_j^*\big|^2\Big) + \mathcal{O}\Big(\sum_{j=1}^N \big|c_j(t) - c_j^*\big|^2\Big) \\ &+ \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big). \end{aligned}$$

Plugging this estimate into (57), combining with (56), and using (35), we are led to the bound

$$\sum_{j=2}^N \big|R_j(t) - R_j(0)\big| \leq \mathcal{O}\big(|\alpha^0|^2\big) + \mathcal{O}\Big(\sum_{j=1}^N \big|c_j(t) - c_j^*\big|^2\Big) + \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big), \tag{58}$$

where α^0 is defined in (17).

Controlling the evolution of the momentum $R_j(t)$ at the right of all the positions $(a_{j-1}(t) + a_j(t))/2$ is enough to control the evolution of the momentum $P_j(t)$ between two of these positions. This follows from definitions (37), (40) and (53), which can be combined with (58) to obtain

$$\begin{aligned} \big|P_j(t) - P_j(0)\big| &\leq \big|R_j(t) - R_j(0)\big| + \big|R_{j+1}(t) - R_{j+1}(0)\big| \\ &\leq \mathcal{O}\big(|\alpha^0|^2\big) + \mathcal{O}\Big(\sum_{j=1}^N \big|c_j(t) - c_j^*\big|^2\Big) + \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big), \end{aligned} \tag{59}$$

for any $2 \leq j \leq N$. The same estimate holds for $j = 1$ due to the conservation of momentum.

Recall now that, due to the exponential decay of the solitons, the quantities $P_j(t)$ are essentially equal to the momentum of the soliton $\mathfrak{v}_{c_j(t)}$, when $\varepsilon(\cdot, t)$ is small. This claim is a consequence of the Taylor formula, which can be applied as in the proof of Proposition 3 (see Claim 2 in Subsection 2.3) to obtain

$$P_j(t) = P(\mathfrak{v}_{c_j(t)}) + \mathcal{O}\big(\|\varepsilon(\cdot, t)\|_{H^1 \times L^2}^2\big) + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{16}\Big)\Big).$$

In view of (59), we are led to

$$\begin{aligned}\big|P(\mathfrak{v}_{c_j(t)}) - P(\mathfrak{v}_{c_j(0)})\big| \leq \mathcal{O}\big(|\alpha^0|^2\big) + \mathcal{O}\big(\|\varepsilon(\cdot, t)\|_{H^1 \times L^2}^2\big) + \mathcal{O}\Big(\sum_{j=1}^N \big|c_j(t) - c_j^*\big|^2\Big)\\ + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big).\end{aligned} \tag{60}$$

At this stage, we make use of the explicit formula (28) of the momentum $P(\mathfrak{v}_c)$ to control the evolution with respect to time of the speeds $c_j(t)$. Combining (29) and (45), we write

$$\big|c_j(t) - c_j(0)\big| = \mathcal{O}\Big(\big|P(\mathfrak{v}_{c_j(t)}) - P(\mathfrak{v}_{c_j(0)})\big|\Big),$$

so that, by (35) and (60),

$$\begin{aligned}\sum_{j=1}^N \big|c_j(t) - c_j^*\big| \leq & \sum_{j=1}^N \Big(\big|c_j(t) - c_j(0)\big| + \big|c_j^* - c_j(0)\big|\Big),\\ \leq & \mathcal{O}\big(\alpha^0\big) + \mathcal{O}\big(\|\varepsilon(\cdot, t)\|_{H^1 \times L^2}^2\big) + \mathcal{O}\Big(\sum_{j=1}^N \big|c_j(t) - c_j^*\big|^2\Big) + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big).\end{aligned}$$

In view of (44), we can decrease the value of α_4^*, if necessary, so that

$$\sum_{j=1}^N \big|c_j(t) - c_j^*\big| \leq \mathcal{O}\big(\alpha^0\big) + \mathcal{O}\big(\|\varepsilon(\cdot, t)\|_{H^1 \times L^2}^2\big) + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big). \tag{61}$$

In order to bound $\varepsilon(\cdot, t)$, we next combine the coercivity formula in Proposition 3 and the monotonicity formula in Proposition 5 to obtain

$$\begin{aligned}\Lambda^* \big\|\varepsilon(\cdot, t)\big\|_{H^1 \times L^2}^2 \leq & \mathcal{F}(t) - \mathcal{F}(0) + \mathcal{O}\big(|\alpha^0|^2\big) + \mathcal{O}\big(\|\varepsilon(\cdot, t)\|_{H^1 \times L^2}^4\big) + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big)\\ \leq & \mathcal{O}\big(|\alpha^0|^2\big) + \mathcal{O}\big(\|\varepsilon(\cdot, t)\|_{H^1 \times L^2}^4\big) + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big).\end{aligned}$$

Decreasing again α_4^*, if necessary, we infer from (44) that

$$\big\|\varepsilon(\cdot, t)\big\|_{H^1 \times L^2}^2 \leq \mathcal{O}\big(|\alpha^0|^2\big) + \mathcal{O}\Big(\exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{32}\Big)\Big), \tag{62}$$

for any $t \in [0, T]$.

In order to conclude the proof, we finally apply a continuation argument. We set

$$T^* = \sup\big\{t \in \mathbb{R}_+,\ \text{s.t. } \mathfrak{v}(\cdot, s) \in \mathcal{V}(\alpha_4^*, L^0 - 2),\ \forall s \in [0, t]\big\}.$$

When the numbers α^* and L^* in Theorem 2 are chosen such that $\alpha^* < \alpha_4^*$ and $L^* > L_4^* + 2$, it follows from the continuity of the flow that T^* is positive. Moreover, since $\mathcal{V}(\alpha_4^*, L^0 - 2)$ is included into $\mathcal{NV}(\mathbb{R})$ by (33), we also have $T^* \leq T_{\max}$.

We next invoke (61) and (62) to guarantee the existence of a positive number K^* such that

$$\big\|\mathfrak{v}(\cdot,t) - S_{\mathfrak{c}^*,\boldsymbol{a}(t),\mathfrak{s}^*}\big\|_{H^1\times L^2} \leq K^*\Big(\alpha^0 + \exp\Big(-\frac{\nu_{\mathfrak{c}^*}L^0}{65}\Big)\Big), \tag{63}$$

for any $t \in [0, T^*)$. On the other hand, we combine the definition of L^0 with (36) to check that

$$\min\big\{a_{j+1}(0) - a_j(0), 1 \leq j \leq N-1\big\} \geq L^0 - 1.$$

In view of (50), this is enough to prove that

$$\min\big\{a_{j+1}(t) - a_j(t), 1 \leq j \leq N-1\big\} \geq L^0 - 2,$$

for any $t \in [0, T^*)$. It then remains to choose numbers α^* and L^* so that

$$K^*\Big(\alpha^* + \exp\Big(-\frac{\nu_{\mathfrak{c}^*}L^*}{65}\Big)\Big) \leq \alpha_4^*,$$

to guarantee that $T^* = T_{\max} = +\infty$. As a consequence, the solution $\mathfrak{v}$ is globally defined on $\mathbb{R}_+$, and it satisfies (19) due to (63). We finally derive (18) from (48), (61) and (63). This completes the proof of Theorem 2. □

1.6 Proof of Corollary 2

The proof is almost completely contained in the proof of Theorem 2. For the sake of completeness, we detail the following differences.

When $N = 1$, we do not take into account any longer the distances between the solitons. As a consequence, the sets $\mathcal{V}(\alpha, L)$ are replaced by the sets

$$\mathcal{W}(\alpha) := \Big\{\mathfrak{v} = (v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \inf_{a\in\mathbb{R}} \big\|\mathfrak{v} - \mathfrak{v}_{c^*,a,s^*}\big\|_{H^1\times L^2} < \alpha\Big\}.$$

When $\alpha^0 < \alpha_4^*$, we can invoke the continuity with respect to the initial datum of the (HLL) flow to find a positive time T such that the solution $\mathfrak{v}(\cdot,t)$ remains in $\mathcal{W}(\alpha_4^*)$ for any $t \in [0,T]$. In this situation, we can check that the proof of Proposition 2 provides the existence of modulation functions a and c of class $\mathcal{C}^1$ on $[0,T]$ such that the perturbation $\boldsymbol{\varepsilon}(\cdot,t) := \mathfrak{v}(\cdot,t) - \mathfrak{v}_{c(t),a(t),s^*}$ satisfies the orthogonality conditions in (34) for any $t \in [0,T]$, as well as the estimates

$$\|\boldsymbol{\varepsilon}(\cdot,t)\|_{H^1\times L^2} + |c(t) - c^*| + |a(t) - a^*| \leq A^*\alpha(t), \quad \mu_{c(t)} \geq \frac{1}{2}\mu_{c^*}, \quad \text{and} \quad \nu_{c(t)} \geq \frac{1}{2}\nu_{c^*}, \tag{64}$$

where $\alpha(t) := \inf_{a\in\mathbb{R}} \|\mathfrak{v}(\cdot,t) - \mathfrak{v}_{c^*,a,s^*}\|_{H^1\times L^2}$. Similarly, we derive from the proof of Proposition 3 that

$$F_{c^*}(\mathfrak{v}(\cdot,t)) \geq F_{c*}(\mathfrak{v}_{c^*}) + \Lambda^*\big\|\boldsymbol{\varepsilon}(\cdot,t)\big\|_{H^1\times L^2}^2 + \mathcal{O}\Big(|c(t) - c^*|^2\Big).$$

The quantity F_{c^*} is now conserved along the (HLL) flow. In particular, it is enough to control the difference $|c(t) - c^*|$ in order to provide a control on $\boldsymbol{\varepsilon}(\cdot,t)$ depending only on the initial perturbation. Arguing as in the proof of (61), we obtain the estimate

$$|c(t) - c^*| \leq \mathcal{O}\big(\alpha^0\big) + \mathcal{O}\big(\|\boldsymbol{\varepsilon}(\cdot,t)\|_{H^1\times L^2}^2\big), \tag{65}$$

so that we are led to

$$\Lambda^*\big\|\boldsymbol{\varepsilon}(\cdot,t)\big\|_{H^1\times L^2}^2 \leq F_{c^*}(\mathfrak{v}^0) - F_{c*}(\mathfrak{v}_{c^*}) + \mathcal{O}\big(|\alpha^0|^2\big).$$

In view of (64) and (65), we conclude that there exists a positive number A^* such that

$$|c(t) - c^*| + \|\boldsymbol{\varepsilon}(\cdot,t)\big\|_{H^1\times L^2} \leq A^*\alpha^0,$$

for any $t\in[0,T]$. Hence, we obtain (21) on $[0,T]$, possibly for a further choice of A^*. Choosing α^* such that $(A^*+1)\alpha^* < \alpha_4^*$, and applying a continuation argument as in the proof of Theorem 2, we derive (21) on $[0,+\infty)$, as soon as $\alpha^0 < \alpha^*$. This estimate also holds on $(-\infty,0]$ due to the time reversibility of the Landau-Lifshitz equation.

It then remains to observe that estimate (20) is a direct consequence of the proof of Proposition 4 to conclude the proof of Corollary 2. □

1.7 Outline of the paper

The paper is organized as follows. Section 2 is devoted to the proof of the stabilizing properties of a chain of solitons, more precisely, the proofs of Propositions 1, 2 and 3. In Section 3, we consider dynamical aspects: the control on the evolution of the modulation parameters in Proposition 4, and the derivation of the monotonicity formulae in Proposition 5. In Section 4, we provide the proof of Corollaries 3 and 4 concerning orbital stability in the original framework of Landau-Lifshitz equation. Finally, we give further details on the Cauchy problems for (LL) and (HLL) in a separate appendix.

2 Stabilizing properties of a chain of solitons

2.1 Proof of Proposition 1

The proof is reminiscent of the one in [3, Proposition 1]. For the sake of completeness, we provide the following details. In view of (1) and (2), the quadratic form Q_c is given by

$$\begin{split} Q_c(\boldsymbol{\varepsilon}) = \int_{\mathbb{R}} \bigg(\frac{(\partial_x \varepsilon_1)^2}{1-v_c^2} - \Big(2v_c\partial_{xx}v_c + (\partial_x v_c)^2 + 4\frac{v_c^2(\partial_x v_c)^2}{1-v_c^2}\Big)\frac{\varepsilon_1^2}{(1-v_c^2)^2} \\ + \big(1-w_c^2\big)\varepsilon_1^2 - 2\big(c+2v_cw_c\big)\varepsilon_1\varepsilon_2 + \big(1-v_c^2\big)\varepsilon_2^2\bigg), \end{split} \tag{2.1}$$

for any pair $\boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2)\in H^1(\mathbb{R})\times L^2(\mathbb{R})$. This quantity is well-defined when $c\neq 0$, in particular, due to the identity

$$\min_{\mathbb{R}}\big\{1-v_c^2\big\} = c^2, \tag{2.2}$$

which is a consequence of (5). Recall also that v_c solves the equation

$$\partial_{xx}v_c = \big(1-c^2-2v_c^2\big)v_c,$$

while w_c is given by

$$w_c = \frac{c\,v_c}{1-v_c^2}.$$

Therefore, we can rewrite the expression in (2.1) under the form

$$Q_c(\boldsymbol{\varepsilon}) = \big\langle \mathcal{L}_c(\varepsilon_1),\varepsilon_1\big\rangle_{L^2} + \int_{\mathbb{R}}\big(1-v_c^2\big)\Big(\varepsilon_2 - c\frac{1+v_c^2}{(1-v_c^2)^2}\varepsilon_1\Big)^2, \tag{2.3}$$

where $\mathcal{L}_c$ refers to the Sturm-Liouville operator defined by

$$\mathcal{L}_c(\varepsilon_1) = -\partial_x\Big(\frac{\partial_x\varepsilon_1}{1-v_c^2}\Big) + \Big(1 - c^2 - (5+c^2)v_c^2 + 2v_c^4\Big)\frac{\varepsilon_1}{(1-v_c^2)^2}.$$

The unbounded operator $\mathcal{L}_c$ is self-adjoint on $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$, and, by the Weyl criterion, with essential spectrum $[1-c^2, +\infty)$. Due to the invariance with respect to translations of (HLL), the derivative $\partial_x v_c$ lies in the kernel of $\mathcal{L}_c$. Since this function has exactly one zero, it follows from the Sturm-Liouville theory (see e.g. [10]) that $\mathcal{L}_c$ owns a unique negative eigenvalue $-\lambda_c$. Moreover, the corresponding eigenspace, as well as the kernel of $\mathcal{L}_c$, have dimension one. We denote by χ_c an eigenfunction of $\mathcal{L}_c$ for the eigenvalue $-\lambda_c$.

In view of (2.3), the unbounded operator $\mathcal{Q}_c$ corresponding to the quadratic form Q_c is given by

$$\mathcal{Q}_c(\boldsymbol{\varepsilon}) = \Big(\mathcal{L}_c(\varepsilon_1) + c^2\frac{(1+v_c^2)^2}{(1-v_c^2)^3}\varepsilon_1 - c\frac{1+v_c^2}{1-v_c^2}\varepsilon_2, -c\frac{1+v_c^2}{1-v_c^2}\varepsilon_1 + (1-v_c^2)\varepsilon_2\Big).$$

It is self-adjoint on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, with domain $H^2(\mathbb{R}) \times L^2(\mathbb{R})$, and by the Weyl criterion, with essential spectrum $[\min\{1-|c|, 3/4-c^2\}, +\infty)$. In view of (2.3), the quadratic form $Q_c(\boldsymbol{\varepsilon})$ is positive when $\boldsymbol{\varepsilon} \neq 0$ satisfies the two orthogonality conditions $\langle \chi_c, \varepsilon_1\rangle_{L^2} = \langle \partial_x v_c, \varepsilon_1\rangle_{L^2} = 0$. Moreover, the pair $\partial_x \mathfrak{v}_c$ lies in the kernel of $\mathcal{Q}_c$, while

$$Q_c\Big(\chi_c, c\frac{1+v_c^2}{(1-v_c^2)^2}\chi_c\Big) = \langle \mathcal{L}_c(\chi_c), \chi_c\rangle_{L^2\times L^2} < 0.$$

As a consequence, $\mathcal{Q}_c$ has exactly one negative eigenvalue $-\mu_c$, and its kernel is spanned by the derivative $\partial_x \mathfrak{v}_c$. In particular, there exists a positive number A_c, depending continuously on c (due to the analytic dependence on c of the operator $\mathcal{Q}_c$), such that

$$Q_c(\boldsymbol{\varepsilon}) \geq A_c \|\boldsymbol{\varepsilon}\|_{L^2\times L^2}^2, \tag{2.4}$$

when $\boldsymbol{\varepsilon}$ satisfies the two orthogonality conditions

$$\langle \mathfrak{u}_c, \boldsymbol{\varepsilon}\rangle_{L^2\times L^2} = \langle \partial_x \mathfrak{v}_c, \boldsymbol{\varepsilon}\rangle_{L^2\times L^2} = 0, \tag{2.5}$$

where $\mathfrak{u}_c$ refers to a $L^2 \times L^2$-normalized eigenfunction of $\mathcal{Q}_c$ for the eigenvalue $-\mu_c$.

We now check that inequality (2.4) remains valid, up to a possible further choice of the positive number A_c, when we replace the orthogonality conditions in (2.5) by the one in (31). With this goal in mind, we consider a pair $\boldsymbol{\varepsilon}$, which satisfies the orthogonality conditions in (31), and we decompose it as $\boldsymbol{\varepsilon} = a\mathfrak{u}_c + \mathfrak{r}$, where $\mathfrak{r}$ satisfies (2.5). Similarly, we decompose the derivative $\partial_c \mathfrak{v}_c = \alpha \mathfrak{u}_c + \mathfrak{r}_c$, with $\mathfrak{r}_c$ satisfying (2.5) [2]. We next compute

$$Q_c(\partial_c \mathfrak{v}_c) = -\mu_c \alpha^2 + Q_c(\mathfrak{r}_c).$$

On the other hand, differentiating (27) with respect to c, we obtain the identity

$$\langle \mathcal{Q}_c(\partial_c \mathfrak{v}_c) - P'(\mathfrak{v}_c), \mathfrak{w}\rangle_{L^2\times L^2} = 0, \tag{2.6}$$

for any $\mathfrak{w} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. When $\mathfrak{w} = \partial_c \mathfrak{v}_c$, we infer from (29) that

$$Q_c(\partial_c \mathfrak{v}_c) = \langle P'(\mathfrak{v}_c), \partial_c \mathfrak{v}_c\rangle_{L^2\times L^2} = -\frac{1}{(1-c^2)^{\frac{1}{2}}} < 0.$$

[2] Since $\partial_c \mathfrak{v}_c$ is an even pair, whereas $\partial_x \mathfrak{v}_c$ is odd, they are orthogonal in $L^2(\mathbb{R})^2$.

As a consequence, there exists a number $0 \leq \delta < 1$ such that

$$Q_c(\mathfrak{r}_c) = \delta \mu_c \alpha^2. \tag{2.7}$$

At this stage, two situations can occur. When $\delta = 0$, $\mathfrak{r}_c$ is equal to 0, and $\partial_c \mathfrak{v}_c$ is an eigenfunction of $\mathcal{Q}_c$ for the eigenvalue $-\mu_c$. In view of (2.6), the orthogonality conditions in (31) and (2.5) are equivalent, so that inequality (2.4) remains valid under the conditions in (31). In contrast, when $\delta > 0$, we write

$$Q_c(\boldsymbol{\varepsilon}) = -\mu_c a^2 + Q_c(\mathfrak{r}). \tag{2.8}$$

Since $\mathcal{Q}_c$ is positive under the orthogonality conditions in (2.5), we can apply the Cauchy-Schwarz inequality to get

$$\langle \mathcal{Q}_c(\mathfrak{r}_c), \mathfrak{r} \rangle_{L^2 \times L^2}^2 \leq Q_c(\mathfrak{r}) Q_c(\mathfrak{r}_c).$$

Since

$$\langle \mathcal{Q}_c(\mathfrak{r}_c), \mathfrak{r} \rangle_{L^2 \times L^2} = \langle \mathcal{Q}_c(\partial_c \mathfrak{v}_c), \boldsymbol{\varepsilon} \rangle_{L^2 \times L^2} + \mu_c a \alpha = \mu_c a \alpha,$$

by (31) and (2.6), we deduce from (2.7) that $Q_c(\mathfrak{r}) \geq \mu_c a^2/\delta$. In view of (2.4) and (2.8), we are led to

$$Q_c(\boldsymbol{\varepsilon}) \geq \frac{1-\delta}{2} \Big(\mu_c a^2 + A_c \|\mathfrak{r}\|_{L^2 \times L^2}^2 \Big) \geq B_c \|\boldsymbol{\varepsilon}\|_{L^2 \times L^2}^2,$$

with $B_c = (1-\delta) \min\{\mu_c, A_c\}/2$. In this case again, inequality (2.4) remains valid under conditions (31).

In order to complete the proof of (30), it remains to replace the $L^2 \times L^2$-norm in the right-hand side of (2.4) by an $H^1 \times L^2$-norm. In this direction, we observe that

$$Q_c(\boldsymbol{\varepsilon}) \geq \int_{\mathbb{R}} \frac{(\partial_x \varepsilon_1)^2}{1 - v_c^2} - K_c \|\boldsymbol{\varepsilon}\|_{L^2 \times L^2}^2 \geq \frac{1}{c^2} \int_{\mathbb{R}} (\partial_x \varepsilon_1)^2 - K_c \|\boldsymbol{\varepsilon}\|_{L^2 \times L^2}^2,$$

by (5) and (2.2), where K_c depends continuously on c. Given a number $0 < \tau < 1$, we deduce that

$$Q_c(\boldsymbol{\varepsilon}) \geq \frac{\tau}{c^2} \int_{\mathbb{R}} (\partial_x \varepsilon_1)^2 + \big(A_c(1-\tau) - K_c \tau \big) \|\boldsymbol{\varepsilon}\|_{L^2 \times L^2}^2,$$

under the orthogonality conditions in (31). It remains to choose $\tau = A_c/2(A_c + K_c)$, to obtain (30) for a positive number Λ_c depending continuously on c. □

2.2 Proof of Proposition 2

The proof is similar to the one performed for establishing [3, Proposition 2] (see also [24, 25]). For the sake of completeness, we recall the following elements.

The first ingredient is the exponential decay of the soliton $\mathfrak{v}_c = (v_c, w_c)$. In view of (5), given any integer p, there exists a positive number A_p, depending only on p, such that

$$\sum_{0 \leq 2j \leq k \leq p} \Big(\big|\partial_c^j \partial_x^k v_c(x)\big| + c^{1+2j+2k} \big|\partial_c^j \partial_x^k w_c(x)\big| \Big) \leq A_p (1-c^2)^{\frac{1}{2}} \exp\Big(- \frac{(1-c^2)^{\frac{1}{2}}}{2} |x| \Big), \tag{2.9}$$

for any $0 < |c| < 1$ and $x \in \mathbb{R}$. As a consequence, we can derive as in [3, Lemma 2.1] that two solitons with same speed and orientation, as well as two sums of solitons with same speeds and orientations, cannot be close in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, except if their centers of mass are close. More quantitatively, let us fix a set of speeds $\mathfrak{c}$, with $c_i \neq 0$ as usual, and a set of orientations $\mathfrak{s}$. Given a positive number ρ, there exist positive numbers β and M such that, if

$$\big\| S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} - S_{\mathfrak{c},\mathfrak{b},\mathfrak{s}} \big\|_{H^1 \times L^2} < \beta, \tag{2.10}$$

for positions $\mathfrak{a} \in \mathrm{Pos}(M)$ and $\mathfrak{b} \in \mathrm{Pos}(M)$, we have

$$\sum_{i=1}^{N} \big|a_i - b_i\big| < \rho. \tag{2.11}$$

The second crucial ingredient is related to the map

$$\Xi(\mathfrak{v}, \boldsymbol{\sigma}, \mathfrak{b}) := \Big(\langle \partial_x \mathfrak{v}_{\sigma_1, b_1, s_1^*}, \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}, \dots, \langle \partial_x \mathfrak{v}_{\sigma_N, b_N, s_N^*}, \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}, \langle P'(\mathfrak{v}_{\sigma_1, b_1, s_1^*}), \boldsymbol{\varepsilon}\rangle_{L^2\times L^2},$$
$$\dots, \langle P'(\mathfrak{v}_{\sigma_N, b_N, s_N^*}), \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}\Big),$$

where we have set, as usual, $\boldsymbol{\varepsilon} = \mathfrak{v} - S_{\boldsymbol{\sigma},\mathfrak{b},\mathfrak{s}^*}$. The map Ξ is well-defined for, and depends smoothly on $\mathfrak{v} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $\boldsymbol{\sigma} \in (-1, 1)$, with $\sigma_i \neq 0$, and $\mathfrak{b} \in \mathbb{R}^N$. In order to construct the mappings $\boldsymbol{c}$ and $\boldsymbol{a}$ in Proposition 2, we apply the quantitative version of the implicit function theorem in [3, Appendix A] to the map Ξ. This is possible due to the exponential decay in (2.9).

Indeed, set

$$\boldsymbol{\Sigma}(\tau) := \big\{\boldsymbol{\sigma} \in (-1, 1)^N, \text{ s.t. } \mu_{\boldsymbol{\sigma}} > \tau \text{ and } \nu_{\boldsymbol{\sigma}} > \tau\big\},$$

for a fixed number $0 < \tau < 1$. Given $\mathfrak{c} \in \boldsymbol{\Sigma}(\tau)$ and $\mathfrak{a} \in \mathbb{R}^N$, we check that

$$\Xi(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = 0,$$

and we compute

$$\begin{cases} \partial_{\sigma_j} \Xi_k(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = -\langle \partial_x \mathfrak{v}_{c_k, a_k, s_k^*}, \partial_c \mathfrak{v}_{c_j, a_j, s_j^*}\rangle_{L^2\times L^2}, \\ \partial_{b_j} \Xi_k(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = \langle \partial_x \mathfrak{v}_{c_k, a_k, s_k^*}, \partial_x \mathfrak{v}_{c_j, a_j, s_j^*}\rangle_{L^2\times L^2}, \end{cases}$$

as well as

$$\begin{cases} \partial_{\sigma_j} \Xi_{N+k}(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = -\langle P'(\mathfrak{v}_{c_k, a_k, s_k^*}), \partial_c \mathfrak{v}_{c_j, a_j, s_j^*}\rangle_{L^2\times L^2}, \\ \partial_{b_j} \Xi_{N+k}(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = \langle P'(\mathfrak{v}_{c_k, a_k, s_k^*}), \partial_x \mathfrak{v}_{c_j, a_j, s_j^*}\rangle_{L^2\times L^2}, \end{cases}$$

for any $1 \leq j, k \leq N$.

When $j = k$, we rely on (5) to derive

$$\partial_{\sigma_k} \Xi_k(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = \partial_{b_k} \Xi_{N+k}(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = 0, \tag{2.12}$$

while

$$\partial_{b_k} \Xi_k(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = \big\|\partial_x \mathfrak{v}_{c_k}\big\|_{L^2}^2 = 2(1 - c_k^2)^{\frac{1}{2}} > 0,$$

and, by (29),

$$\partial_{\sigma_k} \Xi_{N+k}(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a}) = -\frac{d}{dc}\big(P(\mathfrak{v}_c)\big)_{|c=c_k} = \frac{1}{(1 - c_k^2)^{\frac{1}{2}}} > 0.$$

Therefore, the diagonal matrix $D_{\mathfrak{c}}$ with the same diagonal elements as $d_{\boldsymbol{\sigma},\boldsymbol{b}}\Xi(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a})$ is a continuous isomorphism from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N}$, with operator norm bounded from below by 2τ.

When $j \neq k$, we invoke the exponential decay in (2.9) to check the existence of a positive number A_τ, depending only on τ, such that

$$\big|\partial_{\sigma_j} \Xi_k(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a})\big| + \big|\partial_{b_j} \Xi_k(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a})\big|$$
$$+ \big|\partial_{\sigma_j} \Xi_{N+k}(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a})\big| + \big|\partial_{b_j} \Xi_{N+k}(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*}, \mathfrak{c}, \mathfrak{a})\big| \leq A_\tau \exp\Big(-\frac{\nu_{\mathfrak{c}} L}{4}\Big),$$

when $a \in \mathrm{Pos}(L)$ for some positive number L. Combining with (2.12), we can write the differential $d_{\boldsymbol{\sigma},\boldsymbol{b}}\Xi(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\mathfrak{c},\mathfrak{a})$ as

$$d_{\boldsymbol{\sigma},\boldsymbol{b}}\Xi(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\mathfrak{c},\mathfrak{a}) = D_{\mathfrak{c}}\big(Id + T_1(\mathfrak{c},\mathfrak{a})\big),$$

where $T_1(\mathfrak{c},\mathfrak{a})$ has an operator norm less than $1/2$, at least, when L is large enough.

On the other hand, since the operator norm of $D_{\mathfrak{c}}$ is bounded from below by 2τ, we can write

$$d_{\mathfrak{v}}\Xi(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\mathfrak{c},\mathfrak{a})(\mathfrak{w}) = \Big(\langle\partial_x\mathfrak{v}_{c_1,a_1,s_1^*},\mathfrak{w}\rangle_{L^2\times L^2},\dots,\langle P'(\mathfrak{v}_{c_N,a_N,s_N^*}),\mathfrak{w}\rangle_{L^2\times L^2}\Big) = D_{\mathfrak{c}}T_2(\mathfrak{c},\mathfrak{a}),$$

where $T_2(\mathfrak{c},\mathfrak{a})$ is a continuous linear mapping from $H^1(\mathbb{R})\times L^2(\mathbb{R})$ to $\mathbb{R}^{2N}$ with an operator norm depending only on τ. In view of (2.9), the operator norm of the second order differential $d^2\Xi(\mathfrak{v},\mathfrak{c},\mathfrak{a})$ is also bounded by a positive number A_τ, depending only on τ, when $\mathfrak{v}\in H^1(\mathbb{R})\times L^2(\mathbb{R})$, $\mathfrak{c}\in\boldsymbol{\Sigma}(\tau/2)$ and $\mathfrak{a}\in\mathbb{R}^N$.

This is enough to apply [3, Proposition A.1] to the map Ξ. We set $\tau^* := \min\{\mu_{\mathfrak{c}^*}/2,\nu_{\mathfrak{c}^*}/2\}$. Then, there exist positive numbers ρ^*, Λ^* and L^* such that, for any $(\mathfrak{c},\mathfrak{a})\in\boldsymbol{\Sigma}(\tau^*)\times\mathrm{Pos}(L^*)$, there exists a map $\gamma_{\mathfrak{c},\mathfrak{a}}\in\mathcal{C}^1(B(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\rho^*),\boldsymbol{\Sigma}(\tau^*/2)\times\mathbb{R}^N)$ such that, given any $\mathfrak{w}\in B(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\rho^*)$, the pair $(\boldsymbol{\sigma},\mathfrak{b}) = \gamma_{\mathfrak{c},\mathfrak{a}}(\mathfrak{w})$ is the unique solution in $B(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\Lambda^*\rho^*)$ to the equation

$$\Xi(\mathfrak{w},\boldsymbol{\sigma},\mathfrak{b}) = 0. \tag{2.13}$$

Moreover, the map $\gamma_{\mathfrak{c},\mathfrak{a}}$ is Lipschitz on $B(S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}^*},\rho^*)$, with Lipschitz constant at most Λ^*.

We next denote by β^* the positive number such that (2.11) holds, when condition (2.10) is satisfied for $\rho := \Lambda^*\rho^*/3$, and we set $\alpha^* := \min\{\rho^*/3,\beta^*/4\}$. When $\mathfrak{v}\in\mathcal{V}(\alpha^*,L^*)$, there exists $\mathfrak{b}\in\mathrm{Pos}(L^*)$ such that $\mathfrak{v}\in B(S_{\mathfrak{c}^*,\mathfrak{b},\mathfrak{s}^*},2\alpha^*)$. Since $2\alpha^*\le\rho^*$, the numbers $\mathfrak{c}$ and $\mathfrak{a}$ given by

$$(\mathfrak{c},\mathfrak{a}) = \gamma_{\mathfrak{c}^*,\mathfrak{b}}(\mathfrak{v}),$$

are well-defined. We set $\boldsymbol{c}(\mathfrak{v}) = \mathfrak{c}$ and $\boldsymbol{a}(\mathfrak{v}) = \mathfrak{a}$, and we show that the functions $\boldsymbol{c}$ and $\boldsymbol{a}$ satisfy all the statements in Proposition 2.

Combining (2.11), the Lipschitz continuity of the maps $\gamma_{\mathfrak{c},\mathfrak{a}}$, and the local uniqueness of the solution to (2.13), we first check that $\mathfrak{c}$ and $\mathfrak{a}$ do not depend on the choice of $\mathfrak{b}\in\mathrm{Pos}(L^*)$ such that $\mathfrak{v}\in B(S_{\mathfrak{c}^*,\mathfrak{b},\mathfrak{s}^*},2\alpha^*)$. Hence, the functions $\boldsymbol{c}$ and $\boldsymbol{a}$ are well-defined from $\mathcal{V}(\alpha^*,L^*)$ to $\boldsymbol{\Sigma}(\tau^*/2)$, resp. $\mathbb{R}^N$. Moreover, they are of class $\mathcal{C}^1$ on $\mathcal{V}(\alpha^*,L^*)$ due to the $\mathcal{C}^1$ nature of the maps $\gamma_{\mathfrak{c},\mathfrak{a}}$, and again the local uniqueness of the solution to (2.13) (see the proof of [3, Proposition 2] for more details). Statement (ii) follows combining the definition of the map Ξ, and the identity $\Xi(\mathfrak{v},\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v})) = 0$, which holds for any $\mathfrak{v}\in\mathcal{V}(\alpha^*,L^*)$.

Concerning (iii), we deduce from the Lipschitz continuity of the map $\gamma_{\mathfrak{c}^*,\mathfrak{a}^*}$ that

$$\big|\boldsymbol{c}(\mathfrak{v})-\mathfrak{c}^*\big| + \big|\boldsymbol{a}(\mathfrak{v})-\mathfrak{a}^*\big| \le \big|\gamma_{\mathfrak{c}^*,\mathfrak{a}^*}(\mathfrak{v}) - \gamma_{\mathfrak{c}^*,\mathfrak{a}^*}(S_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*})\big| \le \Lambda^*\alpha, \tag{2.14}$$

when $\mathfrak{v}\in B(S_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*},\alpha)$ for some positive number $\alpha<\alpha^*$. As a consequence, we can decrease, if necessary, the value of α^* so that $\mu_{\boldsymbol{c}(\mathfrak{v})}\ge\mu_{\mathfrak{c}^*}/2$ and $\nu_{\boldsymbol{c}(\mathfrak{v})}\ge\nu_{\mathfrak{c}^*}/2$. Similarly, when $\mathfrak{a}^*\in\mathrm{Pos}(L)$ for $L>L^*$, we can assume that $\boldsymbol{a}(\mathfrak{v})\in\mathrm{Pos}(L-1)$.

On the other hand, we also have the following estimate of $\boldsymbol{\varepsilon} = \mathfrak{v} - S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*}$,

$$\big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2} \le \big\|\mathfrak{v}-S_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*}\big\|_{H^1\times L^2} + \big\|S_{\mathfrak{c}^*,\mathfrak{a}^*,\mathfrak{s}^*}-S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*}\big\|_{H^1\times L^2} \le \alpha + K\Big(\big|\mathfrak{c}^*-\boldsymbol{c}(\mathfrak{v})\big| + \big|\mathfrak{a}^*-\boldsymbol{a}(\mathfrak{v})\big|\Big),$$

due to the explicit formulae for $\mathfrak{v}_{\mathfrak{c}}$ in (5). In view of (2.14), this gives

$$\big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2} \le A^*\alpha,$$

for $A^* = 1 + K\Lambda^*$. In particular, we can again decrease, if necessary, the value of α^* in order to obtain

$$\big\|\varepsilon_1\big\|_{L^\infty} \leq \frac{1}{8}\mu_{\mathfrak{c}^*}^2, \tag{2.15}$$

by using the Sobolev embedding theorem. In view of (2.2) and (2.9), we can also increase, if necessary, the value of L^* in order to have

$$\min_{\mathbb{R}} \big\{1 - v_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*}^2\big\} \geq \frac{1}{2}\mu_{\mathfrak{c}^*}^2.$$

Combining with (2.15), this proves that $\mathcal{V}(\alpha^*, L^*)$ is included into $\mathcal{NV}(\mathbb{R})$, with inequality (33). It only remains to set $\alpha_1^* = \alpha^*$ and $L_1^* = L^*$ to conclude the proof of Proposition 2. □

2.3 Proof of Proposition 3

In order to establish the two inequalities in Proposition 3, we refine the partition of unity in (38). Given a positive parameter $\tau < \nu_{\mathfrak{c}^*}/16$ to be fixed later, we set

$$\chi_j(x) := \frac{1}{2}\Big(\tanh\Big(\tau\Big(x - a_j(\mathfrak{v}) + \frac{L_1^*}{4}\Big)\Big) - \tanh\Big(\tau\Big(x - a_j(\mathfrak{v}) - \frac{L_1^*}{4}\Big)\Big)\Big), \tag{2.16}$$

for $1 \leq j \leq N$, as well as

$$\begin{aligned}
\chi_{0,1}(x) &:= \frac{1}{2}\Big(1 - \tanh\Big(\tau\Big(x - a_1(\mathfrak{v}) + \frac{L_1^*}{4}\Big)\Big)\Big),\\
\chi_{j,j+1}(x) &:= \frac{1}{2}\Big(\tanh\Big(\tau\Big(x - a_j(\mathfrak{v}) - \frac{L_1^*}{4}\Big)\Big) - \tanh\Big(\tau\Big(x - a_{j+1}(\mathfrak{v}) + \frac{L_1^*}{4}\Big)\Big)\Big),\\
\chi_{N,N+1}(x) &:= \frac{1}{2}\Big(1 + \tanh\Big(\tau\Big(x - a_N(\mathfrak{v}) - \frac{L_1^*}{4}\Big)\Big)\Big),
\end{aligned}$$

with $1 \leq j \leq N-1$, so that we have the partition of unity

$$\sum_{j=1}^N \chi_j + \sum_{j=0}^N \chi_{j,j+1} = 1. \tag{2.17}$$

Since $|1 - \operatorname{sign}(x)\tanh(x)| \leq 2\exp(-2|x|)$ for any $x \in \mathbb{R}$, we check that

$$0 \leq \chi_j(x) \leq \exp\Big(-2\tau\Big(|x - a_j(\mathfrak{v})| - \frac{L_1^*}{4}\Big)^+\Big), \tag{2.18}$$

and

$$\big|1 - \chi_j(x)\big| \leq \exp\Big(-2\tau\Big(x - a_j(\mathfrak{v}) + \frac{L_1^*}{4}\Big)^+\Big) + \exp\Big(-2\tau\Big(x - a_j(\mathfrak{v}) - \frac{L_1^*}{4}\Big)^-\Big). \tag{2.19}$$

Here, we have set $y^\pm := \max\{\pm y, 0\}$. Similarly, we obtain

$$\begin{aligned}
0 \leq &\chi_{0,1}(x) \leq \exp\Big(-2\tau\Big(x - a_1(\mathfrak{v}) + \frac{L_1^*}{4}\Big)^+\Big),\\
0 \leq &\chi_{j,j+1}(x) \leq \exp\Big(-2\tau\Big(x - a_j(\mathfrak{v}) - \frac{L_1^*}{4}\Big)^-\Big)\exp\Big(-2\tau\Big(x - a_{j+1}(\mathfrak{v}) + \frac{L_1^*}{4}\Big)^+\Big),\\
0 \leq &\chi_{N,N+1}(x) \leq \exp\Big(-2\tau\Big(x - a_N(\mathfrak{v}) - \frac{L_1^*}{4}\Big)^-\Big).
\end{aligned}$$

We next set

$$\varepsilon_j := \chi_j(\cdot + a_j(\mathfrak{v}))^{\frac{1}{2}}\, \varepsilon(\cdot + a_j(\mathfrak{v})), \tag{2.20}$$

for $1 \leq j \leq N$, as well as

$$\varepsilon_{j,j+1} := \chi_{j,j+1}^{\frac{1}{2}}\, \varepsilon,$$

for $0 \leq j \leq N$. In order to clarify the presentation and when this does not lead to any confusion, we drop the dependence with respect to $\boldsymbol{c}(\mathfrak{v})$, $\boldsymbol{a}(\mathfrak{v})$ and $\mathfrak{s}^*$ of $S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*}$ by setting $S_{\boldsymbol{c}(\mathfrak{v}),\boldsymbol{a}(\mathfrak{v}),\mathfrak{s}^*} := S := (V, W)$. Similarly, we set $\mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*} := \mathfrak{v}_j := (v_j, w_j)$, so that $S = \mathfrak{v}_1 + \ldots + \mathfrak{v}_N$.

We now expand the quantity $\mathcal{F}(\mathfrak{v})$ in terms of the localized perturbations ε_j and $\varepsilon_{j,j+1}$. Concerning the energy $E(\mathfrak{v})$, we apply the Taylor formula to write

$$E(\mathfrak{v}) = E(S + \varepsilon) = E(S) + \big\langle E'(S), \varepsilon \big\rangle_{L^2\times L^2} + \frac{1}{2}\big\langle E''(S)(\varepsilon), \varepsilon \big\rangle_{L^2\times L^2} + R(S, \varepsilon), \tag{2.21}$$

where we recall that

$$\big\langle E'(S), \varepsilon \big\rangle_{L^2\times L^2} := \int_{\mathbb{R}} \Big(\frac{\partial_x V}{1 - V^2}\, \partial_x \varepsilon_1 + \frac{V(\partial_x V)^2}{(1 - V^2)^2}\, \varepsilon_1 + (1 - V^2) W \varepsilon_2 - V W^2 \varepsilon_1 + \varepsilon_1 V \Big), \tag{2.22}$$

and

$$\begin{aligned} \big\langle E''(S)(\varepsilon), \varepsilon \big\rangle_{L^2\times L^2} := \int_{\mathbb{R}} \mathfrak{E}(S, \varepsilon) := \int_{\mathbb{R}} \Big(& \frac{(\partial_x \varepsilon_1)^2}{1 - V^2} + \frac{4V(\partial_x V)}{(1 - V^2)^2}\, \varepsilon_1 (\partial_x \varepsilon_1) + \frac{(1 + 3V^2)(\partial_x V)^2}{(1 - V^2)^3}\, \varepsilon_1^2 \\ & + (1 - V^2)\varepsilon_2^2 - 4\varepsilon_1\varepsilon_2 V W + (1 - W^2)\varepsilon_1^2 \Big). \end{aligned} \tag{2.23}$$

In view of this decomposition, we have

Claim 1.

$$\begin{aligned} E(\mathfrak{v}) = & \sum_{j=1}^N E(\mathfrak{v}_{c_j(\mathfrak{v})}) + \frac{1}{2} \sum_{j=1}^N \big\langle E''(\mathfrak{v}_{c_j(\mathfrak{v})})(\varepsilon_j), \varepsilon_j \big\rangle_{L^2\times L^2} + \frac{1}{2} \sum_{j=0}^N \big\langle E''(0)(\varepsilon_{j,j+1}), \varepsilon_{j,j+1} \big\rangle_{L^2\times L^2} \\ & + \mathcal{O}\Big(\Big(\tau + \exp\Big(- \frac{\tau L_1^*}{2} \Big) \Big) \big\| \varepsilon \big\|_{H^1\times L^2}^2 \Big) + \mathcal{O}\Big(\big\| \varepsilon \big\|_{H^1\times L^2}^3 \Big) + \mathcal{O}\Big(L \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{2} \Big) \Big). \end{aligned}$$

Proof. The main tool in order to show Claim 1 is the following inequality

$$\Big\| \exp\big(-\sigma_a(\cdot - a)^+ \big) \exp\big(-\sigma_b(\cdot - b)^- \big) \Big\|_{L^p} \leq \Big(\frac{2}{p \min\{\sigma_a, \sigma_b\}} + b - a \Big)^{\frac{1}{p}} \exp\big(-\min\{\sigma_a, \sigma_b\}(b - a) \big), \tag{2.24}$$

which holds for $1 \leq p \leq +\infty$, $(a, b) \in \mathbb{R}^2$, with $a < b$, and $(\sigma_a, \sigma_b) \in (0, +\infty)^2$. We first apply (2.24) with $p = 1$ for estimating the quantity $E(S)$. Since

$$|\partial_x v_j(x)| + |v_j(x)| + |w_j(x)| = \mathcal{O}\Big(\exp\Big(- \frac{\nu_{\mathfrak{c}^*}|x - a_j(\mathfrak{v})|}{2} \Big) \Big), \tag{2.25}$$

by (5) and (36), and since $a_{j+1}(\mathfrak{v}) - a_j(\mathfrak{v}) > L - 1$, again by (36), we obtain

$$E(S) = \sum_{j=1}^N E(\mathfrak{v}_{c_j(\mathfrak{v})}) + \mathcal{O}\Big(L \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{2} \Big) \Big), \tag{2.26}$$

at least, when $L \geq 2/\nu_{\mathfrak{c}^*}$. Similarly, we combine (2.24), with $p = 2$, (2.25) and the Hölder inequality to estimate (2.22) as

$$\big\langle E'(S), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} = \sum_{j=1}^{N} \big\langle E'(\mathfrak{v}_j), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} + \mathcal{O}\Big(L^{\frac{1}{2}} \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{2}\Big) \big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}\Big).$$

In view of (27) and the orthogonality conditions in (34), we notice that

$$\big\langle E'(\mathfrak{v}_j), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} = \big\langle E'(\mathfrak{v}_j) - c_j(\mathfrak{v}) P'(\mathfrak{v}_j), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} = 0.$$

Since $\|\boldsymbol{\varepsilon}\|_{H^1\times L^2} \leq A^* \alpha_1^*$ by (35), we are led to

$$\big\langle E'(S), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} = \mathcal{O}\Big(L^{\frac{1}{2}} \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L}{2}\Big)\Big). \tag{2.27}$$

We next turn to (2.23). We decompose this quantity according to the partition of unity in (2.17) as

$$\big\langle E''(S)(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} := \sum_{j=1}^{N} \int_{\mathbb{R}} \mathfrak{E}(S, \boldsymbol{\varepsilon})\, \chi_j + \sum_{j=0}^{N} \int_{\mathbb{R}} \mathfrak{E}(S, \boldsymbol{\varepsilon})\, \chi_{j,j+1}. \tag{2.28}$$

Combining (2.24), with $p = +\infty$, and (2.25), and using the fact that $2\tau < \nu_{\mathfrak{c}^*}/2$, we simplify the first integral in the right-hand side of (2.28) as

$$\int_{\mathbb{R}} \mathfrak{E}(S, \boldsymbol{\varepsilon})\, \chi_j = \int_{\mathbb{R}} \mathfrak{E}(\mathfrak{v}_j, \boldsymbol{\varepsilon})\, \chi_j + \mathcal{O}\Big(\exp\big(- \tau L_1^*\big) \big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2\Big).$$

Since $\partial_x(\chi_j^{1/2} \varepsilon_1) = (\partial_x \varepsilon_1)\, \chi_j^{1/2} + \varepsilon_1\, (\partial_x \chi_j)/(2\chi_j^{1/2})$, we obtain, after a translation by $a_j(\mathfrak{v})$,

$$\int_{\mathbb{R}} \mathfrak{E}(\mathfrak{v}_j, \boldsymbol{\varepsilon})\, \chi_j = \int_{\mathbb{R}} \mathfrak{E}(\mathfrak{v}_{c_j(\mathfrak{v})}, \boldsymbol{\varepsilon}_j) - \int_{\mathbb{R}} \partial_x \chi_j \Big(\frac{\varepsilon_1 \partial_x \varepsilon_1}{1 - v_j^2} + \frac{2 v_j \partial_x v_j}{(1 - v_j^2)^2} \varepsilon_1^2 + \frac{\partial_x \chi_j}{4 \chi_j (1 - v_j^2)} \varepsilon_1^2 \Big).$$

In view of (2.16), we check that $|\partial_x \chi_j| = \mathcal{O}(\tau)$ and $|\partial_x \chi_j|/\chi_j = \mathcal{O}(\tau)$. Hence, we obtain

$$\int_{\mathbb{R}} \mathfrak{E}(S, \boldsymbol{\varepsilon})\, \chi_j = \langle E''(\mathfrak{v}_{c_j(\mathfrak{v})})(\boldsymbol{\varepsilon}_j), \boldsymbol{\varepsilon}_j\rangle_{L^2\times L^2} + \mathcal{O}\Big(\Big(\tau + \exp\big(- \tau L_1^*\big)\Big) \big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2\Big). \tag{2.29}$$

Similar computations provide the estimate

$$\int_{\mathbb{R}} \mathfrak{E}(S, \boldsymbol{\varepsilon})\, \chi_{j,j+1} = \langle E''(0)(\boldsymbol{\varepsilon}_{j,j+1}), \boldsymbol{\varepsilon}_{j,j+1}\rangle_{L^2\times L^2} + \mathcal{O}\Big(\Big(\tau + \exp\Big(- \frac{\tau L_1^*}{2}\Big)\Big) \big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^2\Big). \tag{2.30}$$

It remains to notice that

$$\big|R(S, \boldsymbol{\varepsilon})\big| = \mathcal{O}\Big(\big\|\boldsymbol{\varepsilon}\big\|_{H^1\times L^2}^3\Big),$$

due to (5) and (36), and to collect (2.21), (2.26), (2.27), (2.28), (2.29) and (2.30) to complete the proof of Claim 1. □

We now turn to the quantities $P_j(\mathfrak{v})$, which we decompose as

$$P_j(\mathfrak{v}) = P_j(S + \boldsymbol{\varepsilon}) = P_j(S) + \big\langle P_j'(S), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} + \frac{1}{2}\big\langle P_j''(S)(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2},$$

where

$$\big\langle P_j'(S), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} = \int_{\mathbb{R}} \big(\phi_j - \phi_{j+1}\big)\big(V \varepsilon_2 + W \varepsilon_1\big),$$

and

$$\big\langle P_j''(S)(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2} = 2 \int_{\mathbb{R}} \big(\phi_j - \phi_{j+1}\big) \varepsilon_1 \varepsilon_2.$$

For this decomposition, we show

Claim 2.

$$P_j(\mathfrak{v}) = P(\mathfrak{v}_{c_j(\mathfrak{v})}) + \frac{1}{2}\big\langle P''(\mathfrak{v}_{c_j(\mathfrak{v})})(\boldsymbol{\varepsilon}_j), \boldsymbol{\varepsilon}_j\big\rangle_{L^2\times L^2} + \frac{1}{2}\big\langle P_j''(0)(\boldsymbol{\varepsilon}_{j-1,j}), \boldsymbol{\varepsilon}_{j-1,j}\big\rangle_{L^2\times L^2}$$
$$+ \frac{1}{2}\big\langle P_j''(0)(\boldsymbol{\varepsilon}_{j,j+1}), \boldsymbol{\varepsilon}_{j,j+1}\big\rangle_{L^2\times L^2} + \mathcal{O}\Big(\exp\Big(-\frac{\tau L_1^*}{2}\Big)\|\boldsymbol{\varepsilon}\|^2_{H^1\times L^2}\Big) + \mathcal{O}\Big(L\exp\Big(-\frac{\nu_{\mathfrak{c}^*}L}{16}\Big)\Big).$$

Proof. We derive Claim 2 following the proof of Claim 1. The only difference lies in the fact that we also handle with the cut-off functions $\phi_j - \phi_{j+1}$. Since they satisfy the pointwise estimates

$$0 \le \phi_1(x) - \phi_2(x) \le \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_1(\mathfrak{v})+a_2(\mathfrak{v})}{2}\Big)^+\Big),$$
$$0 \le \phi_j(x) - \phi_{j+1}(x) \le \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_{j-1}(\mathfrak{v})+a_j(\mathfrak{v})}{2}\Big)^-\Big)\exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_j(\mathfrak{v})+a_{j+1}(\mathfrak{v})}{2}\Big)^+\Big),$$
$$0 \le \phi_N(x) - \phi_{N+1}(x) \le \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_{N-1}(\mathfrak{v})+a_N(\mathfrak{v})}{2}\Big)^-\Big),$$

and

$$1 - \phi_1(x) + \phi_2(x) \le \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_1(\mathfrak{v})+a_2(\mathfrak{v})}{2}\Big)^-\Big),$$
$$1 - \phi_j(x) + \phi_{j+1}(x) \le \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_{j-1}(\mathfrak{v})+a_j(\mathfrak{v})}{2}\Big)^+\Big) + \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_j(\mathfrak{v})+a_{j+1}(\mathfrak{v})}{2}\Big)^-\Big),$$
$$1 - \phi_N(x) + \phi_{N+1}(x) \le \exp\Big(-\tfrac{\nu_{\mathfrak{c}}^*}{8}\Big(x - \tfrac{a_{N-1}(\mathfrak{v})+a_N(\mathfrak{v})}{2}\Big)^+\Big),$$

by (37), we can again rely on (2.24) to bound the exponentially small interactions between the solitons $\mathfrak{v}_j$, and the cut-off functions ϕ_j, χ_j and $\chi_{j,j+1}$. This leads to Claim 2. We refer to the proof of [3, Proposition 3] for more technical details. □

We now write $\mathcal{F}(\mathfrak{v})$ along the decompositions in Claims 1 and 2. Before collecting the two identities, we observe that the Taylor formula provides

$$E(\mathfrak{v}_{c_j(\mathfrak{v})}) - c_j^* P(\mathfrak{v}_{c_j(\mathfrak{v})}) = E(\mathfrak{v}_{c_j^*}) - c_j^* P(\mathfrak{v}_{c_j^*}) + \mathcal{O}\big(|c_j(\mathfrak{v}) - c_j^*|^2\big),$$

due to (27). We also compute

$$\begin{aligned}\big|c_j(\mathfrak{v}) - c_j^*\big|\,\big|\langle P''(\mathfrak{v}_{c_j(\mathfrak{v})})(\boldsymbol{\varepsilon}_j), \boldsymbol{\varepsilon}_j\rangle_{L^2\times L^2}\big| \le& 2|c_j(\mathfrak{v}) - c_j^*| \int_{\mathbb{R}} |\varepsilon_1||\varepsilon_2| \\ =& \mathcal{O}\big(|c_j(\mathfrak{v}) - c_j^*|^2\big) + \mathcal{O}\big(\|\boldsymbol{\varepsilon}\|^4_{H^1\times L^2}\big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\big|c_j(\mathfrak{v}) - c_j^*\big|\,\Big(\big|\langle P_j''(0)(\boldsymbol{\varepsilon}_{j-1,j}), \boldsymbol{\varepsilon}_{j-1,j}\rangle_{L^2\times L^2}\big| + \big|\langle P_j''(0)(\boldsymbol{\varepsilon}_{j,j+1}),\boldsymbol{\varepsilon}_{j,j+1}\rangle_{L^2\times L^2}\big|\Big) \\ &\qquad = \mathcal{O}\big(|c_j(\mathfrak{v}) - c_j^*|^2\big) + \mathcal{O}\big(\|\boldsymbol{\varepsilon}\|^4_{H^1\times L^2}\big).\end{aligned}$$

In view of Claims 1 and 2, this gives

$$\begin{aligned}\mathcal{F}(\mathfrak{v}) =& \sum_{j=1}^N F_{c_j^*}(\mathfrak{v}_{c_j^*}) + \frac{1}{2}\sum_{j=1}^N Q_{c_j(\mathfrak{v})}(\boldsymbol{\varepsilon}_j) + \frac{1}{2}\sum_{j=0}^N Q_0^j(\boldsymbol{\varepsilon}_{j,j+1}) + \mathcal{O}\Big(\sum_{j=1}^N |c_j(\mathfrak{v}) - c_j^*|^2\Big) \\ &+ \mathcal{O}\Big(\Big(\tau + \exp\Big(-\frac{\tau L_1^*}{2}\Big)\Big)\|\boldsymbol{\varepsilon}\|^2_{H^1\times L^2}\Big) + \mathcal{O}\Big(\|\boldsymbol{\varepsilon}\|^3_{H^1\times L^2}\Big) + \mathcal{O}\Big(L\exp\Big(-\frac{\nu_{\mathfrak{c}^*}L}{16}\Big)\Big),\end{aligned} \tag{2.31}$$

where we have set

$$
\begin{aligned}
Q_0^0 &= E''(0) - c_1(\mathfrak{v})P_1''(0),\\
Q_0^j &= E''(0) - c_j(\mathfrak{v})P_j''(0) - c_{j+1}(\mathfrak{v})P_{j+1}''(0),\\
Q_0^N &= E''(0) - c_N(\mathfrak{v})P_N''(0),
\end{aligned}
$$

for $1 \leq j \leq N-1$. In order to establish inequality (41), we are reduced to show some coercivity for the quadratic forms $Q_{c_j(\mathfrak{v})}$ and Q_0^j. We deduce from Proposition 1 the following claim.

Claim 3. *There exists a positive number Λ_1^*, depending only on $\mathfrak{c}^*$, such that*

$$
Q_{c_j(\mathfrak{v})}(\boldsymbol{\varepsilon}_j) \geq \Lambda_1^* \|\boldsymbol{\varepsilon}_j\|_{H^1\times L^2} + \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2}^2 \Big),
$$

for any $1 \leq j \leq N$.

Proof. In view of the orthogonality conditions in (34), and of definition (2.20), we know that

$$
\big\langle \partial_x \mathfrak{v}_{c_j(\mathfrak{v})}, \boldsymbol{\varepsilon}_j \big\rangle_{L^2\times L^2} = s_j^* \big\langle \partial_x \mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}, (\chi_j^{\frac{1}{2}} - 1)\boldsymbol{\varepsilon} \big\rangle_{L^2\times L^2},
$$

and

$$
\big\langle P'(\mathfrak{v}_{c_j(\mathfrak{v})}), \boldsymbol{\varepsilon}_j \big\rangle_{L^2\times L^2} = s_j^* \big\langle P'(\mathfrak{v}_{c_j(\mathfrak{v}),a_j(\mathfrak{v}),s_j^*}), (\chi_j^{\frac{1}{2}} - 1)\boldsymbol{\varepsilon} \big\rangle_{L^2\times L^2}.
$$

Combining inequality (2.24), with $p = 2$, and estimates (2.18), (2.19) and (2.25), we infer that

$$
\big\langle \partial_x \mathfrak{v}_{c_j(\mathfrak{v})}, \boldsymbol{\varepsilon}_j \big\rangle_{L^2\times L^2} = \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2} \Big),
$$

and similarly,

$$
\big\langle P'(\mathfrak{v}_{c_j(\mathfrak{v})}), \boldsymbol{\varepsilon}_j \big\rangle_{L^2\times L^2} = \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2} \Big).
$$

Invoking formula (5) as well as the bounds on $c_j(\mathfrak{v})$ in (36), we can decompose the pair $\boldsymbol{\varepsilon}_j$ as $\boldsymbol{\varepsilon}_j := \alpha_j \, \partial_x \mathfrak{v}_{c_j(\mathfrak{v})} + \beta_j \, P'(\mathfrak{v}_{c_j(\mathfrak{v})}) + \boldsymbol{r}_j$, with $\boldsymbol{r}_j$ satisfying the orthogonality conditions in (31),

$$
\alpha_j = \frac{\langle \partial_x \mathfrak{v}_{c_j(\mathfrak{v})}, \boldsymbol{\varepsilon}_j \rangle_{L^2\times L^2}}{\|\partial_x \mathfrak{v}_{c_j(\mathfrak{v})}\|_{L^2\times L^2}^2} = \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2} \Big), \tag{2.32}
$$

and the same estimate for β_j. This ensures that

$$
Q_{c_j(\mathfrak{v})}(\boldsymbol{r}_j) \geq \Lambda_{c_j(\mathfrak{v})} \|r_j\|_{H^1\times L^2}^2,
$$

by Proposition 1. Since $\partial_x \mathfrak{v}_{c_j(\mathfrak{v})}$ lies in the kernel of $Q_{c_j(\mathfrak{v})}$, we also have

$$
Q_{c_j(\mathfrak{v})}(\boldsymbol{\varepsilon}_j) = -\beta_j^2 Q_{c_j(\mathfrak{v})}\big(P'(\mathfrak{v}_{c_j(\mathfrak{v})})\big) + 2\beta_j \big\langle \mathcal{Q}_{c_j(\mathfrak{v})}\big(P'(\mathfrak{v}_{c_j(\mathfrak{v})})\big), \boldsymbol{\varepsilon}_j \big\rangle_{L^2\times L^2} + Q_{c_j(\mathfrak{v})}(\boldsymbol{r}_j),
$$

so that we are led to

$$
Q_{c_j(\mathfrak{v})}(\boldsymbol{\varepsilon}_j) \geq \Lambda_{c_j(\mathfrak{v})} \|\boldsymbol{r}_j\|_{H^1\times L^2}^2 + \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2}^2 \Big),
$$

using (2.20) and (2.32). It remains to check that

$$
\big| \|\boldsymbol{r}_j\|_{H^1\times L^2} - \|\boldsymbol{\varepsilon}_j\|_{H^1\times L^2} \big| = \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2} \Big),
$$

by (5), (35), (36) and (2.32) in order to obtain

$$
Q_{c_j(\mathfrak{v})}(\boldsymbol{\varepsilon}_j) \geq \frac{\Lambda_{c_j(\mathfrak{v})}}{2} \|\boldsymbol{\varepsilon}_j\|_{H^1\times L^2}^2 + \mathcal{O}\Big(\big(L_1^*\big)^{\frac{1}{2}} \exp\Big(- \frac{\tau L_1^*}{2}\Big) \, \|\boldsymbol{\varepsilon}\|_{H^1\times L^2}^2 \Big).
$$

Claim 3 follows combining (35) with the property that the numbers Λ_c in Proposition 1 are uniformly bounded from below for c lying in a compact subset of $(-1, 1) \setminus \{0\}$. □

For the quadratic form Q_0^j, we similarly show

Claim 4. *There exists a positive number Λ_2^*, depending only on $\mathfrak{c}^*$, such that*

$$Q_0^j(\varepsilon_{j,j+1}) \geq \Lambda_2^* \|\varepsilon_{j,j+1}\|_{H^1\times L^2},$$

for any $0 \leq j \leq N$.

Proof. In view of (1), (37) and (40), we have

$$\begin{aligned} Q_0^j(\mathfrak{w}) = & \int_{\mathbb{R}} \Big((\partial_x w_1)^2 + w_1^2 + w_2^2 - 2\big(c_j(\mathfrak{v})(\phi_j - \phi_{j+1}) + c_{j+1}(\mathfrak{v})(\phi_{j+1} - \phi_{j+2})\big) w_1 w_2 \Big) \\ \geq & \int_{\mathbb{R}} \Big((\partial_x w_1)^2 + \big(1 - \max\{|c_j(\mathfrak{v})|, |c_{j+1}(\mathfrak{v})|\}\big)\big(w_1^2 + w_2^2\big) \Big), \end{aligned}$$

for any $\mathfrak{w} = (w_1, w_2) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, and $1 \leq j \leq N-1$. Similarly, we obtain

$$Q_0^0(\mathfrak{w}) \geq \int_{\mathbb{R}} \Big((\partial_x w_1)^2 + \big(1 - |c_1(\mathfrak{v})|\big)\big(w_1^2 + w_2^2\big) \Big),$$

and

$$Q_0^N(\mathfrak{w}) \geq \int_{\mathbb{R}} \Big((\partial_x w_1)^2 + \big(1 - |c_N(\mathfrak{v})|\big)\big(w_1^2 + w_2^2\big) \Big).$$

Claim 4 follows combining with the equality

$$\max\big\{|c_1(\mathfrak{v})|, \ldots, |c_N(\mathfrak{v})|\big\} = (1 - \nu_{\mathfrak{c}(\mathfrak{v})}^2)^{\frac{1}{2}},$$

and the bounds in (36). □

We are now in position to complete the proof of Proposition 3.

End of the proof of Proposition 3. Concerning inequality (41), we derive from Claims 3 and 4 the inequality

$$\begin{aligned} \frac{1}{2}\sum_{j=1}^N Q_{c_j(\mathfrak{v})}(\varepsilon_j) + \frac{1}{2}\sum_{j=0}^N Q_0^j(\varepsilon_{j,j+1}) \geq & 2\Lambda^* \Big(\sum_{j=1}^N \|\varepsilon_j\|_{H^1\times L^2}^2 + \sum_{j=0}^N \|\varepsilon_{j,j+1}\|_{H^1\times L^2}^2 \Big) \\ & + \mathcal{O}\Big((L_1^*)^{\frac{1}{2}} \exp\Big(-\frac{\tau L_1^*}{2} \Big) \|\varepsilon\|_{H^1\times L^2}^2 \Big), \end{aligned}$$

where we have set $\Lambda^* = \min\{\Lambda_1^*, \Lambda_2^*\}/4$. On the other hand, it was proved in [3, Lemma 1] that

$$\sum_{j=1}^N \|\varepsilon_j\|_{H^1\times L^2}^2 + \sum_{j=0}^N \|\varepsilon_{j,j+1}\|_{H^1\times L^2}^2 \geq \|\varepsilon\|_{H^1\times L^2}^2.$$

Therefore, we can estimate (2.31) from below by

$$\begin{aligned} \mathcal{F}(\mathfrak{v}) \geq & \sum_{j=1}^N F_{c_j^*}(\mathfrak{v}_{c_j^*}) + 2\Lambda^* \big\|\varepsilon\big\|_{H^1\times L^2}^2 + \mathcal{O}\Big(\big\|\varepsilon\big\|_{H^1\times L^2}^3 \Big) + \mathcal{O}\Big(\sum_{j=1}^N |c_j(\mathfrak{v}) - c_j^*|^2 \Big) \\ & + \mathcal{O}\Big(\Big(\tau + \big((L_1^*)^{\frac{1}{2}} + 1 \big) \exp\Big(-\frac{\tau L_1^*}{2} \Big) \Big) \big\|\varepsilon\big\|_{H^1\times L^2}^2 \Big) + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{16} \Big) \Big). \end{aligned}$$

At this stage, we can fix the value of τ small enough, and then decrease the value of α_1^* and increase the value of L_1^*, if necessary, so that

$$\mathcal{O}\Big(\big\|\varepsilon\big\|_{H^1\times L^2}^3 \Big) + \mathcal{O}\Big(\Big(\tau + \big((L_1^*)^{\frac{1}{2}} + 1 \big) \exp\Big(-\frac{\tau L_1^*}{2} \Big) \Big) \big\|\varepsilon\big\|_{H^1\times L^2}^2 \Big) \leq \Lambda^* \big\|\varepsilon\big\|_{H^1\times L^2}^2.$$

This is enough to obtain inequality (41). Similarly, inequality (42) results from (2.31) using the property that the quadratic forms $Q_{c_j(\mathfrak{v})}$ and Q_0^j are continuous on $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, with continuous bounds depending only on $\mathfrak{c}^*$ by (36). This concludes the proof of Proposition 3. □

3 Dynamical properties of a chain of solitons

3.1 Proof of Proposition 4

Coming back to Proposition 2, we notice that the modulation functions $\boldsymbol{a}(t)$ and $\boldsymbol{c}(t)$ own a $\mathcal{C}^1$ dependence on the variations of the solution $\mathfrak{v}(\cdot,t)$ in $H^1(\mathbb{R})\times L^2(\mathbb{R})$. On the other hand, the solution $\mathfrak{v}$ belongs to $\mathcal{C}^0([0,T],H^3(\mathbb{R})\times H^2(\mathbb{R}))$, when the initial datum $\mathfrak{v}^0$ belongs to $H^5(\mathbb{R})\times H^4(\mathbb{R})$ (see Proposition A.2). In this situation, it belongs to $\mathcal{C}^1([0,T],H^1(\mathbb{R})\times L^2(\mathbb{R}))$ by (HLL), so that we can apply the chain rule in order to guarantee that $\boldsymbol{a}$ and $\boldsymbol{c}$ are of class $\mathcal{C}^1$ on $[0,T]$. Moreover, we are allowed to differentiate with respect to time the orthogonality conditions in (34) and to invoke equations (46) and (47) to write

$$M\begin{pmatrix}\boldsymbol{c}'\\ \boldsymbol{a}'-\boldsymbol{c}\end{pmatrix}=\begin{pmatrix}Y\\ Z\end{pmatrix}. \tag{3.1}$$

Here, M refers to the matrix of size $2N$ given by

$$\begin{aligned}
M_{k,\ell}&=-\langle P'(\mathfrak{v}_k),\partial_c\mathfrak{v}_\ell\rangle_{L^2\times L^2}+\delta_{k,\ell}\langle P'(\partial_c\mathfrak{v}_k),\boldsymbol{\varepsilon}\rangle_{L^2\times L^2},\\
M_{k,\ell+N}&=\langle P'(\mathfrak{v}_k)),\partial_x\mathfrak{v}_\ell\rangle_{L^2\times L^2}-\delta_{k,\ell}\langle P'(\partial_x\mathfrak{v}_k),\boldsymbol{\varepsilon}\rangle_{L^2\times L^2},\\
M_{k+N,\ell}&=-\langle\partial_x\mathfrak{v}_k,\partial_c\mathfrak{v}_\ell\rangle_{L^2\times L^2}+\delta_{k,\ell}\langle\partial_c\partial_x\mathfrak{v}_k,\boldsymbol{\varepsilon}\rangle_{L^2\times L^2},\\
M_{k+N,\ell+N}&=\langle\partial_x\mathfrak{v}_k,\partial_x\mathfrak{v}_\ell\rangle_{L^2\times L^2}-\delta_{k,\ell}\langle\partial_{xx}\mathfrak{v}_k,\boldsymbol{\varepsilon}\rangle_{L^2\times L^2},
\end{aligned}$$

for $0\leq k,\ell\leq N$, where $\mathfrak{v}_k(\cdot,t)=\mathfrak{v}_{c_k(t),a_k(t),s_k^*}(\cdot)$. The vectors Y and Z are defined by

$$\begin{aligned}
Y_k=&\Big\langle\partial_x w_k,\big((V+\varepsilon_1)^2-1\big)(W+\varepsilon_2)-\sum_{\ell=1}^N\big(v_\ell^2-1\big)w_\ell\Big\rangle_{L^2}\\
&+\Big\langle\partial_x v_k,\big((W+\varepsilon_2)^2-1\big)(V+\varepsilon_1)-\sum_{\ell=1}^N\big(w_\ell^2-1\big)v_\ell\Big\rangle_{L^2}\\
&-\Big\langle\partial_{xx}v_k,\frac{\partial_x V+\partial_x\varepsilon_1}{1-(V+\varepsilon_1)^2}-\sum_{\ell=1}^N\frac{\partial_x v_\ell}{1-v_\ell^2}\Big\rangle_{L^2}+c_k\big\langle P'(\partial_x\mathfrak{v}_k),\boldsymbol{\varepsilon}\big\rangle_{L^2\times L^2},
\end{aligned}$$

and

$$\begin{aligned}
Z_k=&\Big\langle\partial_{xx}v_k,\big((V+\varepsilon_1)^2-1\big)(W+\varepsilon_2)-\sum_{\ell=1}^N\big(v_\ell^2-1\big)w_\ell\Big\rangle_{L^2}\\
&+\Big\langle\partial_{xx}w_k,\big((W+\varepsilon_2)^2-1\big)(V+\varepsilon_1)-\sum_{\ell=1}^N\big(w_\ell^2-1\big)v_\ell\Big\rangle_{L^2}\\
&-\Big\langle\partial_{xxx}w_k,\frac{\partial_x V+\partial_x\varepsilon_1}{1-(V+\varepsilon_1)^2}-\sum_{\ell=1}^N\frac{\partial_x v_\ell}{1-v_\ell^2}\Big\rangle_{L^2}+c_k\langle\partial_{xx}\mathfrak{v}_k,\boldsymbol{\varepsilon}\rangle_{L^2\times L^2},
\end{aligned}$$

for $1\leq k\leq N$, where $V=\sum_{k=1}^N v_k$ and $W=\sum_{k=1}^N w_k$ as in the introduction.

We next decompose the matrix M as $M=D+H$, where D is the diagonal matrix of size $2N$ with diagonal coefficients

$$D_{k,k}=-\langle P'(\mathfrak{v}_k),\partial_c\mathfrak{v}_k\rangle_{L^2\times L^2}=-\frac{d}{dc}\Big(P(\mathfrak{v}_{c_k(t)})\Big)=\frac{1}{(1-c_k(t)^2)^{\frac{1}{2}}},$$

and

$$D_{k+N,k+N} = \|\partial_x \mathfrak{v}_k\|_{L^2}^2 = 2(1 - c_k(t)^2)^{\frac{1}{2}}.$$

As a consequence of (45), we deduce that D is invertible, with the operator norm of its inverse bounded by some number depending only on $\mathfrak{c}^*$.

Concerning the matrix H, we check that

$$\langle P'(\mathfrak{v}_k), \partial_x \mathfrak{v}_k\rangle_{L^2\times L^2} = \langle \partial_x \mathfrak{v}_k, \partial_c \mathfrak{v}_k\rangle_{L^2\times L^2} = 0,$$

whereas we can invoke (45), (2.9) and (2.24), and then argue as in the proof of Proposition 3 to obtain

$$\begin{aligned}&\big|\langle P'(\mathfrak{v}_k), \partial_c \mathfrak{v}_\ell\rangle_{L^2\times L^2}\big| + \big|\langle P'(\mathfrak{v}_k), \partial_x \mathfrak{v}_\ell\rangle_{L^2\times L^2}\big|\\ &+\big|\langle \partial_x \mathfrak{v}_k, \partial_c \mathfrak{v}_\ell\rangle_{L^2\times L^2}\big| + \big|\langle \partial_x \mathfrak{v}_k, \partial_x \mathfrak{v}_\ell\rangle_{L^2\times L^2}\big| = \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{2}\Big)\Big),\end{aligned}$$

for $\ell \neq k$. On the other hand, it follows from (45) and (2.9) that

$$\begin{aligned}&\big|\langle P'(\partial_c \mathfrak{v}_k), \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}\big| + \big|\langle P'(\partial_x \mathfrak{v}_k), \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}\big|\\ &+\big|\langle \partial_c\partial_x \mathfrak{v}_k, \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}\big| + \big|\langle \partial_{xx} \mathfrak{v}_k, \boldsymbol{\varepsilon}\rangle_{L^2\times L^2}\big| = \mathcal{O}\big(\|\boldsymbol{\varepsilon}\|_{L^2\times L^2}\big).\end{aligned}$$

As a consequence, we can make a further choice of positive numbers $\alpha_3^* \leq \alpha_2^*$ and $L_3^* \geq L_2^*$ such that, for $\alpha \leq \alpha_3^*$ and $L \geq L_3^*$, the operator norm of the matrix $D^{-1}H$ is less than $1/2$. In this case, the matrix M is invertible and the operator norm of its inverse is uniformly bounded with respect to t. Coming back to (3.1), we are led to the estimate

$$\sum_{k=1}^N \Big(\big|c_k'(t)\big| + \big|a_k'(t) - c_k(t)\big|\Big) \leq \mathcal{O}\Big(\sum_{k=1}^N \Big(\big|Y_k(t)\big| + \big|Z_k(t)\big|\Big)\Big). \tag{3.2}$$

It remains to estimate the quantities Y_k and Z_k. We write

$$\begin{aligned}&\big((V+\varepsilon_1)^2 - 1\big)(W+\varepsilon_2) - \sum_{\ell=1}^N \big(v_\ell^2 - 1\big)w_\ell\\ &= \big((V+\varepsilon_1)^2 - 1\big)(W+\varepsilon_2) - \big(V^2 - 1\big)W + \big(V^2 - 1\big)W - \sum_{\ell=1}^N \big(v_\ell^2 - 1\big)w_\ell.\end{aligned}$$

Combining the Sobolev embedding theorem, (45) and (2.9), we compute

$$\big\langle \partial_x w_k, \big((V+\varepsilon_1)^2 - 1\big)(W+\varepsilon_2) - \big(V^2 - 1\big)W\big\rangle_{L^2} = \mathcal{O}\big(\|\boldsymbol{\varepsilon}\|_{L^2\times L^2}\big).$$

Similarly, we rely on (45), (2.9) and (2.24) to derive

$$\big\langle \partial_x w_k, \big(V^2 - 1\big)W - \sum_{\ell=1}^N \big(v_\ell^2 - 1\big)w_\ell\big\rangle_{L^2} = \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{2}\Big)\Big).$$

Arguing in the same way for the other terms in Y_k and Z_k, we conclude that

$$\sum_{k=1}^N \Big(\big|Y_k(t)\big| + \big|Z_k(t)\big|\Big) = \mathcal{O}\big(\|\boldsymbol{\varepsilon}\|_{L^2\times L^2}\big) + \mathcal{O}\Big(L \exp\Big(-\frac{\nu_{\mathfrak{c}^*} L}{2}\Big)\Big),$$

which is enough to deduce (48) from (3.2).

Finally, we apply a density argument to extend (48) to any solution $\mathfrak{v}(\cdot,t)$ in $\mathcal{C}^0([0,T],H^1(\mathbb{R})\times L^2(\mathbb{R}))$. Recall that the modulation functions $\boldsymbol{a}(t)$ and $\boldsymbol{c}(t)$ depend continuously on $\mathfrak{v}(\cdot,t)$ in $H^1(\mathbb{R})\times L^2(\mathbb{R})$, which in turn depends continuously on the initial data $\mathfrak{v}^0$ by Theorem 1. As a consequence, the matrices $M(\cdot,t)$ and the vectors $(Y(\cdot,t),Z(\cdot,t))$ also depend continuously on $\mathfrak{v}^0$ in $H^1(\mathbb{R})\times L^2(\mathbb{R})$. Since the operator norm of the inverse matrices $M(\cdot,t)^{-1}$ is bounded by some positive number depending only on $\mathfrak{c}^*$, we can apply a density argument to derive from (3.1) the $\mathcal{C}^1$ nature of the modulation functions $t\mapsto\boldsymbol{a}(t)$ and $t\mapsto\boldsymbol{c}(t)$, as well as estimate (48). We refer to [3] for more details. This concludes the proof of Proposition 4. □

3.2 Proof of Proposition 5

The monotonicity formulae in Proposition 5 are based on the conservation law for the momentum in (52). In order to perform the derivation of this conservation law rigorously, we introduce the spaces

$$\mathcal{NV}^k(\mathbb{R}) := \Big\{\mathfrak{v}=(v,w)\in H^{k+1}(\mathbb{R})\times H^k(\mathbb{R}),\ \text{s.t. } \max_{\mathbb{R}}|v|<1\Big\}, \tag{3.3}$$

for any $k\in\mathbb{N}$, and we endow them with the metric structure provided by the norm

$$\|\mathfrak{v}\|_{\mathcal{NV}^k} := \|\mathfrak{v}\|_{H^{k+1}\times H^k} = \big(\|v\|_{H^{k+1}}^2+\|w\|_{H^k}^2\big)^{\frac{1}{2}}.$$

Notice in particular that $\mathcal{NV}^0(\mathbb{R})=\mathcal{NV}(\mathbb{R})$.

Lemma 3.1. *Let $\mathfrak{v}=(v,w)$ be a solution to* (HLL) *in $\mathcal{C}^0([0,T],\mathcal{NV}^2(\mathbb{R}))$. Then, the map vw belongs to $\mathcal{C}^1([0,T],L^1(\mathbb{R}))$ and satisfies* (52)*, i.e.*

$$\partial_t\big(vw\big) = -\frac{1}{2}\partial_x\Big(v^2+w^2-3v^2w^2+\frac{3-v^2}{(1-v^2)^2}(\partial_x v)^2\Big)-\frac{1}{2}\partial_{xxx}\Big(\ln\big(1-v^2\big)\Big).$$

Proof. In view of (HLL), the function vw is in $\mathcal{C}^1([0,T],L^1(\mathbb{R}))$, so that we are authorized to derive from (HLL) that

$$\begin{aligned}\partial_t(vw) &= w\partial_x\Big(\big(v^2-1\big)w\Big)+v\partial_x\Big(\frac{\partial_{xx}v}{1-v^2}+v\frac{(\partial_x v)^2}{(1-v^2)^2}+v\big(w^2-1\big)\Big)\\ &= \frac{1}{2}\partial_x\big(3v^2w^2-v^2-w^2\big)+\partial_x\Big(\frac{v\,\partial_{xx}v}{1-v^2}+\frac{v^2(\partial_x v)^2}{(1-v^2)^2}-\frac{(\partial_x v)^2}{2(1-v^2)}\Big).\end{aligned}$$

Equation (52) then follows from the computation

$$\frac{v\,\partial_{xx}v}{1-v^2} = -\frac{1}{2}\partial_{xx}\Big(\ln\big(1-v^2\big)\Big)-\frac{1+v^2}{(1-v^2)^2}(\partial_x v)^2.$$

□

Using Lemma 3.1, we can provide the

Proof of Proposition 5. When $\mathfrak{v}$ is a solution to (HLL) in $\mathcal{C}^0([0,T],\mathcal{NV}(\mathbb{R}))$, the quantity R_j is well-defined and continuous on $[0,T]$. When $\mathfrak{v}$ additionally belongs to $\mathcal{C}^0([0,T],\mathcal{NV}^2(\mathbb{R}))$, the function R_j becomes of class $\mathcal{C}^1$ on $[0,T]$ in view of the continuous differentiability of the position parameters $\boldsymbol{a}$ in Proposition 4. In this case, we derive from (52) that the derivative of R_j is equal to

$$\begin{aligned}R_j'(t) =& \frac{1}{2}\int_{\mathbb{R}}\partial_x\phi_j\,\Big(v^2+w^2-\big(a_j'(t)+a_{j-1}'(t)\big)vw-3v^2w^2+\frac{3-v^2}{(1-v^2)^2}(\partial_x v)^2\Big)\\ &+\frac{1}{2}\int_{\mathbb{R}}\partial_{xxx}\phi_j\,\ln\big(1-v^2\big),\end{aligned}\tag{3.4}$$

for any $t \in [0, T]$.

At this stage, we recall that the position parameters $a_j(t)$, as well as their derivatives $a_j'(t)$, depend continuously on $\mathfrak{v}$ in $\mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$ due to Proposition 2 on the one hand, and formula (3.1) on the other hand. As a consequence, the right-hand side of (3.4) depends continuously on $\mathfrak{v}$ in $\mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$. In view of the Cauchy theory for (HLL) in Theorem 1, we can apply a density argument to conclude that the function R_j remains of class $\mathcal{C}^1$ on $[0, T]$ when $\mathfrak{v}$ is only in $\mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$. Moreover, its derivative remains given by (3.4). In particular, in view of definition (39), and the conservation of the energy E and the momentum P, the function $\mathcal{F}$ is also of class $\mathcal{C}^1$ on $[0, T]$, when $\mathfrak{v}$ belongs to $\mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$.

In order to estimate the derivative $R_j'(t)$, we remark that the integrand in the first integral of the right-hand side of (3.4) is positive when v is small enough. In our context of a perturbation of a sum of solitons, this quantity is positive far away from the positions $a_k(t)$. On the other hand, in areas close to the positions $a_k(t)$, the integrand is exponentially small due to the decay of the derivatives $\partial_x \phi_j$ and $\partial_{xxx} \phi_j$. As a matter of fact, we can compute

$$0 \leq \partial_x \phi_j(x,t) \leq \frac{\nu_{\mathfrak{c}^*}}{8} \exp\Big(- \frac{\nu_{\mathfrak{c}^*}}{8} \Big| x - \frac{a_{j-1}(t) + a_j(t)}{2} \Big| \Big). \tag{3.5}$$

Similarly, we have

$$\big| \partial_{xxx} \phi_j(x,t) \big| \leq \frac{\nu_{\mathfrak{c}^*}^2}{64} \partial_x \phi_j(x,t) \leq \frac{\nu_{\mathfrak{c}^*}^3}{512} \exp\Big(- \frac{\nu_{\mathfrak{c}^*}}{8} \Big| x - \frac{a_{j-1}(t) + a_j(t)}{2} \Big| \Big). \tag{3.6}$$

Following the remark above, we decompose the derivative $R_j'(t)$ according to the two areas given by the interval

$$I_j(t) = \Big[\frac{a_{j-1}(t) + a_j(t)}{2} - \frac{1}{4}\big(L + \delta_{\mathfrak{c}^*} t\big), \frac{a_{j-1}(t) + a_j(t)}{2} + \frac{1}{4}\big(L + \delta_{\mathfrak{c}^*} t\big) \Big],$$

and its complementary set. More precisely, we set

$$R_j'(t) = \mathfrak{R}_1(t) + \mathfrak{R}_2(t),$$

where we denote

$$\begin{aligned} \mathfrak{R}_1(t) =& \frac{1}{2} \int_{I_j(t)} \partial_x \phi_j \Big(v^2 + w^2 - \big(a_j'(t) + a_{j-1}'(t)\big) v w - 3 v^2 w^2 + \frac{3 - v^2}{(1 - v^2)^2} (\partial_x v)^2 \Big) \\ & + \frac{1}{2} \int_{I_j(t)} \partial_{xxx} \phi_j \ln\big(1 - v^2\big). \end{aligned}$$

Concerning $\mathfrak{R}_2(t)$, we deduce from (33), (51), (3.5) and (3.6) that

$$\big| \mathfrak{R}_2(t) \big| \leq A^* \exp\Big(- \frac{1}{32}\big(L + \delta_{\mathfrak{c}^*} t\big) \Big) \int_{\mathbb{R}} \Big(v^2 + w^2 + (\partial_x v)^2 - \ln\big(1 - v^2\big) \Big),$$

where A^* denotes, here as in the sequel, a positive number depending only on $\mathfrak{c}^*$ and $\mathfrak{s}^*$. On the other hand, since $1 - v^2 \geq \mu_{\mathfrak{c}^*}^2/8$ by (33), there exists a further positive number A^*, depending only on $\mu_{\mathfrak{c}^*}$, such that

$$- \ln\big(1 - v^2\big) \leq A^* v^2.$$

As a consequence, we obtain

$$\big| \mathfrak{R}_2(t) \big| \leq A^* \exp\Big(- \frac{1}{32}\big(L + \delta_{\mathfrak{c}^*} t\big) \Big). \tag{3.7}$$

We next turn to $\mathfrak{R}_1(t)$, which we bound from below by

$$\mathfrak{R}_1(t) \geq \frac{1}{2} \int_{I_j(t)} \partial_x \phi_j \, \Big(v^2 + w^2 - 2 \Big(1 - \frac{\nu_{\mathfrak{c}^*}^2}{4} \Big)^{\frac{1}{2}} |v||w| - 3 v^2 w^2 + \frac{\nu_{\mathfrak{c}^*}^2}{64} \ln \big(1 - v^2 \big) \Big), \tag{3.8}$$

using (33) and (3.6). When $x \in I_j(t)$, we deduce from (50) that

$$\big| x - a_k(t) \big| \geq \Big| a_k(t) - \frac{a_{j-1}(t) + a_j(t)}{2} \Big| - \frac{1}{4} \big(L + \delta_{\mathfrak{c}^*} t \big) \geq \frac{1}{4} \big(L - 2 + \delta_{\mathfrak{c}^*} t \big),$$

for any $1 \leq k \leq N$. In view of (43), (44) (and the Sobolev embedding theorem), (45) and (2.9), this gives

$$\big| v(x,t) \big| \leq \big| \varepsilon_1(x,t) \big| + \sum_{k=1}^N \big| v_{c_k(t)}(x - a_k(t)) \big| \leq A^* \Big(\alpha + \exp \Big(- \frac{\nu_{\mathfrak{c}^*}}{16} \big(L + \delta_{\mathfrak{c}^*} t \big) \Big) \Big),$$

for any $x \in I_j(t)$. We now decrease α and increase L, if necessary, so that

$$v^2 \leq \min \Big\{ \frac{1}{2}, \frac{\nu_{\mathfrak{c}^*}^2}{96} \Big\}, \tag{3.9}$$

on the interval $I_j(t)$. Since $\ln(1 - s) \geq -2s$ for $0 \leq s \leq 1/2$, we deduce from (3.8) and (3.9) that

$$\mathfrak{R}_1(t) \geq \frac{1}{2} \Big(1 - \Big(1 - \frac{\nu_{\mathfrak{c}^*}^2}{4} \Big)^{\frac{1}{2}} - \frac{\nu_{\mathfrak{c}^*}^2}{32} \Big) \int_{I_j(t)} \partial_x \phi_j \, \big(v^2 + w^2 \big).$$

Since $1 - (1 - s)^{1/2} \geq s/2$, for $0 \leq s \leq 1$, we obtain

$$\mathfrak{R}_1(t) \geq \frac{1}{2} \Big(1 - \Big(1 - \frac{\nu_{\mathfrak{c}^*}^2}{4} \Big)^{\frac{1}{2}} - \frac{\nu_{\mathfrak{c}^*}^2}{32} \Big) \int_{I_j(t)} \partial_x \phi_j \, \big(v^2 + w^2 \big) \geq \frac{3 \nu_{\mathfrak{c}^*}^2}{64} \int_{I_j(t)} \partial_x \phi_j \, \big(v^2 + w^2 \big) \geq 0.$$

Combining with (3.7), we are led to (54). In order to conclude the proof of Proposition 5, it remains to use the conservation of the energy E and the momentum P to obtain (55). □

4 Rephrasing orbital stability in the original framework

4.1 Proof of Corollary 3

In order to rephrase orbital stability in the original setting of the Landau-Lifshitz equation, the main difficulty lies in defining properly the phase θ of the function $\check{m}$ corresponding to an hydrodynamical pair $\mathfrak{v}$. When $\mathfrak{v}$ is close to a sum $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ in the space $\mathcal{NV}(\mathbb{R})$, we can rely on the following lemma.

Lemma 4.1. *Let $\mathfrak{s} \in \{\pm 1\}^N$, $\mathfrak{a} \in \mathbb{R}^N$, with $a_1 < \ldots < a_N$, and $\mathfrak{c} \in (-1,1)^N$, with $c_1 < \ldots < 0 < \ldots < c_N$. Set*

$$I_1 := \Big(- \infty, \frac{a_1 + a_2}{2} \Big], \quad I_j = \Big[\frac{a_{j-1} + a_j}{2}, \frac{a_j + a_{j+1}}{2} \Big], \quad \text{and} \quad I_N = \Big[\frac{a_{N-1} + a_N}{2}, +\infty \Big),$$

for $2 \leq j \leq N-1$. Given any positive number ϵ, there exist positive numbers α and L, depending only on $\mathfrak{c}$ and ϵ, such that, if a pair $\mathfrak{v} \in \mathcal{NV}(\mathbb{R})$ satisfies

$$\big\| \mathfrak{v} - S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}} \big\|_{H^1 \times L^2} \leq \alpha, \tag{4.1}$$

with

$$\min\big\{a_{j+1}-a_j, 1\leq j\leq N-1\big\}\geq L, \tag{4.2}$$

then, given any function $m\in\mathcal{E}(\mathbb{R})$ *corresponding to the pair* $\mathfrak{v}$*, there exist numbers* $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_N)\in\mathbb{R}^N$ *such that*

$$\sum_{j=1}^N\Big(\big|\check{m}(a_j)-\check{u}_j(a_j)\big|+\big\|m'-u_j'\big\|_{L^2(I_j)}+\big\|m_3-[u_j]_3\big\|_{L^2(I_j)}\Big)\leq\epsilon, \tag{4.3}$$

where we have set $u_j=u_{c_j,a_j,\theta_j,s_j}$ *for any* $1\leq j\leq N$.

Proof. Let $1\leq j\leq N$ be fixed. Given any positive number α, we can rely on the exponential decay of the solitons in (5) to guarantee that the sum $S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}$ belongs to the space $\mathcal{NV}(\mathbb{R})$, when the positions a_j satisfy condition (4.2) for L large enough. For a possible further choice of L, we can also derive the estimate

$$\big\|S_{\mathfrak{c},\mathfrak{a},\mathfrak{s}}-\mathfrak{v}_j\big\|_{H^1(I_j)\times L^2(I_j)}\leq\alpha,$$

where we have set $\mathfrak{v}_j:=\mathfrak{v}_{c_j,a_j,s_j}$. When a pair $\mathfrak{v}$ satisfies condition (4.1), we conclude that

$$\big\|\mathfrak{v}-\mathfrak{v}_j\big\|_{H^1(I_j)\times L^2(I_j)}\leq 2\alpha. \tag{4.4}$$

We now consider a function $m\in\mathcal{E}(\mathbb{R})$ corresponding to the hydrodynamical pair $\mathfrak{v}=(v,w)$. By definition, we have $m_3=v$, so that (4.4) directly provides

$$\big\|m_3-[u_j]_3\big\|_{H^1(I_j)}=\big\|v-v_j\big\|_{H^1(I_j)}\leq 2\alpha. \tag{4.5}$$

Here, we have set $\mathfrak{v}_j:=(v_j,w_j)$ in order to simplify the notation.

Similarly, we can write the function $\check{m}=m_1+im_2$ under the form $\check{m}=(1-v^2)^{1/2}e^{i\theta}$, with $\theta'=w$. Setting $\theta_j=\theta(a_j)$, we deduce from (5) that

$$\check{m}(a_j)-\check{u}_j(a_j)=\Big((1-v(a_j)^2)^{\frac{1}{2}}-|c_j|\Big)e^{i\theta_j}.$$

Combining (5) and (4.4) with the Sobolev embedding theorem, we obtain

$$\big|v(a_j)-s_j(1-c_j^2)^{\frac{1}{2}}\big|\leq K\big\|v-v_j\big\|_{H^1(I_j)}\leq K\alpha,$$

where K refers, here as in the sequel, to a universal constant. As a consequence, there exists a positive number A_j, depending only on c_j, such that, decreasing, if necessary, the value of α, we have

$$\big|\check{m}(a_j)-\check{u}_j(a_j)\big|\leq A_j\alpha. \tag{4.6}$$

We finally turn to the derivative of the function $\check{m}$, which is equal to

$$\check{m}'=\Big(-\frac{vv'}{(1-v^2)^{\frac{1}{2}}}+iw(1-v^2)^{\frac{1}{2}}\Big)e^{i\theta}.$$

This identity provides the estimate

$$\begin{aligned}\big|\check{m}'-\check{u}_j'\big|\leq&\bigg|\frac{vv'}{(1-v^2)^{\frac{1}{2}}}-\frac{v_jv_j'}{(1-v_j^2)^{\frac{1}{2}}}\bigg|+\Big|w(1-v^2)^{\frac{1}{2}}-w_j(1-v_j^2)^{\frac{1}{2}}\Big|\\&+\bigg|-\frac{v_jv_j'}{(1-v_j^2)^{\frac{1}{2}}}+iw_j(1-v_j^2)^{\frac{1}{2}}\bigg|\Big|e^{i\theta}-e^{i\vartheta_j}\Big|.\end{aligned} \tag{4.7}$$

In this expression, the phase function ϑ_j is defined as

$$\vartheta_j(x) = \theta_j + \int_{a_j}^x w_j(y)\, dy,$$

so that

$$\big|\theta(x) - \vartheta_j(x)\big| \leq \int_{a_j}^x \big|w(y) - w_j(y)\big|\, dy \leq |x - a_j|^{\frac{1}{2}} \|w - w_j\|_{L^2}, \tag{4.8}$$

for any $x \in I_j$. At this stage, we can combine (5) with the Sobolev embedding theorem to find a positive number R_j, depending only on c_j, such that

$$\int_{|x-a_j| \geq R_j} \Big[\frac{v_j^2 (v_j')^2}{(1 - v_j^2)} + w_j^2 (1 - v_j^2)\Big](x)\, dx \leq \frac{\epsilon^2}{32 N^2}.$$

In view of (4.8), we are led to the bound

$$\int_{I_j} \Big| - \frac{v_j v_j'}{(1 - v_j^2)^{\frac{1}{2}}} + i w_j (1 - v_j^2)^{\frac{1}{2}}\Big|^2 \Big|e^{i\theta} - e^{i\vartheta_j}\Big|^2 \leq A_j R_j \|w - w_j\|_{L^2}^2 + \frac{\epsilon^2}{8N^2}.$$

On the other hand, we can again invoke the Sobolev embedding theorem to write

$$\int_{I_j} \Big|\frac{v v'}{(1 - v^2)^{\frac{1}{2}}} - \frac{v_j v_j'}{(1 - v_j^2)^{\frac{1}{2}}}\Big|^2 + \int_{I_j} \Big|w (1 - v^2)^{\frac{1}{2}} - w_j (1 - v_j^2)^{\frac{1}{2}}\Big|^2 \leq A_j \big\|\mathfrak{v} - \mathfrak{v}_j\big\|_{H^1(I_j) \times L^2(I_j)}^2.$$

In view of (4.4), (4.5), (4.6) and (4.7), we can decrease the value of α to obtain

$$\big|\check{m}(a_j) - \check{u}_j(a_j)\big| + \big\|m_3 - [u_j]_3\big\|_{H^1(I_j)} + \big\|\check{m}' - \check{u}_j'\big\|_{L^2(I_j)} \leq \frac{\epsilon}{N}.$$

This is enough to derive (4.3), and complete the proof of Lemma 4.1. □

With Lemma 4.1 at hand, we are in position to provide the

Proof of Corollary 3. The proof is a direct application of Corollary 1, Theorem 2, and Lemmas 4.1 and A.4. For the sake of completeness, we provide the following details.

We denote by α^* the positive number provided by Theorem 2, and we apply Lemma A.4 with $m_* = R_{\mathfrak{c}^*, \mathfrak{a}^0, \mathfrak{s}^*}$. This provides the existence of a positive number ρ^* such that, under condition (22), the hydrodynamical pair $\mathfrak{v}^0 = (v^0, w^0)$ corresponding to m^0 is well-defined, and satisfies the estimate

$$\big\|\mathfrak{v}^0 - S_{\mathfrak{c}^*, \mathfrak{a}^0, \mathfrak{s}^*}\big\|_{H^1 \times L^2} \leq \alpha^*.$$

Assuming that the number L^* in (23) is larger than the one provided by Theorem 2, we conclude that the solution $\mathfrak{v}$ to (HLL) with initial datum $\mathfrak{v}^0$ is globally well-defined on $\mathbb{R}_+$. In view of Corollary 1, this is enough to guarantee that the solution m to (LL) with initial datum m^0 is also globally well-defined on $\mathbb{R}_+$.

Moreover, there exists a function $\boldsymbol{a} \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$ such that we have the bounds

$$\sum_{j=1}^N \big|a_j'(t) - c_j^*\big| \leq A^* \Big(\alpha^* + \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L^*}{65}\Big)\Big), \tag{4.9}$$

and

$$\big\|\mathfrak{v}(\cdot, t) - S_{\mathfrak{c}^*, \boldsymbol{a}(t), \mathfrak{s}^*}\big\|_{H^1 \times L^2} \leq A^* \Big(\alpha^* + \exp\Big(- \frac{\nu_{\mathfrak{c}^*} L^*}{65}\Big)\Big), \tag{4.10}$$

for any $t \in \mathbb{R}_+$. Observe also that we can derive from the proof of Theorem 2 that

$$\min \Big\{ a_{j+1}(t) - a_j(t), 1 \leq j \leq N-1 \Big\} \geq L^* - 2, \tag{4.11}$$

for any $t \in \mathbb{R}_+$.

At this stage, we can decrease the value of α^* and increase the value of L^*, if necessary, such that (4.9) provides (24). Similarly, we can assume that α^* is small enough and L^* large enough so that (4.10) and (4.11) are enough to apply Lemma 4.1, with $\epsilon = \epsilon^*$. In this case, estimate (25) is exactly (4.3), which is enough to conclude the proof of Corollary 3. □

4.2 Proof of Corollary 4

We now assume that the sum of solitons reduces to a single soliton u_{c^*,a^0,θ^0,s^*}. In this situation, following the lines of the proof of Lemma 4.1 is enough to provide the existence of a positive number α, depending only on c^* and ϵ^*, such that, if a pair $\mathfrak{v} \in \mathcal{NV}(\mathbb{R})$ satisfies

$$\big\| \mathfrak{v} - \mathfrak{v}_{c^*,a,s^*} \big\|_{H^1 \times L^2} \leq \alpha,$$

for a point $a \in \mathbb{R}$, then, there exists a number $\theta \in \mathbb{R}$ such that any function $m \in \mathcal{E}(\mathbb{R})$ corresponding to the pair $\mathfrak{v}$ satisfies the estimates

$$\big| \check{m}(a) - \check{u}_{c^*,a,\theta,s^*}(a) \big| + \big\| m' - u'_{c^*,a,\theta,s^*} \big\|_{L^2} + \big\| m_3 - [u_{c^*,a,\theta,s^*}]_3 \big\|_{L^2} \leq \epsilon^*.$$

With this statement at hand, we can argue as in the proof of Corollary 3 (replacing the use of Theorem 2 by the use of Corollary 2) to complete the proof of Corollary 4. We refer to the proof of Corollary 3 for more details. □

A The Cauchy problem for the Landau-Lifshitz equation

This appendix is mainly devoted to the proof of Theorem 1, in other words, to the local well-posedness of (HLL) in the space $\mathcal{NV}(\mathbb{R})$. In Subsection A.1, we establish the existence of smooth solutions by following the strategy developed by Sulem, Sulem and Bardos in [30] (see also [15]) for the Schrödinger map equation (see Proposition A.2 below). We then control the smooth solutions by controlling the solutions v and Ψ to the system of equations (13) and (15) (see Proposition A.3 below). This provides the statements in Theorem 1. We complete this analysis by the proof of Corollary 1 in Subsection A.3.

A.1 Construction of smooth solutions

Before addressing the Cauchy problem for the Landau-Lifshitz equation, we establish a useful density result concerning the energy space $\mathcal{E}(\mathbb{R})$.

Lemma A.1. *Let $m \in \mathcal{E}(\mathbb{R})$. There exists a sequence of smooth functions $m_n \in \mathcal{E}(\mathbb{R})$, with $\partial_x m_n \in H^\infty(\mathbb{R})$, and such that*

$$m_n - m \to 0 \quad \text{in } H^1(\mathbb{R}), \tag{A.1}$$

as $n \to +\infty$. If the derivative $\partial_x m$ is moreover in $H^k(\mathbb{R})$ for an integer $k \geq 1$, then

$$\partial_x m_n \to \partial_x m \quad \text{in } H^k(\mathbb{R}). \tag{A.2}$$

Proof. The proof is standard (see e.g. [29]). For the sake of completeness, we recall the following details. Consider a function $\chi \in \mathcal{C}^\infty(\mathbb{R})$, with a compactly supported Fourier transform, and such that $|\widehat{\chi}| \leq 1$, $\widehat{\chi} = 1$ on $(-1, 1)$, and $\widehat{\chi} = 0$ outside $(-2, 2)$. Denote by μ_n the maps given by

$$\mu_n(x) = n \int_\mathbb{R} \chi(n(x-y))\, m(y)\, dy,$$

for any $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$. Since χ belongs to the Schwartz class, we can combine its decay at infinity with the fact that $|m| = 1$ almost everywhere to guarantee that μ_n is well-defined and smooth on $\mathbb{R}$. On the other hand, the Fourier transform of μ_n is equal to

$$\widehat{\mu_n}(\xi) = \widehat{\chi}\Big(\frac{\xi}{n}\Big)\widehat{m}(\xi).$$

Since $\partial_x m$ is square integrable, and $\widehat{\chi}$ has compact support in $(-2, 2)$, the Plancherel formula provides

$$\big\|\partial_x \mu_n\big\|_{H^k}^2 = \frac{1}{2\pi} \int_\mathbb{R} (1 + |\xi|^2)^k \big|\widehat{\partial_x m}(\xi)\big|^2 \Big|\widehat{\chi}\Big(\frac{\xi}{n}\Big)\Big|^2\, d\xi \leq \big(1 + 4n^2\big)^k \big\|\partial_x m\big\|_{L^2}^2.$$

Hence, $\partial_x \mu_n$ belongs to $H^\infty(\mathbb{R})$. We also check that

$$\big\|\mu_n - m\big\|_{L^2}^2 = \frac{1}{2\pi} \int_\mathbb{R} \big|\widehat{m}(\xi)\big|^2 \Big|\widehat{\chi}\Big(\frac{\xi}{n}\Big) - 1\Big|^2\, d\xi \leq \frac{1}{n^2} \int_\mathbb{R} \big|\widehat{\partial_x m}(\xi)\big|^2\, d\xi \to 0,$$

while, by the dominated convergence theorem,

$$\big\|\partial_x \mu_n - \partial_x m\big\|_{L^2}^2 = \frac{1}{2\pi} \int_\mathbb{R} \big|\widehat{\partial_x m}(\xi)\big|^2 \Big|\widehat{\chi}\Big(\frac{\xi}{n}\Big) - 1\Big|^2\, d\xi \to 0,$$

as $n \to +\infty$. This proves (A.1). The convergence in (A.2) follows similarly. As a conclusion, the maps μ_n satisfy all the statements in Lemma A.1, except that they are not valued into $\mathbb{S}^2$.

In order to complete the proof, we infer from (A.1) and the Sobolev embedding theorem that

$$\big\|\mu_n - m\big\|_{L^\infty} \to 0,$$

as $n \to +\infty$. In particular, we have

$$\big\||\mu_n| - 1\big\|_{L^\infty} \to 0.$$

For n large enough, we can assume that $|\mu_n| \geq 1/2$ on $\mathbb{R}$, so that we can define the map $m_n = \mu_n/|\mu_n|$. It is then enough to apply the chain rule formula for Sobolev functions to check that the maps m_n are smooth from $\mathbb{R}$ to $\mathbb{S}^2$, belong to the energy space $\mathcal{E}(\mathbb{R})$, with $\partial_x m_n \in H^\infty(\mathbb{R})$, and satisfy the convergences in (A.1) and (A.2). □

We now turn to the well-posedness of (LL) when the prescribed initial data m^0 is smooth enough. We recall that the Landau-Lifshitz equation is integrable in dimension one by means of the inverse scattering method (see e.g. [11]). In particular, it owns an infinite number of invariant quantities, among which the energy E and the second order energy

$$E_2(m) := \int_\mathbb{R} \Big(|\partial_t m|^2 + |\partial_{xx} m|^2 - \frac{3}{2}|\partial_x m|^4 - m_3^2 |\partial_x m|^2 - m_3^2 + \frac{1}{2} m_3^4\Big).$$

Lemma A.2. *Let $T > 0$. Given a smooth function $m^0 \in \mathcal{E}(\mathbb{R})$, with $\partial_x m^0 \in H^\infty(\mathbb{R})$, we consider a solution $m \in \mathcal{C}^\infty(\mathbb{R} \times [0,T], \mathbb{S}^2)$ to* (LL) *with initial datum m^0, and we assume that m_3 and $\partial_x m$ are in $\mathcal{C}^0([0,T], H^\ell(\mathbb{R}))$ for any $\ell \in \mathbb{N}$. Then, we have*

$$E\big(m(\cdot,t)\big) = E(m^0), \quad \text{and} \quad E_2\big(m(\cdot,t)\big) = E_2\big(m(\cdot,0)\big),$$

for any $t \in [0,T]$. In particular, there exists a positive number A such that

$$\|\partial_t m(\cdot,t)\|_{H^{k-1}}^2 + \|m_3(\cdot,t)\|_{H^k}^2 + \|\partial_x m(\cdot,t)\|_{H^k}^2 \leq A\big(\|m_3^0\|_{H^k}^2 + \|\partial_x m^0\|_{H^k}^2\big), \tag{A.3}$$

for any $t \in [0,T]$ and $k \in \{0,1\}$.

Proof. The conservation of the energy E follows from the direct computation

$$\frac{d}{dt}\big(E(m)\big) = -\int_{\mathbb{R}} \langle \partial_t m, \partial_{xx} m - m_3 e_3\rangle_{\mathbb{R}^3} = 0,$$

using (LL). This conservation provides the control of the L^2-norm of m_3 and $\partial_x m$ in (A.3) with $k = 0$. We next use the identity $\partial_t m = m \times m_3 e_3 - \partial_x(m \times \partial_x m)$ to bound the derivative $\partial_t m$ according to (A.3).

Concerning the second order energy, we derive from (LL) the second order equation

$$\begin{aligned}\partial_{tt} m + \partial_{xxxx} m - (\partial_{xx} m_3) e_3 = &- \partial_x\big(|\partial_x m|^2 \partial_x m + 4\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3} m\big) - 2 m_3^2 \partial_{xx} m\\ &- 2 m_3 (\partial_x m_3) \partial_x m + m_3 (\partial_{xx} m_3) m + m_3^3 e_3 - m_3^2\, m\\ &- \langle m \times \partial_{xx} m, e_3\rangle_{\mathbb{R}^3}\, m \times e_3.\end{aligned} \tag{A.4}$$

This equation appears as a consequence of the pointwise identities

$$\langle m, \partial_x m\rangle_{\mathbb{R}^3} = \langle m, \partial_{xx} m\rangle_{\mathbb{R}^3} + |\partial_x m|^2 = \langle m, \partial_{xxx} m\rangle_{\mathbb{R}^3} + 3\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3} = 0,$$

which follow from the condition $|m| = 1$, and of the algebraic identities

$$a \times (b \times c) = -\langle a, b\rangle_{\mathbb{R}^3}\, c + \langle a, c\rangle_{\mathbb{R}^3}\, b, \quad \text{and} \quad (a \times b) \times c = -\langle b, c\rangle_{\mathbb{R}^3}\, a + \langle a, c\rangle_{\mathbb{R}^3}\, b.$$

Taking the L^2-product of (A.4) with $\partial_t m$, using the identity $\langle m, \partial_t m\rangle_{\mathbb{R}^3} = 0$, and integrating by parts, we compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \Big(|\partial_t m|^2 + |\partial_{xx} m|^2 - \frac{3}{2}|\partial_x m|^4\Big) = 2\int_{\mathbb{R}} \Big(&\partial_t m_3 \big(\partial_{xx} m_3 + m_3^3\big) + m_3^2 \langle \partial_t \partial_x m, \partial_x m\rangle_{\mathbb{R}^3}\\ &+ \big(2|\partial_x m|^2 - m_3^2\big)\langle \partial_t m, \partial_{xx} m\rangle_{\mathbb{R}^3} - \langle \partial_t m, m \times e_3\rangle_{\mathbb{R}^3}\, \langle m \times \partial_{xx} m, e_3\rangle_{\mathbb{R}^3}\Big).\end{aligned}$$

The conservation of E_2 then follows from the identities $\langle \partial_t m, \partial_{xx} m\rangle_{\mathbb{R}^3} = m_3\, \partial_t m_3$, and

$$\langle m \times \partial_{xx} m, e_3\rangle_{\mathbb{R}^3}\, \langle \partial_t m, m \times e_3\rangle_{\mathbb{R}^3} = \big(m_3 |\partial_x m|^2 + \partial_{xx} m_3 + m_3^3 - m_3\big)\partial_t m_3.$$

Combining this conservation with the Gagliardo-Nirenberg inequality

$$\|\partial_x f\|_{L^4}^4 \leq K \|\partial_{xx} f\|_{L^2} \|\partial_x f\|_{L^2}^3,$$

and bound (A.3) for $k = 0$, we obtain (A.3) for $k = 1$. □

We next derive the following higher order estimates.

Lemma A.3. *Let $k \geq 2$ and $T > 0$. Given a smooth function $m^0 \in \mathcal{E}(\mathbb{R})$, with $\partial_x m^0 \in H^\infty(\mathbb{R})$, we consider a solution $m \in \mathcal{C}^\infty(\mathbb{R} \times [0,T], \mathbb{S}^2)$ to (LL) with initial datum m^0, and we assume that m_3 and $\partial_x m$ are in $\mathcal{C}^0([0,T], H^\ell(\mathbb{R}))$ for any $\ell \in \mathbb{N}$. Then, there exists a positive number A, depending only on k, such that*

$$\begin{aligned} \|\partial_t m(\cdot,t)\|_{H^{k-1}}^2 + &\|m_3(\cdot,t)\|_{H^k}^2 + \|\partial_x m(\cdot,t)\|_{H^k}^2 \\ &\leq \big(\|m_3^0\|_{H^k}^2 + \|\partial_x m^0\|_{H^k}^2\big) \exp\Big(A\big(1 + \|m_3^0\|_{H^1}^3 + \|\partial_x m^0\|_{H^1}^3\big)\, t\Big), \end{aligned} \tag{A.5}$$

for any $t \in [0,T]$.

Proof. The proof relies on standard energy estimates. Set

$$I_k(t) := \frac{1}{2} \int_{\mathbb{R}} \Big(|\partial_t \partial_x^{k-1} m|^2 + |\partial_x^{k+1} m|^2 + |\partial_x^k m_3|^2\Big).$$

We deduce from (A.4) the formula

$$I_k'(t) = \int_{\mathbb{R}} \langle \partial_t \partial_x^{k-1} m, \partial_x^{k-1} F(m) \rangle_{\mathbb{R}^3}. \tag{A.6}$$

In this identity, $F(m)$ refers to the right-hand side of (A.4), which we rewrite as

$$\begin{aligned} F(m) = &- \partial_x\Big(|\partial_x m|^2 \partial_x m + 4\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3} m + 2m_3^2 \partial_x m - m_3(\partial_x m_3) m \\ &+ \langle m \times \partial_x m, e_3\rangle_{\mathbb{R}^3}\, m \times e_3\Big) + m_3(\partial_x m_3)\partial_x m - (\partial_x m_3)^2 m + m_3^3 e_3 - m_3^2\, m \\ &+ \langle m \times \partial_x m, e_3\rangle_{\mathbb{R}^3}\, \partial_x m \times e_3. \end{aligned}$$

Recall next the Gagliardo-Nirenberg inequalities

$$\big\|\partial_x^j f\big\|_{L^{\frac{2k}{j}}} \leq A \big\|f\big\|_{L^\infty}^{1-\frac{j}{k}} \big\|\partial_x^k f\big\|_{L^2}^{\frac{j}{k}}, \tag{A.7}$$

which hold for $f \in H^k(\mathbb{R})$, for $0 \leq j \leq k$ and for some positive number A, depending only on k. In view of the expression above for $F(m)$, we infer from (A.7) and the Leibniz rule that

$$\big\|\partial_x^{k-1} F(m) + 4\partial_x^k\big(\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3}\big) m\big\|_{L^2} \leq A\big(1 + \|m_3\|_{L^\infty}^2 + \|\partial_x m\|_{L^\infty}^2\big)\big(\|m_3\|_{H^k}^2 + \|\partial_x m\|_{H^k}^2\big)^{\frac{1}{2}}. \tag{A.8}$$

On the other hand, we derive from an integration by parts, and the Leibniz rule that

$$\begin{aligned} \int_{\mathbb{R}} \langle \partial_t \partial_x^{k-1} m, &m\rangle_{\mathbb{R}^3}\, \partial_x^k\Big(\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3}\Big) \\ = &\sum_{j=1}^{k} \binom{k}{j} \partial_x\Big(\langle \partial_x^{j-1}(\partial_x m) \times \partial_x^{k-j}(\partial_x m), m\rangle_{\mathbb{R}^3}\Big)\, \partial_x^{k-1}\Big(\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3}\Big) \\ &- \int_{\mathbb{R}} \partial_x\Big(\langle \partial_x^{k-1}(m \times m_3 e_3), m\rangle_{\mathbb{R}^3}\Big)\, \partial_x^{k-1}\Big(\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3}\Big), \end{aligned}$$

so that, again by (A.7) and the Leibniz rule,

$$\int_{\mathbb{R}} \langle \partial_t \partial_x^{k-1} m, m\rangle_{\mathbb{R}^3}\, \partial_x^k\Big(\langle \partial_x m, \partial_{xx} m\rangle_{\mathbb{R}^3}\Big) \leq A\big(1 + \|m_3\|_{L^\infty}^3 + \|\partial_x m\|_{L^\infty}^3\big)\big(\|m_3\|_{H^k}^2 + \|\partial_x m\|_{H^k}^2\big).$$

Combining with (A.6) and (A.8), and applying the Sobolev embedding theorem, we are led to

$$I_k'(t) \leq A\big(1 + \|m_3\|_{H^1}^3 + \|\partial_x m\|_{H^1}^3\big)\big(\|\partial_t \partial_x^{k-1} m\|_{L^2}^2 + \|m_3\|_{H^k}^2 + \|\partial_x m\|_{H^k}^2\big).$$

It remains to invoke the uniform bound on $\|m_3\|_{H^1}$ and $\|\partial_x m\|_{H^1}$ in Lemma A.2, and to apply the Gronwall lemma to obtain (A.5). This completes the proof of Lemma A.3. □

We are now in position to address the Cauchy problem for the Landau-Lifshitz equation. The energy estimates in (A.5) provide a natural functional framework to solve this problem. We shall look for solutions m with $\partial_t m \in \mathcal{C}^0([0,T], H^{k-1}(\mathbb{R}))$, $m_3 \in \mathcal{C}^0([0,T], H^k(\mathbb{R}))$ and $\partial_x m \in \mathcal{C}^0([0,T], H^k(\mathbb{R}))$, for some integer k.

This approach has the drawback of not providing any functional setting for the function m itself. However, we observe that

$$m(\cdot,t) = m^0(\cdot) + \int_0^t \partial_t m(\cdot,s)\, ds, \tag{A.9}$$

lies in $\mathcal{C}^0([0,T], m^0 + H^{k-1}(\mathbb{R}))$. Again, it is natural to look for the solution m in this functional space, or equivalently, in the space $\mathcal{C}^0([0,T], \boldsymbol{m} + H^{k-1}(\mathbb{R}))$, where, according to Lemma A.1, $\boldsymbol{m}$ refers to a smooth function in $\mathcal{E}(\mathbb{R})$, with $\partial_x \boldsymbol{m} \in H^\infty(\mathbb{R})$, and $m^0 - \boldsymbol{m} \in H^{k-1}(\mathbb{R})$.

At this stage, recall that Lemma A.1 guarantees that any function in the energy space $\mathcal{E}(\mathbb{R})$ belongs to some space of the form $\boldsymbol{m} + H^1(\mathbb{R})$. In other words, solving the Cauchy problem for (LL) in $\mathcal{E}(\mathbb{R})$ amounts to solve it in all the sets $\boldsymbol{m} + H^1(\mathbb{R})$. An advantage of the sets $\boldsymbol{m} + H^1(\mathbb{R})$ is that they are naturally endowed with the metric structure corresponding to the H^1-norm (see [12] for similar results in the context of the Gross-Pitaevskii equation).

As a consequence, we fix from now on a smooth map $\boldsymbol{m} \in \mathcal{E}(\mathbb{R})$, with $\partial_x \boldsymbol{m} \in H^\infty(\mathbb{R})$. Following the arguments developed in [30], we show the following statement for the Cauchy problem for (LL).

Proposition A.1. *Let $k \geq 3$ and $m^0 \in \boldsymbol{m} + H^{k+1}(\mathbb{R})$, with $|m^0| = 1$ a.e. There exists a unique solution $m : \mathbb{R} \times [0,+\infty) \to \mathbb{S}^2$ to* (LL)*, with initial datum m^0, such that $\partial_t m \in L^\infty([0,T], H^{k-1}(\mathbb{R}))$, $m_3 \in L^\infty([0,T], H^k(\mathbb{R}))$ and $\partial_x m \in L^\infty([0,T], H^k(\mathbb{R}))$ for any positive number T. In particular, m belongs to $\mathcal{C}^0([0,+\infty), \boldsymbol{m} + H^{k-1}(\mathbb{R}))$. Moreover, the energy E is constant along the flow.*

When $k \in \{0,1,2\}$, the existence of such a weak solution remains true. There still exists a solution $m : \mathbb{R} \times [0,+\infty) \to \mathbb{S}^2$ to (LL), with initial datum m^0, such that $\partial_t m \in L^\infty([0,T], H^{k-1}(\mathbb{R}))$, $m_3 \in L^\infty([0,T], H^k(\mathbb{R}))$ and $\partial_x m \in L^\infty([0,T], H^k(\mathbb{R}))$ for any $T \in (0,+\infty)$. We refer to [30] for the construction of this solution in the context of the Schrödinger map equation (see also [15]). However, its uniqueness is not immediate. We refer to [18] for a discussion about this subject (again for the Schrödinger map equation).

In the sequel, we solve this issue in the context of the hydrodynamical Landau-Lifshitz equation by establishing the uniqueness of the (HLL) flow when the initial datum $\mathfrak{v}^0$ belongs to $\mathcal{NV}(\mathbb{R})$. This turns out to be sufficient in order to establish the stability of (well-prepared) sums of solitons for the Landau-Lifshitz equation, which is the main focus of this paper.

Proof of Proposition A.1. Concerning the existence of a weak solution, we rely on the strategy developed by Sulem, Sulem and Bardos [30] in the context of the Schrödinger map equation (see also [15, Chapter 3]). We discretize the Landau-Lifshitz equation according to the finite-difference scheme in [30], and we check that the a priori bounds in (A.5) remain available for the discretized equation. We refer to [30] for more details about these computations.

Combining these a priori bounds with standard weak compactness and local strong compactness results, we obtain the existence of a weak solution $m : \mathbb{R} \times [0,T] \to \mathbb{S}^2$ to (LL), with initial datum m^0, and such that $\partial_t m \in L^\infty([0,T], H^{k-1}(\mathbb{R}))$, $m_3 \in L^\infty([0,T], H^k(\mathbb{R}))$ and $\partial_x m \in L^\infty([0,T], H^k(\mathbb{R}))$ for any fixed positive time T. This solution satisfies the a priori bounds in (A.5). Moreover, in view of (A.9), it lies in $m^0 + \mathcal{C}^0([0,T], H^{k-1}(\mathbb{R}))$, or equivalently, in $\boldsymbol{m} + \mathcal{C}^0([0,T], H^{k-1}(\mathbb{R}))$. We now turn to the uniqueness of this solution.

In this direction, we rely on the arguments developed in [9] (see also [18, Appendix]). We consider a similar solution $\tilde{m}$ for a possible different initial datum $\tilde{m}^0 \in \boldsymbol{m} + H^{k+1}(\mathbb{R})$, and we set $f := m - \tilde{m}$ and $g := (m + \tilde{m})/2$. The functions f and g belong to $\mathcal{C}^0([0,T], H^2(\mathbb{R}))$, resp. $\boldsymbol{m} + \mathcal{C}^0([0,T], H^2(\mathbb{R}))$, and they satisfy

$$\partial_t f = -\partial_x\big(f \times \partial_x g + g \times \partial_x f\big) + g_3\big(f \times e_3\big) + f_3\big(g \times e_3\big). \tag{A.10}$$

Hence, $\partial_t f$ lies in $\mathcal{C}^0([0,T], L^2(\mathbb{R}))$, and we are allowed to compute after an integration by parts,

$$\frac{d}{dt}\int_{\mathbb{R}} |f|^2 = 2\int_{\mathbb{R}} \Big(\langle \partial_x g \times \partial_x f, f\rangle_{\mathbb{R}^3} + f_3 \langle g \times e_3, f\rangle_{\mathbb{R}^3}\Big). \tag{A.11}$$

Similarly, $\partial_t\partial_x f$ belongs to $\mathcal{C}^0([0,T], H^{-1}(\mathbb{R}))$, while $\partial_x f$ is in $\mathcal{C}^0([0,T], H^1(\mathbb{R}))$. As a consequence, we can write

$$\frac{d}{dt}\int_{\mathbb{R}} |\partial_x f|^2 = 2\langle \partial_t\partial_x f, \partial_x f\rangle_{H^{-1},H^1} = -2\int_{\mathbb{R}} \langle \partial_t f, \partial_{xx} f\rangle_{\mathbb{R}^3},$$

so that, by (A.10),

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |\partial_x f|^2 =& 2\int_{\mathbb{R}} \Big(- \langle f \times \partial_{xxx} g, \partial_x f\rangle_{\mathbb{R}^3} + \partial_x g_3 \, \langle \partial_x f, f \times e_3\rangle_{\mathbb{R}^3} \\ & + \partial_x f_3 \, \langle \partial_x f, g \times e_3\rangle_{\mathbb{R}^3} + f_3 \langle \partial_x f, \partial_x g \times e_3\rangle_{\mathbb{R}^3}\Big).\end{aligned} \tag{A.12}$$

Combining (A.11) and (A.12) with the a priori bound (A.5) (and the Sobolev embedding theorem), we deduce the existence of a positive number A, depending only on T, and the H^2-norms of m_3^0, $\tilde{m}_3^0$, $\partial_x m^0$ and $\partial_x \tilde{m}^0$, such that

$$\frac{d}{dt}\big(\|f\|_{H^1}^2\big) \leq A \|f\|_{H^1}^2.$$

The uniqueness for any positive time T then follows from the Gronwall inequality. As a consequence of uniqueness, the solution is actually global.

It finally remains to show the conservation of the energy. Since m_3 and $\partial_x m$ belong to $\mathcal{C}^0([0,+\infty), H^2(\mathbb{R}))$, resp. $\mathcal{C}^0([0,+\infty), H^1(\mathbb{R}))$, while $\partial_t m_3$ and $\partial_t\partial_x m$ are in $\mathcal{C}^0([0,T], L^2(\mathbb{R}))$, resp. $\mathcal{C}^0([0,T], H^{-1}(\mathbb{R}))$, we are allowed to differentiate the energy with respect to time. In view of the computations in Lemma A.2, this is enough to guarantee that the energy is constant along the flow. □

We now turn to the hydrodynamical Landau-Lifshitz equation. Our goal is to prove that it is locally well-posed in the spaces $\mathcal{NV}^k(\mathbb{R})$, which were previously defined in (3.3). When k is large enough, we can show

Proposition A.2. *Let $k \geq 4$ and $\mathfrak{v}^0 = (v^0, w^0) \in \mathcal{NV}^k(\mathbb{R})$. There exists a positive maximal time $T_{\max}$, and a unique solution $\mathfrak{v} = (v, w)$ to* (HLL)*, with initial datum $\mathfrak{v}^0$, such that $\mathfrak{v}$ belongs to $\mathcal{C}^0([0, T_{\max}), \mathcal{NV}^{k-2}(\mathbb{R}))$, and $L^\infty([0,T], \mathcal{NV}^k(\mathbb{R}))$ for any $0 < T < T_{\max}$. The maximal time $T_{\max}$ is characterized by the condition*

$$\lim_{t \to T_{\max}} \|v(\cdot, t)\|_{\mathcal{C}^0} = 1 \quad \text{if} \quad T_{\max} < +\infty. \tag{A.13}$$

Moreover, the energy E and the momentum P are constant along the flow.

Proof. Let us set

$$\varphi^0(x) := \int_0^x w^0(y)\, dy,$$

for any $x \in \mathbb{R}$, and

$$m^0 := \Big(\big(1 - (v^0)^2\big)^{\frac{1}{2}} \cos\big(\varphi^0\big), \big(1 - (v^0)^2\big)^{\frac{1}{2}} \sin\big(\varphi^0\big), v^0 \Big).$$

The function m^0 is well-defined and belongs to $\mathcal{E}(\mathbb{R})$. Moreover, $\partial_x m^0$ and m_3^0 are in $H^k(\mathbb{R})$, resp. $H^{k+1}(\mathbb{R})$. In particular, we deduce from Lemma A.1 the existence of a smooth map $\boldsymbol{m} \in \mathcal{E}(\mathbb{R})$, with $\partial_x \boldsymbol{m} \in H^\infty(\mathbb{R})$, and such that $m^0 \in \boldsymbol{m} + H^{k+1}(\mathbb{R})$.

Concerning the existence of a solution $\mathfrak{v}$, we apply Proposition A.1. It provides the existence of a unique solution m to (LL), with initial condition m^0, such that $m \in \mathcal{C}^0(\mathbb{R}, \boldsymbol{m} + H^{k-1}(\mathbb{R}))$, while $\partial_t m \in L^\infty([0,T], H^{k-1}(\mathbb{R}))$, $m_3 \in L^\infty([0,T], H^k(\mathbb{R}))$ and $\partial_x m \in L^\infty([0,T], H^k(\mathbb{R}))$ for any $T \in (0, +\infty)$. At this stage, we introduce the number

$$T^* := \inf \big\{ t \in [0, +\infty) \text{ s.t. } \|m_3(\cdot, t)\|_{\mathcal{C}^0} = 1 \big\} \in [0, +\infty].$$

Since $m_3 \in \mathcal{C}^0([0,T], H^1(\mathbb{R}))$ and $\|m_3^0\|_{\mathcal{C}^0} = \|v^0\|_{\mathcal{C}^0} < 1$, we infer from the Sobolev embedding theorem that $T^* > 0$. In particular, we can set

$$\mathfrak{v}(x,t) := \Big(m_3(x,t), \frac{m_1(x,t)\partial_x m_2(x,t) - m_2(x,t)\partial_x m_1(x,t)}{1 - m_3(x,t)^2} \Big), \tag{A.14}$$

for any $(x,t) \in \mathbb{R} \times [0, T^*)$. The function $\mathfrak{v}$ is well-defined, with $\mathfrak{v}(\cdot, 0) = \mathfrak{v}^0$. Moreover, it belongs to $\mathcal{C}^0([0,T^*), \mathcal{NV}^{k-2}(\mathbb{R}))$ and $L^\infty([0,T], \mathcal{NV}^k(\mathbb{R}))$ for any $0 < T < T^*$. Finally, since m satisfies (LL), $\mathfrak{v}$ is solution to (HLL). This completes the proof of the existence.

Concerning the uniqueness, we consider a solution $\tilde{\mathfrak{v}} = (\tilde{v}, \tilde{w})$ to (HLL), with the same initial datum $\mathfrak{v}^0$, which belongs to $\mathcal{C}^0([0,T_*), \mathcal{NV}^{k-2}(\mathbb{R}))$ for some positive number T_*, and $L^\infty([0,T], \mathcal{NV}^k(\mathbb{R}))$ for any $0 < T < T_*$. We introduce the solution $\tilde{\varphi}$ to the equation

$$\partial_t \tilde{\varphi} = \frac{1}{(1 - \tilde{v}^2)^{\frac{1}{2}}} \partial_x \Big(\frac{\partial_x \tilde{v}}{(1 - \tilde{v}^2)^{\frac{1}{2}}} \Big) + \tilde{v}\big(\tilde{w}^2 - 1\big), \tag{A.15}$$

with initial datum φ^0, and we consider the map

$$\tilde{m} := \Big((1 - \tilde{v}^2)^{\frac{1}{2}} \cos(\tilde{\varphi}), (1 - \tilde{v}^2)^{\frac{1}{2}} \sin(\tilde{\varphi}), \tilde{v} \Big).$$

Since $\tilde{\mathfrak{v}}$ is in $\mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$, there exists, for any fixed number $0 < T < T_*$, a positive number δ_T such that

$$1 - \tilde{v}(x,t)^2 \geq \delta_T, \tag{A.16}$$

for any $(x,t) \in \mathbb{R} \times [0,T]$. Therefore, the quantity in the right-hand side of (A.15) is well-defined on $\mathbb{R} \times [0,T]$, and it belongs to $L^\infty([0,T], H^{k-1}(\mathbb{R}))$. It follows from the Sobolev embedding theorem that the function $\tilde{\varphi}$ is also well-defined as a continuous function on $\mathbb{R} \times [0,T]$. Moreover, in view of the second equation in (HLL), we have

$$\partial_t \big(\partial_x \tilde{\varphi} - \tilde{w} \big) = 0,$$

in $L^\infty([0,T], H^{k-2}(\mathbb{R}))$. Since $\partial_x \tilde{\varphi}(\cdot, 0) = \partial_x \varphi^0 = w^0 = \tilde{w}(\cdot, 0)$, it follows that $\tilde{w} = \partial_x \tilde{\varphi}$. In particular, $\partial_x \tilde{\varphi}$ belongs to $L^\infty([0,T], H^k(\mathbb{R}))$.

As a result, the map $\tilde{m}$ is well-defined, at least, as a continuous map from $\mathbb{R} \times [0, T]$ to $\mathbb{S}^2$, with $\tilde{m}(\cdot, 0) = m^0$. In addition, $\tilde{m}_3 = \tilde{v}$ is in $L^\infty([0, T], H^{k+1}(\mathbb{R}))$, while, by the first equation in (HLL), $\partial_t \tilde{m}_3$ lies in $L^\infty([0, T], H^{k-1}(\mathbb{R}))$. Concerning the other two components, we can write

$$\partial_t \tilde{m}_1 = -\frac{\tilde{v}\, \partial_t \tilde{v}}{(1 - \tilde{v}^2)^{\frac{1}{2}}} \cos(\tilde{\varphi}) - (1 - \tilde{v}^2)^{\frac{1}{2}}\, \partial_t \tilde{\varphi}\, \sin(\tilde{\varphi}).$$

Recall that $\tilde{v} \in L^\infty([0, T], H^{k+1}(\mathbb{R}))$, with the lower bound in (A.16), and $\partial_t \tilde{v} \in L^\infty([0, T], H^{k-1}(\mathbb{R}))$ by (HLL), while $\tilde{\varphi}$ is continuous on $\mathbb{R} \times [0, T]$, with $\partial_t \tilde{\varphi} \in L^\infty([0, T], H^{k-1}(\mathbb{R}))$ by (A.15), and $\partial_x \tilde{\varphi} \in L^\infty([0, T], H^k(\mathbb{R}))$. As a consequence, $\partial_t \tilde{m}_1$, and $\partial_t \tilde{m}_2$ as well, are in $L^\infty([0, T], H^{k-1}(\mathbb{R}))$. Similarly, $\partial_x \tilde{m}_1$ and $\partial_x \tilde{m}_2$ are in $L^\infty([0, T], H^k(\mathbb{R}))$

In view of (HLL), (A.15), and the identity $\tilde{w} = \partial_x \tilde{\varphi}$, we also observe that $\tilde{m}$ is solution to (LL). Using Proposition A.1, we conclude that $\tilde{m}$ is equal to the unique solution m with initial datum m^0, which was considered in the existence part of this proof. Therefore, the pair $\tilde{\mathfrak{v}}$ is equal to the pair $\mathfrak{v}$ in (A.14). This proves the uniqueness of the solution $\mathfrak{v}$.

In addition, the maximal time of existence $T_{\max}$ is necessarily larger or equal to T^*. Since we cannot continue the solution $\mathfrak{v}$ corresponding to m beyond T^* due to the fact that $\|m_3(\cdot, T^*)\|_{\mathcal{C}^0} = 1$, when T^* is finite, $T_{\max}$ is necessarily equal to T^*. Hence, it is characterized by condition (A.13).

Finally, the conservation of the energy for $\mathfrak{v}$ follows from the conservation of the energy for m in Proposition A.1. The conservation of the momentum is a consequence of the conservation law in Lemma 3.1, which is available since $k \geq 4$. This completes the proof of Proposition A.2. $\square$

The smooth solutions to (HLL) constructed in Proposition A.2 depend continuously on their initial datum in some high order space $\mathcal{NV}^k(\mathbb{R})$ with k large enough (see e.g. [18, Appendix]). However, there is no evidence, at least with the arguments developed in the proof of Proposition A.2, that this continuity can hold in the energy space $\mathcal{NV}(\mathbb{R})$. This is a major obstacle in the construction of solutions in the energy space by taking the limit of smooth solutions.

In order to by-pass this obstacle, we introduce the system of equations (13)-(15), for which it is possible to establish continuity with respect to the initial datum in the energy space (see Proposition A.3 below). This in turn provides a similar continuity dependence for (HLL). We finally show the local well-posed of (HLL) in the energy space by taking limits of the smooth solutions built in Proposition A.2 (see the proof of Theorem 1 below).

Before considering this limit, we justify the derivation of the system of equations (13)-(15) satisfied by the variables v and Ψ, when $\mathfrak{v}$ is a smooth solution to (HLL).

Corollary A.1. *Let $k \geq 4$ and $\mathfrak{v}^0 \in \mathcal{NV}^k(\mathbb{R})$. Consider the unique solution $\mathfrak{v}$ to* (HLL) *with initial datum $\mathfrak{v}^0$, which is given by Proposition A.2. Then, the maps Ψ in* (11) *and $F(v, \Psi)$ in* (14) *are well-defined and continuous on $\mathbb{R} \times [0, T_{\max})$, with $\Psi \in \mathcal{C}^0([0, T_{\max}), H^{k-2}(\mathbb{R}))$ and $\partial_x F(v, \Psi) \in \mathcal{C}^0([0, T_{\max}), H^{k-2}(\mathbb{R}))$. Moreover, they solve the system of equations* (13)-(15).

Proof. Let $0 < T < T_{\max}$. Since $\mathfrak{v} \in \mathcal{C}^0([0, T], \mathcal{NV}^{k-2}(\mathbb{R}))$, we deduce from (12) that the function θ is well-defined, bounded and continuous on $\mathbb{R} \times [0, T]$, with $\partial_x \theta = -vw \in \mathcal{C}^0([0, T], H^{k-2}(\mathbb{R}))$. On the other hand, since $v \in \mathcal{C}^0([0, T], H^1(\mathbb{R}))$, we deduce from the Sobolev embedding theorem the existence of a positive number δ_T such that we have inequality (A.16) for the function $1 - v^2$. As a consequence, the map Ψ in (11) is well-defined on $\mathbb{R} \times [0, T]$, with $\Psi \in \mathcal{C}^0([0, T], H^{k-2}(\mathbb{R}))$. Similarly, $F(v, \Psi)$ is well-defined, bounded and continuous on $\mathbb{R} \times [0, T]$, with $\partial_x F(v, \Psi) = v\Psi \in \mathcal{C}^0([0, T], H^{k-2}(\mathbb{R}))$. This extends to the interval $[0, T_{\max})$ due to the arbitrary choice of $T \in (0, T_{\max})$.

Concerning the proof of (15), we observe that

$$v\Psi = -\frac{1}{2}\partial_x\Big((1-v^2)^{\frac{1}{2}}\exp i\theta\Big).$$

Since $(1-v(x,t)^2)^{1/2}\exp i\theta(x,t)\to 1$ as $x\to-\infty$ for any $t\in[0,T_{\max})$, we obtain the formula

$$2F(v,\Psi) = 1-(1-v^2)^{\frac{1}{2}}\exp i\theta. \tag{A.17}$$

In particular, it follows from (11) that

$$\partial_x v = \operatorname{Re}\Big(2\Psi(1-v^2)^{\frac{1}{2}}\exp i\theta\Big) = 2\operatorname{Re}\Big(\big(1-2F(v,\overline{\Psi})\big)\Psi\Big).$$

Similarly, we deduce the first equation in (15) from the first equation in (HLL), (11) and (A.17).

Finally, we turn to (13). Given a number $0<T<T_{\max}$, we can uniformly bound from below the function $1-v^2$ on $\mathbb{R}\times[0,T]$ according to (A.16). In view of (HLL), we deduce that $\partial_t v$ belongs to $\mathcal{C}^0([0,T],H^{k-3}(\mathbb{R}))$, while $\partial_t w$ is in $\mathcal{C}^0([0,T],H^{k-4}(\mathbb{R}))$. Since $k\geq 4$, the derivative $\partial_t\theta$ is in $\mathcal{C}^0([0,T],\mathcal{C}^0_b(\mathbb{R}))$, with

$$\partial_t\theta(x,t) = -\int_{-\infty}^x \big(v(y,t)\partial_t w(y,t)+\partial_t v(y,t)w(y,t)\big)\,dy.$$

Going back to Lemma 3.1, and using the fact that $|v(x,t)|+|\partial_x v(x,t)|+|\partial_{xx}v(x,t)|+|w(x,t)|\to 0$ as $x\to-\infty$ for any $t\in[0,T]$, we obtain the expression

$$\partial_t\theta = \frac{1}{2}\Big(v^2+w^2-3v^2w^2-2v\frac{\partial_{xx}v}{1-v^2}+\frac{(1-3v^2)(\partial_x v)^2}{(1-v^2)^2}\Big).$$

Differentiating (11) with respect to t and using (HLL), we get

$$\begin{aligned}2i\partial_t\Psi =&\Big(-\frac{\partial_{xxx}v}{(1-v^2)^{\frac{1}{2}}}-\frac{3v(\partial_x v)\partial_{xx}v}{(1-v^2)^{\frac{3}{2}}}-\frac{3(1+v^2)(\partial_x v)^3}{2(1-v^2)^{\frac{5}{2}}}+\frac{\partial_x v}{2(1-v^2)^{\frac{1}{2}}}\Big(7v^2w^2-v^2-w^2\Big)\\ &+(1-v^2)^{\frac{1}{2}}\big((1-w^2)\partial_x v-3vw\partial_x w\big)+i\Big(\frac{3vw\partial_{xx}v}{(1-v^2)^{\frac{1}{2}}}+\frac{3(1+v^2)w(\partial_x v)^2}{2(1-v^2)^{\frac{3}{2}}}+\frac{3v(\partial_x v)\partial_x w}{(1-v^2)^{\frac{1}{2}}}\\ &-\frac{1}{2}(1-v^2)^{\frac{1}{2}}\big(2\partial_{xx}w+v^2w+w^3-3v^2w^3\big)\Big)\Big)\exp i\theta.\end{aligned}$$

On the other hand, since $\partial_x\theta=-vw$ by (12), we can write

$$\begin{aligned}2\partial_{xx}\Psi =&\Big(\frac{\partial_{xxx}v}{(1-v^2)^{\frac{1}{2}}}+\frac{3v(\partial_x v)\partial_{xx}v}{(1-v^2)^{\frac{3}{2}}}+\frac{(1+2v^2)(\partial_x v)^3}{(1-v^2)^{\frac{5}{2}}}-\frac{3v^2w^2\partial_x v}{(1-v^2)^{\frac{1}{2}}}\\ &+(1-v^2)^{\frac{1}{2}}\big(3vw\partial_x w+w^2\partial_x v\big)+i\Big(-\frac{3vw\partial_{xx}v}{(1-v^2)^{\frac{1}{2}}}-\frac{3v(\partial_x v)\partial_x w}{(1-v^2)^{\frac{1}{2}}}\\ &-\frac{(2+v^2)w(\partial_x v)^2}{(1-v^2)^{\frac{3}{2}}}+(1-v^2)^{\frac{1}{2}}\big(\partial_{xx}w-v^2w^3\big)\Big)\Big)\exp i\theta.\end{aligned}$$

Therefore, we obtain

$$i\partial_t\Psi+\partial_{xx}\Psi+2|\Psi|^2\Psi+\frac{1}{2}v^2\Psi-\frac{1}{2}(1-v^2)^{\frac{1}{2}}(\partial_x v)\exp i\theta=0.$$

Since

$$\frac{1}{2}(1-v^2)^{1/2}(\partial_x v)\exp i\theta = (1-v^2)\operatorname{Re}\big(\Psi\exp(-i\theta)\big)\exp i\theta,$$

we deduce from (A.17) that v and Ψ are solutions to (13). This completes the proof of Corollary A.1. □

A.2 Local well-posedness of the hydrodynamical Landau-Lifshitz equation in the energy space

We now examine the continuous dependence with respect to the initial datum of the solutions to (HLL). We first address this issue for the system of equations (13)-(15).

Proposition A.3. *Let $(v^0, \Psi^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(\tilde{v}^0, \tilde{\Psi}^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be such that*

$$\partial_x v^0 = 2\,\mathrm{Re}\Big(\Psi^0\big(1 - 2F(v^0, \overline{\Psi^0})\big)\Big), \quad \text{and} \quad \partial_x \tilde{v}^0 = 2\,\mathrm{Re}\Big(\tilde{\Psi}^0\Big(1 - 2F\big(\tilde{v}^0, \overline{\tilde{\Psi}^0}\big)\Big)\Big).$$

Given two solutions (v, Ψ) and $(\tilde{v}, \tilde{\Psi})$ in $\mathcal{C}^0([0, T_*], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, with $(\Psi, \tilde{\Psi}) \in L^4([0, T_*], L^\infty(\mathbb{R}))^2$, to (13)-(15) with initial datum (v^0, Ψ^0), resp. $(\tilde{v}^0, \tilde{\Psi}^0)$, for some positive time T_*, there exist a positive number τ, depending only on $\|v^0\|_{L^2}$, $\|\tilde{v}^0\|_{L^2}$, $\|\Psi^0\|_{L^2}$ and $\|\tilde{\Psi}^0\|_{L^2}$, and a universal constant A such that we have

$$\begin{aligned} \big\|v - \tilde{v}\big\|_{\mathcal{C}^0([0,T],L^2)} + \big\|\Psi - \tilde{\Psi}\big\|_{\mathcal{C}^0([0,T],L^2)} + &\big\|\Psi - \tilde{\Psi}\big\|_{L^4([0,T],L^\infty)} \\ &\leq A\Big(\big\|v^0 - \tilde{v}^0\big\|_{L^2} + \big\|\Psi^0 - \tilde{\Psi}^0\big\|_{L^2}\Big), \end{aligned} \tag{A.18}$$

for any $T \in [0, \min\{\tau, T_*\}]$. In addition, there exists a positive number B, depending only on $\|v^0\|_{L^2}$, $\|\tilde{v}^0\|_{L^2}$, $\|\Psi^0\|_{L^2}$ and $\|\tilde{\Psi}^0\|_{L^2}$, such that

$$\big\|\partial_x v - \partial_x \tilde{v}\big\|_{\mathcal{C}^0([0,T],L^2)} \leq B\Big(\big\|v^0 - \tilde{v}^0\big\|_{L^2} + \big\|\Psi^0 - \tilde{\Psi}^0\big\|_{L^2}\Big), \tag{A.19}$$

for any $T \in [0, \min\{\tau, T_*\}]$.

Proof. We split the proof into four steps. We first focus on the L^2-norm of the difference $z = \tilde{v} - v$, which we estimate performing an energy method.

Step 1. *Let $T \in [0, T_*]$ be fixed. Set $\Xi := \tilde{\Psi} - \Psi$, and*

$$\Lambda(T) := \Big(\big\|\tilde{v}\big\|^2_{\mathcal{C}^0([0,T],L^2)} + \big\|v\big\|^2_{\mathcal{C}^0([0,T],L^2)} + \big\|\tilde{\Psi}\big\|^2_{\mathcal{C}^0([0,T],L^2)} + \big\|\Psi\big\|^2_{\mathcal{C}^0([0,T],L^2)}\Big)^{\frac{1}{2}}.$$

There exists a positive number K_1 such that

$$\int_{\mathbb{R}} \big|z(\cdot, T)\big|^2 \leq \int_{\mathbb{R}} \big|z^0\big|^2 + K_1 T\Big(\big\|\Xi\big\|^2_{\mathcal{C}^0([0,T],L^2)} + \Lambda(T)^2\Big(\big\|z\big\|^2_{\mathcal{C}^0([0,T],L^2)} + \big\|\Xi\big\|^2_{\mathcal{C}^0([0,T],L^2)}\Big)\Big). \tag{A.20}$$

We derive from (15) the two equations

$$\begin{cases} \partial_t z = 2\partial_x \,\mathrm{Im}\Big(- \Xi + 2\Xi F\big(\tilde{v}, \overline{\tilde{\Psi}}\big) + 2\Psi F\big(z, \overline{\tilde{\Psi}}\big) + 2\Psi F\big(v, \overline{\Xi}\big)\Big), \\ \partial_x z = 2\,\mathrm{Re}\Big(\Xi - 2\Xi F\big(\tilde{v}, \overline{\tilde{\Psi}}\big) - 2\Psi F\big(z, \overline{\tilde{\Psi}}\big) - 2\Psi F\big(v, \overline{\Xi}\big)\Big). \end{cases} \tag{A.21}$$

In view of (14), the map $F_1(u, \phi)$ is a bounded, continuous function on $\mathbb{R}$, which satisfies the upper bound

$$\|F(u, \phi)\|_{L^\infty} \leq \|u\|_{L^2} \|\phi\|_{L^2}, \tag{A.22}$$

as soon as u and ϕ are in $L^2(\mathbb{R})$. As a consequence, it follows from the first equation in (A.21) that z belongs to $\mathcal{C}^1([0, T_*], H^{-1}(\mathbb{R}))$, when the pairs (v, Ψ), resp. $(\tilde{v}, \tilde{\Psi})$, satisfy the assumptions of Proposition A.3. Since v and $\tilde{v}$ are in $\mathcal{C}^0([0, T_*], H^1(\mathbb{R}))$ under these assumptions, we are allowed to compute the derivative of the L^2-integral of z, and to deduce from the first equation in (A.21) the inequality

$$\frac{d}{dt}\int_{\mathbb{R}} |z|^2 \leq 4\big\|\partial_x z\big\|_{L^2}\Big(\big\|\Xi\big\|_{L^2}\big(1+2\big\|F(\tilde{v},\overline{\tilde{\Psi}})\big\|_{L^\infty}\big)+2\big\|\Psi\big\|_{L^2}\big(\big\|F(z,\overline{\tilde{\Psi}})\big\|_{L^\infty}+\big\|F(v,\Xi)\big\|_{L^\infty}\big)\Big). \tag{A.23}$$

On the other hand, the second equation in (A.21) provides the bound

$$\big\|\partial_x z\big\|_{L^2} \leq 2\big\|\Xi\big\|_{L^2}\Big(1+2\big\|F(\tilde{v},\overline{\tilde{\Psi}})\big\|_{L^\infty}\Big)+4\big\|\Psi\big\|_{L^2}\Big(\big\|F(z,\overline{\tilde{\Psi}})\big\|_{L^\infty}+\big\|F(v,\Xi)\big\|_{L^\infty}\Big). \tag{A.24}$$

It remains to insert (A.22) and (A.24) into (A.23), and to integrate from 0 to T in order to obtain (A.20), with $K=64$.

We now turn to the L^2-norm of the map Ξ. Instead of deriving energy estimates, we rely on the Duhamel formula for the map Ψ, which may be written as

$$\Psi(\cdot,t)=e^{it\partial_{xx}}\Psi^0(\cdot)+i\int_0^t e^{i(t-s)\partial_{xx}}G(v,\Psi)(\cdot,s)\,ds, \tag{A.25}$$

with

$$G(v,\Psi):=2|\Psi|^2\Psi+\frac{1}{2}v^2\Psi-\mathrm{Re}\,\big(\Psi(1-2F(v,\overline{\Psi}))\big)\big(1-2F(v,\Psi)\big),$$

and we apply the Strichartz estimates for the one-dimensional Schrödinger group $(e^{it\partial_{xx}})_{t\in\mathbb{R}}$. Recall (see e.g. [6]) that they write as

$$\|e^{it\partial_{xx}}f\|_{L^p(\mathbb{R},L^q)} \leq K\|f\|_{L^2}, \tag{A.26}$$

for any function $f\in L^2(\mathbb{R})$, and any admissible pair $(p,q)\in[2,+\infty]^2$, i.e. such that $2/p+1/q=1/2$. Given any positive number T, we also have

$$\Big\|\int_0^t e^{i(t-s)\partial_{xx}}g(\cdot,s)\,ds\Big\|_{L^{p_1}([0,T],L^{q_1})} \leq K\|g\|_{L^{p_2'}([0,T],L^{q_2'})}, \tag{A.27}$$

for any admissible pairs (p_1,q_1) and (p_2,q_2), and any function $g\in L^{p_2'}([0,T],L^{q_2'}(\mathbb{R}))$. Applying (A.26) and (A.27) to (A.25), we can show

Step 2. *Let $T\in[0,T_*]$ be fixed. There exists a positive number K_2 such that*

$$\begin{aligned}\big\|\Xi\big\|_{\mathcal{C}^0([0,T],L^2)}+\big\|\Xi\big\|_{L^4([0,T],L^\infty)} \leq &K_2\Big(\big\|\Xi^0\big\|_{L^2}+T^{\frac{2}{3}}\Lambda(T)^2\big\|\Xi\big\|_{L^4([0,T],L^\infty)}\\ &+T\big(1+\Lambda(T)^2\big)^2\big(\big\|z\big\|_{\mathcal{C}^0([0,T],L^2)}+\big\|\Xi\big\|_{\mathcal{C}^0([0,T],L^2)}\big)\Big).\end{aligned} \tag{A.28}$$

Coming back to (A.25) and invoking (A.26) and (A.27), we write

$$\begin{aligned}\big\|\Xi\big\|_{\mathcal{C}^0([0,T],L^2)}+\big\|\Xi\big\|_{L^4([0,T],L^\infty)} \leq &K\Big(\big\|\Xi^0\big\|_{L^2}+\big\||\tilde{\Psi}|^2\tilde{\Psi}-|\Psi|^2\Psi\big\|_{L^{\frac{4}{3}}([0,T],L^1)}\\ &+\big\|G(\tilde{v},\tilde{\Psi})-2|\tilde{\Psi}|^2\tilde{\Psi}-G(v,\Psi)+2|\Psi|^2\Psi\|_{L^1([0,T],L^2)}\Big),\end{aligned} \tag{A.29}$$

for some positive number K. On one hand, we check that

$$\begin{aligned}\big\||\tilde{\Psi}|^2\tilde{\Psi}-|\Psi|^2\Psi\big\|_{L^{\frac{4}{3}}([0,T],L^1)} \leq &T^{\frac{2}{3}}\big\|\Xi\big\|_{L^4([0,T],L^\infty)}\Big(\big\|\tilde{\Psi}\big\|_{\mathcal{C}^0([0,T],L^2)}+\big\|\Psi\big\|_{\mathcal{C}^0([0,T],L^2)}\Big)^2\\ \leq &2T^{\frac{2}{3}}\Lambda(T)^2\big\|\Xi\big\|_{L^4([0,T],L^\infty)}.\end{aligned} \tag{A.30}$$

On the other hand, applying (A.22) to the second equation in (15), we obtain

$$\|\partial_x v\|_{L^2} \leq 2\|\Psi\|_{L^2}\big(1 + 2\|v\|_{L^2}\|\Psi\|_{L^2}\big),$$

so that, by the Sobolev embedding theorem,

$$\|v\|_{L^\infty} \leq 2\|v\|_{L^2}^{\frac{1}{2}}\|\Psi\|_{L^2}^{\frac{1}{2}}\big(1 + 2\|v\|_{L^2}\|\Psi\|_{L^2}\big)^{\frac{1}{2}}. \tag{A.31}$$

The same inequality holds replacing v by $\tilde{v}$, resp. Ψ by $\tilde{\Psi}$. Regarding the function z, we derive similarly from (A.22) and (A.24) that

$$\big\|z\big\|_{L^\infty} \leq 2\big\|z\big\|_{L^2}^{\frac{1}{2}}\big\|\Xi\big\|_{L^2}^{\frac{1}{2}}\Big(1+2\big\|\tilde{v}\big\|_{L^2}\big\|\tilde{\Psi}\big\|_{L^2}+2\big\|v\big\|_{L^2}\big\|\Psi\big\|_{L^2}\Big)^{\frac{1}{2}}+2\sqrt{2}\big\|z\big\|_{L^2}\big\|\tilde{\Psi}\big\|_{L^2}^{\frac{1}{2}}\big\|\Psi\big\|_{L^2}^{\frac{1}{2}}. \tag{A.32}$$

Since

$$\big\|\tilde{v}^2\tilde{\Psi} - v^2\Psi\big\|_{L^2} \leq \big\|z\big\|_{L^\infty}\Big(\big\|v\big\|_{L^\infty} + \big\|\tilde{v}\big\|_{L^\infty}\Big)\big\|\tilde{\Psi}\big\|_{L^2} + \big\|v\big\|_{L^\infty}^2\big\|\Xi\big\|_{L^2},$$

and similarly,

$$\begin{aligned}&\Big\|\operatorname{Re}\big(\tilde{\Psi}\big(1 - 2F\big(\tilde{v}, \overline{\tilde{\Psi}}\big)\big)\big)\big(1 - 2F\big(\tilde{v}, \tilde{\Psi}\big)\big) - \operatorname{Re}\big(\Psi\big(1 - 2F\big(v, \overline{\Psi}\big)\big)\big)\big(1 - 2F\big(v, \Psi\big)\big)\Big\|_{L^2}\\ &\leq \big\|z\big\|_{L^2}\big\|\tilde{\Psi}\big\|_{L^2}\big\|\Psi\big\|_{L^2}\Big(3 + 2\big\|\tilde{v}\big\|_{L^2}\big\|\tilde{\Psi}\big\|_{L^2} + 2\big\|v\big\|_{L^2}\big\|\Psi\big\|_{L^2}\Big)\\ &\quad + \big\|\Xi\big\|_{L^2}\Big(1 + 2\big\|\tilde{v}\big\|_{L^2}\big\|\tilde{\Psi}\big\|_{L^2} + 2\big\|v\big\|_{L^2}\big\|\Psi\big\|_{L^2}\Big)^2,\end{aligned}$$

we conclude from (A.31) and (A.32) that

$$\begin{aligned}&\big\|G\big(\tilde{v}, \tilde{\Psi}\big) - 2|\tilde{\Psi}|^2\tilde{\Psi} - G(v, \Psi) + 2|\Psi|^2\Psi\big\|_{L^1([0,T],L^2)}\\ &\leq 12T\big(1 + \Lambda(T)^2\big)^2\Big(\big\|z\big\|_{\mathcal{C}^0([0,T],L^2)} + \big\|\Xi\big\|_{\mathcal{C}^0([0,T],L^2)}\Big).\end{aligned}$$

Estimate (A.28) follows combining with (A.29) and (A.30).

In order to complete the proof of Proposition A.3, we have to control the quantity $\Lambda(T)$. We introduce the quantity

$$E(T) := \|v\|_{\mathcal{C}^0([0,T],L^2)} + \|\Psi\|_{\mathcal{C}^0([0,T],L^2)} + \|\Psi\|_{L^4([0,T],L^\infty)},$$

for which we can show

Step 3. *There exists a positive number τ_1, depending only on $\|v^0\|_{L^2}$ and $\|\Psi^0\|_{L^2}$, such that*

$$E(T) \leq 2\big(K_2 + 1\big)\big(\|v^0\|_{L^2} + \|\Psi^0\|_{L^2}\big), \tag{A.33}$$

for any $0 \leq T \leq \min\{\tau_1, T_*\}$.

We invoke estimates (A.20) and (A.28) for the solutions (v, Ψ) and $(0, 0)$. This gives the bound

$$E(T) \leq \big(K_2 + 1\big)\big(\|v^0\|_{L^2} + \|\Psi^0\|_{L^2}\big) + KE(T)\big(T^{\frac{1}{2}}(1 + E(T)^2)^{\frac{1}{2}} + T^{\frac{2}{3}}E(T)^2 + T(1 + E(T)^2)^2\big),$$

where the number K depends only on K_1 and K_2. Since $E(0) = \|v^0\|_{L^2} + \|\Psi^0\|_{L^2}$, and the map $T \mapsto E(T)$ is continuous on $[0, T^*]$, it remains to apply a continuation argument in order to obtain (A.33), with $\tau_1 = \min\{1, 1/(36K^2(1 + 4(K_2 + 1)^2E(0)^2)^4)\}$.

We are now in position to complete the proof of Proposition A.3.

Step 4. *End of the proof.*

We first invoke Step 3 to exhibit a time τ_1, depending only on $\|v^0\|_{L^2}$, $\|\tilde{v}^0\|_{L^2}$, $\|\Psi^0\|_{L^2}$ and $\|\tilde{\Psi}^0\|_{L^2}$, such that

$$\Lambda(T) \leq \Lambda_0 := 2(K_2+1)\Big(\big\|v^0\big\|_{L^2} + \big\|\tilde{v}^0\big\|_{L^2} + \big\|\Psi^0\big\|_{L^2} + \big\|\tilde{\Psi}^0\big\|_{L^2}\Big), \tag{A.34}$$

for any $0 \leq T \leq \min\{\tau_1, T_*\}$. Combining with (A.20) and (A.28), we deduce that

$$\begin{aligned}\Big(\big\|z\big\|_{\mathcal{C}^0([0,T],L^2)} + \big\|\Xi\big\|_{\mathcal{C}^0([0,T],L^2)} + \big\|\Xi\big\|_{L^4([0,T],L^\infty)}\Big)\times \\ \times\Big(1 - K\big(T^{\frac{1}{2}}(1+\Lambda_0^2)^{\frac{1}{2}} + T^{\frac{2}{3}}\Lambda_0^2 + T(1+\Lambda_0^2)^2\big)\Big) \leq \big\|z^0\big\|_{L^2} + K_2\big\|\Xi^0\big\|_{L^2},\end{aligned}$$

where the number K depends only on K_1 and K_2. Estimate (A.18) follows letting $A := 2(1+K_2)$, and $\tau = \min\{1, 1/(36K^2(1+\Lambda_0^2)^4), \tau_1\}$.

Finally, we deduce from (A.22) and (A.24) that

$$\big\|\partial_x z\big\|_{\mathcal{C}^0([0,T],L^2)} \leq 2\Big(\big(1+4\Lambda(T)^2\big)\big\|\Xi\big\|_{\mathcal{C}^0([0,T],L^2)} + 4\Lambda(T)^2\big\|z\big\|_{\mathcal{C}^0([0,T],L^2)}\Big).$$

Combining with (A.18) and (A.34), we obtain (A.19). □

We are now in position to complete the proof of Theorem 1.

Proof of Theorem 1. We first show the local existence of a solution $\mathfrak{v}$ to (HLL) corresponding to an initial datum $\mathfrak{v}^0 \in \mathcal{NV}(\mathbb{R})$.

Step 1. *There exist a positive number T, and a solution $\mathfrak{v} \in \mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$ to* (HLL)*, with initial datum $\mathfrak{v}^0$, such that there exists a sequence of solutions $(\mathfrak{v}_n)_{n\in\mathbb{N}}$ to* (HLL)*, which belong to $\mathcal{C}^0([0,T], \mathcal{NV}^k(\mathbb{R}))$ for any $k \in \mathbb{N}$, and which satisfy*

$$\mathfrak{v}_n \to \mathfrak{v} \quad \text{in } \mathcal{C}^0([0,T], H^1(\mathbb{R}) \times L^2(\mathbb{R})), \tag{A.35}$$

as $n \to +\infty$. In addition, the energy and the momentum of $\mathfrak{v}$ are constant along the flow.

We consider a sequence of functions $\mathfrak{v}_n^0 \in \mathcal{C}_c^\infty(\mathbb{R})^2$ such that

$$\mathfrak{v}_n^0 \to \mathfrak{v}^0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}), \tag{A.36}$$

as $n \to +\infty$, and we set

$$\delta := 1 - \|v^0\|_{\mathcal{C}^0} > 0, \quad \text{and} \quad M := \sup_{n\in\mathbb{N}} \|\mathfrak{v}_n^0\|_{H^1\times L^2} \geq 0.$$

We denote by $\mathfrak{v}_n$ the solutions to (HLL), with initial datum $\mathfrak{v}_n^0$, provided by Proposition A.2, and by (v_n, Ψ_n) the solutions to the system of equations (13)-(15) corresponding to $\mathfrak{v}_n$, which are given by Corollary A.1. Recall that $\mathfrak{v}_n$ is in $\mathcal{C}^0([0,T_n), \mathcal{NV}^k(\mathbb{R}))$ for any $k \in \mathbb{N}$, where the maximal time of existence T_n is characterized by (A.13). Similarly, the pair (v_n, Ψ_n) belongs to $\mathcal{C}^0([0,T_n), H^{k+1}(\mathbb{R}) \times H^k(\mathbb{R}))$ for any $k \in \mathbb{N}$. Combining (A.36) with the Sobolev embedding theorem, there exists an integer N such that

$$\|v_n^0\|_{\mathcal{C}^0} < 1 - \frac{3\delta}{4}, \tag{A.37}$$

for any $n \geq N$. In view of (11), this guarantees the existence of a positive number A, depending only on δ and M, such that

$$\|v_n^0\|_{L^2} + \|\Psi_n^0\|_{L^2} \leq A,$$

for any $n \geq N$. As a consequence of Proposition A.3, we derive the existence of a positive number τ_1, depending only on δ and M, such that

$$\|v_n - v_m\|_{\mathcal{C}^0([0,t],H^1)} + \|\Psi_n - \Psi_m\|_{\mathcal{C}^0([0,t],L^2)} \leq A\big(\|v_n^0 - v_m^0\|_{L^2} + \|\Psi_n^0 - \Psi_m^0\|_{L^2}\big), \tag{A.38}$$

for any $n, m \geq N$, and $0 \leq t \leq \min\{\tau_1, T_n, T_m\}$. Here as in the sequel, A refers to a further positive number depending only on δ and M. At this stage, we deduce from (11) and (A.37) that

$$\|\Psi_n^0 - \Psi_m^0\|_{L^2} \leq A\|\mathfrak{v}_n^0 - \mathfrak{v}_m^0\|_{H^1\times L^2}.$$

Hence, we can rephrase (A.38) as

$$\|v_n - v_m\|_{\mathcal{C}^0(\mathbb{R}\times[0,t])} \leq \|v_n - v_m\|_{\mathcal{C}^0([0,t],H^1)} + \|\Psi_n - \Psi_m\|_{\mathcal{C}^0([0,t],L^2)} \leq A\|\mathfrak{v}_n^0 - \mathfrak{v}_m^0\|_{H^1\times L^2}. \tag{A.39}$$

Using (A.36), we now enlarge our choice of N, if necessary, so that the right-hand side of (A.39) satisfies

$$A\|\mathfrak{v}_n^0 - \mathfrak{v}_m^0\|_{H^1\times L^2} \leq \frac{\delta}{4},$$

for any $n, m \geq N$. Since v_N belongs to $\mathcal{C}^0([0, T_N), H^1(\mathbb{R}))$, we deduce from (A.37) the existence of a number $0 < \tau_2 < T_N$ such that

$$\|v_N\|_{\mathcal{C}^0(\mathbb{R}\times[0,\tau_2])} < 1 - \frac{\delta}{2}.$$

Coming back to (A.39), this guarantees that

$$\|v_n\|_{\mathcal{C}^0(\mathbb{R}\times[0,\min\{\tau_1,\tau_2,T_n\}])} < 1 - \frac{\delta}{4}, \tag{A.40}$$

for any $n \geq N$, so that, in view of (A.13),

$$T := \min\big\{\tau_1, \tau_2\big\} < T_n.$$

In particular, the inequalities in (A.39) hold for any $n, m \geq N$, and $t = T$. Coming back to (11) and (A.17), we notice that

$$w_n = 2\,\mathrm{Im}\Big(\frac{\Psi_n\big(1 - 2F(v_n, \overline{\Psi_n})\big)}{1 - v_n^2}\Big).$$

As a consequence of (A.39) and (A.40), this gives

$$\|w_n - w_m\|_{\mathcal{C}^0([0,T],L^2)} \leq A\Big(\|v_n - v_m\|_{\mathcal{C}^0([0,T],H^1)} + \|\Psi_n - \Psi_m\|_{\mathcal{C}^0([0,T],L^2)}\Big) \leq A\|\mathfrak{v}_n^0 - \mathfrak{v}_m^0\|_{H^1\times L^2}. \tag{A.41}$$

In conclusion, $(\mathfrak{v}_n)_{n\geq N}$ is a Cauchy sequence in the complete metric space $\mathcal{C}^0([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. It converges towards a map $\mathfrak{v} \in \mathcal{C}^0([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. In particular, it satisfies the condition

$$\|v\|_{\mathcal{C}^0(\mathbb{R}\times[0,T])} \leq 1 - \frac{\delta}{4},$$

due to (A.40), so that it lies in $\mathcal{C}^0([0, T], \mathcal{NV}(\mathbb{R}))$. In addition, the convergence in (A.35) is enough to guarantee that $\mathfrak{v}$ solves (HLL) for the initial condition $\mathfrak{v}^0$ (in the sense of distributions), and that the energy and the momentum are constant along the flow.

We now turn to the uniqueness of the solution $\mathfrak{v}$.

Step 2. *Assume that* $\mathfrak{v} \in \mathcal{C}^0([0,T],\mathcal{NV}(\mathbb{R}))$ *and* $\tilde{\mathfrak{v}} \in \mathcal{C}^0([0,\tilde{T}],\mathcal{NV}(\mathbb{R}))$ *are two solutions to* (HLL)*, with initial datum* $\mathfrak{v}^0$*, which both satisfy condition* (A.35)*. Then,* $\mathfrak{v}$ *and* $\tilde{\mathfrak{v}}$ *are well-defined and equal on* $\mathbb{R} \times [0, \min\{T, \tilde{T}\}]$.

Set

$$\tau^0 := \max\big\{t \in [0, \min\{T, \tilde{T}\}], \text{ s.t. } \mathfrak{v}(\cdot, s) = \tilde{\mathfrak{v}}(\cdot, s),\ \forall 0 \leq s \leq t\big\},$$

and assume that $\tau_0 \neq \min\{T, \tilde{T}\}$. Due to the continuity of the maps $\mathfrak{v}$ and $\tilde{\mathfrak{v}}$, we have $\mathfrak{v}(\cdot, \tau^0) = \tilde{\mathfrak{v}}(\cdot, \tau^0)$. In particular, if we denote by $(\mathfrak{v}_n)_{n\in\mathbb{N}}$ and $(\tilde{\mathfrak{v}}_n)_{n\in\mathbb{N}}$ two sequences of smooth solutions to (HLL) such that (A.35) holds for $\mathfrak{v}$, resp. $\tilde{\mathfrak{v}}$, then we have

$$\mathfrak{v}_n(\cdot, \tau^0) \to \mathfrak{v}(\cdot, \tau^0), \quad \text{and} \quad \tilde{\mathfrak{v}}_n(\cdot, \tau^0) \to \mathfrak{v}(\cdot, \tau^0) \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

as $n \to +\infty$. Setting

$$\mathfrak{w}_{2n}^0 := \mathfrak{v}_n(\cdot, \tau^0), \quad \text{and} \quad \mathfrak{w}_{2n+1}^0 := \tilde{\mathfrak{v}}_n(\cdot, \tau^0), \tag{A.42}$$

and denoting by $\mathfrak{w}_n$ the corresponding solution to (HLL) provided by Proposition A.2, we deduce as in the proof of Step 1 the existence of a number $0 < \tau < \min\{T, \tilde{T}\} - \tau_0$, and a solution $\mathfrak{w}$ to (HLL), with initial datum $\mathfrak{v}(\cdot, \tau^0)$, such that

$$\mathfrak{w}_n \to \mathfrak{w} \quad \text{in } \mathcal{C}^0([0,\tau], H^1(\mathbb{R}) \times L^2(\mathbb{R})),$$

as $n \to +\infty$. In view of (A.42) and the uniqueness of smooth solutions in Proposition A.2, we also have

$$\mathfrak{w}_{2n} = \mathfrak{v}_n(\cdot, \tau^0 + \cdot) \to \mathfrak{v}(\cdot, \tau^0 + \cdot), \quad \text{and} \quad \mathfrak{w}_{2n+1} = \tilde{\mathfrak{v}}_n(\cdot, \tau^0 + \cdot) \to \tilde{\mathfrak{v}}(\cdot, \tau^0 + \cdot),$$

in $\mathcal{C}^0([0,\tau], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. This proves that $\mathfrak{v} = \mathfrak{w} = \tilde{\mathfrak{v}}$ on $\mathbb{R} \times [\tau_0, \tau_0 + \tau]$, which contradicts the definition of τ^0, and completes the proof of Step 2.

Applying Step 2, we denote by $\mathfrak{v}$ the unique solution in $\mathcal{C}^0([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$ to (HLL), with initial datum $\mathfrak{v}^0$, which satisfies the condition in (A.35) for any $T < T_{\max}$. In particular, the maximal time of existence $T_{\max}$ for $\mathfrak{v}$ is defined as the supremum of the numbers T such that the condition in (A.35) holds for T. We have the following characterization of $T_{\max}$.

Step 3. *Either* $T_{\max} = +\infty$*, or*

$$\lim_{t \to T_{\max}} \|v(\cdot, t)\|_{\mathcal{C}^0} = 1. \tag{A.43}$$

For $T_{\max} \neq +\infty$, we argue by contradiction assuming the existence of a positive number δ, and of an increasing sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to T_{\max}$ as $n \to +\infty$, and

$$\|v(\cdot, s_n)\|_{\mathcal{C}^0} \leq 1 - \delta,$$

for any $n \in \mathbb{N}$. Our first goal in the sake of a contradiction is to establish that

$$\rho := \|v\|_{\mathcal{C}^0(\mathbb{R}\times[0,T_{\max}))} < 1. \tag{A.44}$$

We introduce a positive number ε to be fixed later, and we use the continuity of $\mathfrak{v}$ in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ to exhibit another increasing sequence $(t_n)_{n\in\mathbb{N}}$, with $s_n < t_n < s_{n+1}$, and such that

$$\|\mathfrak{v}(\cdot, t) - \mathfrak{v}(\cdot, s_n)\|_{H^1\times L^2} \leq \varepsilon, \tag{A.45}$$

for any $s_n \leq t \leq t_n$. As soon as $\varepsilon < \delta/4$, we deduce from the Sobolev embedding theorem that

$$\|v\|_{\mathcal{C}^0(\mathbb{R}\times[s_n,t_n])} \leq 1 - \frac{3\delta}{4}. \tag{A.46}$$

We then invoke the definition of $T_{\max}$ to find smooth solutions $\mathfrak{v}_n$ to (HLL) such that

$$\|\mathfrak{v}_n - \mathfrak{v}\|_{\mathcal{C}^0([0,t_n],H^1\times L^2)} \leq \frac{\delta}{4^{n+1}}, \tag{A.47}$$

for any $n \in \mathbb{N}$. We denote by T_n the maximal time of existence of $\mathfrak{v}_n$, and by (v_n, Ψ_n) the corresponding solutions to the system of equations (13)-(15). In view of (A.46) and (A.47), we have

$$\|v_n\|_{\mathcal{C}^0(\mathbb{R}\times[s_p,t_p])} \leq 1 - \frac{\delta}{2}, \tag{A.48}$$

for any $n \geq p$.

On the other hand, the conservation of the energy of $\mathfrak{v}$ implies the existence of a positive number A, depending only on $\mathfrak{v}^0$ and δ, such that

$$\|\mathfrak{v}\|_{\mathcal{C}^0([s_p,t_p],H^1\times L^2)} \leq A.$$

As a consequence of (11), (A.47) and (A.48), we derive the inequality

$$\|v_n\|_{\mathcal{C}^0([s_p,t_p],L^2)} + \|\Psi_n\|_{\mathcal{C}^0([s_p,t_p],L^2)} \leq A,$$

for $n \geq p$, and a further positive number A, depending only, here as in the sequel, of $\mathfrak{v}^0$ and δ. Invoking Proposition A.3, we obtain the existence of a positive number τ, depending only on $\mathfrak{v}^0$ and δ, such that

$$\begin{aligned} &\|v_n(\cdot,\cdot+\sigma) - v_n\|_{\mathcal{C}^0([s_p,s],H^1)} + \|\Psi_n(\cdot,\cdot+\sigma) - \Psi_n\|_{\mathcal{C}^0([s_p,s],L^2)} \\ &\leq A\big(\|v_n(\cdot,s_p+\sigma) - v_n(\cdot,s_p)\|_{L^2} + \|\Psi_n(\cdot,s_p+\sigma) - \Psi_n(\cdot,s_p)\|_{L^2}\big) \\ &\leq A\|\mathfrak{v}_n(\cdot,s_p+\sigma) - \mathfrak{v}_n(\cdot,s_p)\|_{H^1\times L^2}, \end{aligned} \tag{A.49}$$

for any $n \geq p$, $0 \leq \sigma \leq t_p - s_p$ and $s_p \leq s \leq \min\{s_p+\tau, t_n - \sigma\}$. Here, we have also used (11) and (A.48) to derive the second inequality.

At this stage, we fix an integer p such that

$$s_p \geq T_{\max} - \frac{\tau}{2}, \quad \text{and} \quad 0 < t_p - s_p < T_{\max}.$$

For $n \geq p$, we derive from (A.45), (A.47) and (A.49) that

$$\|v_n(\cdot,\cdot+\sigma) - v_n\|_{\mathcal{C}^0(\mathbb{R}\times[s_p,s])} \leq A\Big(\varepsilon + \frac{\delta}{4^n}\Big),$$

for $0 \leq \sigma \leq t_p - s_p$ and $s_p \leq s \leq t_n - \sigma$. Since

$$\|v_n(\cdot,t_n)\|_{\mathcal{C}^0} \leq 1 - \frac{\delta}{2},$$

by (A.48), we can choose $s = t_n - \sigma$, with σ varying between 0 and $t_p - s_p$, and use (A.47) to obtain

$$\|v\|_{\mathcal{C}^0(\mathbb{R}\times[t_n-t_p+s_p,t_n])} \leq 1 - \frac{\delta}{2} + A\Big(\varepsilon + \frac{\delta}{4^n}\Big).$$

Choosing ε small enough, we deduce the existence of an integer N such that

$$\|v\|_{\mathcal{C}^0(\mathbb{R}\times\underset{n\geq N}{\cup}[t_n-t_p+s_p,t_n])} \leq 1 - \frac{\delta}{4}.$$

Since $t_n \to T_{\max}$ as $n \to +\infty$, we conclude that

$$\|v\|_{\mathcal{C}^0(\mathbb{R}\times[T_{\max}-t_p+s_p,T_{\max}))} \leq 1 - \frac{\delta}{4}.$$

Statement (A.44) then follows from the fact that $v \in \mathcal{C}^0([0, T_{\max} - t_p + s_p], \mathcal{NV}(\mathbb{R}))$.

Using (A.44), we now extend the solution $\mathfrak{v}$ up to a time $T > T_{\max}$. This provides a contradiction with the definition of $T_{\max}$, which is enough to complete the proof of Step 3. In order to extend $\mathfrak{v}$, we introduce as before smooth solutions $\mathfrak{v}_n$ to (HLL) such that

$$\|\mathfrak{v}_n - \mathfrak{v}\|_{\mathcal{C}^0([0,s_n],H^1 \times L^2)} \leq \frac{1}{2^n}, \tag{A.50}$$

for any $n \in \mathbb{N}$. We again denote by T_n the maximal time of existence of $\mathfrak{v}_n$, and by (v_n, Ψ_n) the corresponding solutions to the system of equations (13)-(15). Invoking statement (A.44), and arguing as in its proof, we can find further positive numbers δ and M such that

$$\|v_n\|_{\mathcal{C}^0(\mathbb{R} \times [0,s_n])} \leq 1 - \delta, \tag{A.51}$$

and

$$\|v_n\|_{\mathcal{C}^0([0,s_n],L^2)} + \|\Psi_n(\cdot, t)\|_{\mathcal{C}^0([0,s_n],L^2)} \leq M,$$

for any n large enough. As a consequence, we can invoke Proposition A.3 as in the proof of (A.49) to exhibit two positive numbers τ and A, depending only on δ and M, such that

$$\|v_n - v_m\|_{\mathcal{C}^0([\sigma,\sigma+s],H^1)} + \|\Psi_n - \Psi_m\|_{\mathcal{C}^0([\sigma,\sigma+s],L^2)} \leq A \|\mathfrak{v}_n(\cdot, \sigma) - \mathfrak{v}_m(\cdot, \sigma)\|_{H^1 \times L^2}, \tag{A.52}$$

for any $0 \leq \sigma \leq \max\{s_m, s_n\}$, and $0 \leq s \leq \min\{\tau, T_m - \sigma, T_n - \sigma\}$.

We set $\sigma := T_{\max} - \tau/2$, and we assume, up to some subsequence, that $s_n > \sigma$ for any $n \in \mathbb{N}$. In this case, we deduce from (A.50) that $(\mathfrak{v}_n(\cdot, \sigma))_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Up to a further subsequence, we can assume that the right-hand side of (A.52) is less than $\delta/2$. By the Sobolev embedding theorem, this guarantees that

$$\|v_n(\cdot, t)\|_{\mathcal{C}^0} \leq \|v_m(\cdot, t)\|_{\mathcal{C}^0} + \frac{\delta}{2},$$

for any $m \geq n$, and $\sigma \leq t \leq \min\{\sigma + \tau, T_n, s_m\}$. Since $s_m \to T_{\max}$ as $m \to +\infty$, we can invoke (A.13) and (A.51) to conclude that $T_n > T_{\max}$ and

$$\|v_n\|_{\mathcal{C}^0(\mathbb{R} \times [\sigma, T_{\max}])} \leq 1 - \frac{\delta}{2}.$$

Coming back again to (A.13), we next introduce an integer N and a number $T \in (T_{\max}, \sigma + \tau)$ such that

$$\|v_N\|_{\mathcal{C}^0(\mathbb{R} \times [\sigma, T])} \leq 1 - \frac{\delta}{4},$$

and the right-hand side of (A.52) is less than $\delta/8$ for $m, n \geq N$. Using the Sobolev embedding theorem, we obtain as before,

$$\|v_n\|_{\mathcal{C}^0(\mathbb{R} \times [\sigma, \min\{T, T_n\}]} \leq 1 - \frac{\delta}{8},$$

for any $n \geq N$. This guarantees that $T_n \geq T$. Moreover, we can argue as in the proof of Step 1 to derive from (A.52) the fact that $(\mathfrak{v}_n)_{n \geq N}$ is a Cauchy sequence in $\mathcal{C}^0([\sigma, T], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. Since it is also a Cauchy sequence in $\mathcal{C}^0([0, \sigma], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ by (A.50), we can extend the solution $\mathfrak{v}$ up to the time T, and contradict the definition of $T_{\max}$.

In order to conclude the proof of Theorem 1, it remains to analyze the dependence with respect to the initial datum of the solution $\mathfrak{v}$. In this direction, we first establish

Step 4. *Let $0 < T < T_{\max}$. Given any positive number ε, there exists a positive number η such that, if $\tilde{\mathfrak{v}}^0$ is an initial datum in $\mathcal{NV}^k(\mathbb{R})$ for any $k \in \mathbb{N}$, and such that*

$$\big\|\tilde{\mathfrak{v}}^0 - \mathfrak{v}^0\big\|_{H^1\times L^2} < \eta, \tag{A.53}$$

then, the maximal time of existence $\tilde{T}_{\max}$ of the corresponding solution $\tilde{\mathfrak{v}}$ to (HLL) *satisfies*

$$\tilde{T}_{\max} \geq T,$$

and moreover,

$$\big\|\tilde{\mathfrak{v}} - \mathfrak{v}\big\|_{\mathcal{C}^0([0,T],H^1\times L^2)} < \varepsilon. \tag{A.54}$$

Since $\mathfrak{v} \in \mathcal{C}^0([0,T_{\max}),\mathcal{NV}(\mathbb{R}))$, there exists a number $0 < \delta < 1$ such that

$$\|v(\cdot,t)\|_{\mathcal{C}^0} < 1 - \delta,$$

for any $t \in [0,T]$. Using the conservation of the energy, there also exists a positive number M such that

$$\|\mathfrak{v}(\cdot,t)\|_{\mathcal{C}^0([0,T],H^1\times L^2)} \leq M.$$

In particular, given a sequence of solutions $(\mathfrak{v}_n)_{n\in\mathbb{N}}$ to (HLL), which belong to $\mathcal{C}^0([0,T],\mathcal{NV}^k(\mathbb{R}))$ for any $k \in \mathbb{N}$, and which satisfy (A.35), we can assume that

$$\|v_n(\cdot,t)\|_{\mathcal{C}^0} < 1 - \frac{\delta}{2}, \quad \text{and} \quad \|\mathfrak{v}_n(\cdot,t)\|_{H^1\times L^2} \leq 2M, \tag{A.55}$$

for any $t \in [0,T]$.

When η is small enough, we can also assume that the initial datum $\tilde{\mathfrak{v}}^0$ is such that

$$\big\|\tilde{v}^0\big\|_{\mathcal{C}^0} < 1 - \frac{\delta}{4}, \quad \text{and} \quad \big\|\tilde{\mathfrak{v}}^0\big\|_{H^1\times L^2} \leq 4M. \tag{A.56}$$

Arguing as in Step 1, there exist two numbers $0 < \tau < T$ and $A > 0$, depending only on δ and M, such that the corresponding solutions (v_n, Ψ_n) and $(\tilde{v}, \tilde{\Psi})$ to the system of equations (13)-(15) satisfy

$$\big\|v_n - \tilde{v}\big\|_{\mathcal{C}^0(\mathbb{R}\times[0,t])} \leq \big\|v_n - \tilde{v}\big\|_{\mathcal{C}^0([0,t],H^1)} + \big\|\Psi_n - \tilde{\Psi}\big\|_{\mathcal{C}^0([0,t],L^2)} \leq A\big\|\mathfrak{v}_n^0 - \tilde{\mathfrak{v}}^0\big\|_{H^1\times L^2}, \tag{A.57}$$

for any $n \in \mathbb{N}$ and $t \in [0, \min\{\tau, \tilde{T}_{\max}\}]$. For η small enough and n large enough, the right-hand side of (A.57) is less than $\delta/4$. Therefore, we deduce from (A.55) that

$$\big\|\tilde{v}(\cdot,t)\big\|_{\mathcal{C}^0} < 1 - \frac{\delta}{4},$$

when $t \in [0, \min\{\tau, \tilde{T}_{\max}\}]$. As a consequence of (A.13), we know that $\tilde{T}_{\max} > \tau$. Moreover, arguing as in the proof of (A.41), we show that

$$\big\|w_n - \tilde{w}\big\|_{\mathcal{C}^0([0,\tau],L^2)} \leq A\big\|\mathfrak{v}_n^0 - \tilde{\mathfrak{v}}^0\big\|_{H^1\times L^2},$$

so that, by (A.57),

$$\big\|\mathfrak{v}_n - \tilde{\mathfrak{v}}\big\|_{\mathcal{C}^0([0,\tau],H^1\times L^2)} \leq A\big\|\mathfrak{v}_n^0 - \tilde{\mathfrak{v}}^0\big\|_{H^1\times L^2}. \tag{A.58}$$

At this stage, we can use (A.53) to guarantee that

$$\big\|\mathfrak{v}_n(\cdot,\tau) - \tilde{\mathfrak{v}}(\cdot,\tau)\big\|_{H^1\times L^2} \leq \min\Big\{\frac{\delta}{4}, 2M\Big\},$$

for η small enough and n large enough. In view of (A.55), this ensures that (A.56) holds with $\tilde{v}(\cdot,\tau)$ and $\tilde{\mathfrak{v}}(\cdot,\tau)$ replacing $\tilde{v}^0$ and $\tilde{\mathfrak{v}}^0$. In particular, we can repeat the arguments from (A.56) to (A.58) to obtain that $\tilde{T}_{\max} \geq \min\{2\tau, T\}$, and

$$\big\|\mathfrak{v}_n - \tilde{\mathfrak{v}}\big\|_{\mathcal{C}^0([\tau,2\tau],H^1\times L^2)} \leq A\big\|\mathfrak{v}_n(\cdot,\tau) - \tilde{\mathfrak{v}}(\cdot,\tau)\big\|_{H^1\times L^2} \leq A\big\|\mathfrak{v}_n^0 - \tilde{\mathfrak{v}}^0\big\|_{H^1\times L^2},$$

for η small enough and n large enough. Repeating this argument a finite number of times, we establish that $\tilde{T}_{\max} \geq T$, as well as the estimate

$$\big\|\mathfrak{v}_n - \tilde{\mathfrak{v}}\big\|_{\mathcal{C}^0([0,T],H^1\times L^2)} \leq A\big\|\mathfrak{v}_n^0 - \tilde{\mathfrak{v}}^0\big\|_{H^1\times L^2}, \tag{A.59}$$

for a positive number A, depending only on δ, M and T. It now remains to decrease, if necessary, the value of η such that the right-hand side of (A.59) is less than ε for n large enough, and then to take the limit $n \to +\infty$ in (A.59) to obtain (A.54). This completes the proof of Step 4.

We are now in position to complete the proof of Theorem 1.

Step 5. *End of the proof.*

Statements (i) and (iii) in Theorem 1 were shown in Steps 1 and 2, while Step 3 is exactly statement (ii). Concerning (iv), we rely on Step 4. We fix a number $0 < T < T_{\max}$, and we consider a positive number ε such that

$$\|v\|_{\mathcal{C}^0(\mathbb{R}\times[0,T])} \leq 1 - 2\varepsilon. \tag{A.60}$$

We denote by η the positive number such that the conclusions of Step 4 hold when an initial datum $\tilde{\mathfrak{v}}^0$ in $\mathcal{NV}^k(\mathbb{R})$ for any $k \in \mathbb{N}$, satisfies (A.53). In view of (7), there exists an integer N such that we have

$$\|\mathfrak{v}_n^0 - \mathfrak{v}^0\|_{H^1\times L^2} < \frac{\eta}{2}, \tag{A.61}$$

for any $n \geq N$. We claim that the maximal time of existence T_n of the solution $\mathfrak{v}_n$ satisfies

$$T_n > T,$$

for any $n \geq N$.

Indeed, assume for the sake of a contradiction that $T_n \leq T$ for an integer $n \geq N$. In view of (A.43), there would exist a number $0 < \tau < T_n$ such that

$$\|v_n(\cdot,\tau)\|_{\mathcal{C}^0} \geq 1 - \frac{\varepsilon}{2}. \tag{A.62}$$

On the other hand, we can find a sequence of smooth solutions $(\tilde{\mathfrak{v}}_p)_{p\in\mathbb{N}}$ to (HLL) such that

$$\tilde{\mathfrak{v}}_p \to \mathfrak{v}_n \quad \text{in } \mathcal{C}^0([0,\tau], H^1(\mathbb{R}) \times L^2(\mathbb{R})),$$

as $p \to +\infty$. In particular, we deduce from (A.61) that

$$\big\|\tilde{\mathfrak{v}}_p^0 - \mathfrak{v}^0\big\|_{H^1\times L^2} < \eta,$$

when p is large enough. In view of Step 4, the maximal time of existence $\tilde{T}_p$ of the solution $\tilde{\mathfrak{v}}_p$ to (HLL), with initial datum $\tilde{\mathfrak{v}}_p^0$, is larger than T, and we have

$$\big\|\tilde{\mathfrak{v}}_p - \mathfrak{v}\big\|_{\mathcal{C}^0([0,T],H^1\times L^2)} < \varepsilon.$$

Taking the limit $p \to +\infty$, we obtain

$$\|\mathfrak{v}_n - \mathfrak{v}\|_{\mathcal{C}^0([0,\tau],H^1\times L^2)} < \varepsilon, \tag{A.63}$$

so that by (A.60) and the Sobolev embedding theorem,

$$\|v_n(\cdot,\tau)\|_{\mathcal{C}^0} < 1-\varepsilon.$$

This provides a contradiction with (A.62). Therefore, $T_n > T$ for any $n \geq N$, which can be rewritten as in (8).

In addition, we can replace τ by T in the proof of (A.63), and obtain

$$\|\mathfrak{v}_n - \mathfrak{v}\|_{\mathcal{C}^0([0,T],H^1\times L^2)} < \varepsilon,$$

for any $n \geq N$. This completes the proof of (9), and as a consequence, of Theorem 1. □

A.3 Local well-posedness of the Landau-Lifshitz equation in the energy space

We now come back to the Cauchy problem for the original Landau-Lifshitz equation. We first provide some properties concerning the hydrodynamical variables.

Lemma A.4. *Let $0 < \delta < 1$. Consider a function $m_* \in \mathcal{E}(\mathbb{R})$ such that $1-(m_*)_3^2 \geq \delta$, and denote by $\mathfrak{v}_* = (v_*, w_*)$ the corresponding hydrodynamical variables. Given any positive number ϵ, there exist a positive number ρ such that, if a function $m \in \mathcal{E}(\mathbb{R})$ lies in the ball $B_{\mathcal{E}}(m_*,\rho)$, then, the corresponding hydrodynamical variables $\mathfrak{v} = (v,w)$ are well-defined, and satisfy*

$$\big\|\mathfrak{v} - \mathfrak{v}_*\big\|_{H^1\times L^2} \leq \epsilon. \tag{A.64}$$

Proof. Let $\rho > 0$ and $m \in B_{\mathcal{E}}(m_*,\rho)$. Invoking the Sobolev embedding theorem, we obtain the uniform bound

$$\|m_3 - (m_*)_3\|_{L^\infty} \leq \|m_3 - (m_*)_3\|_{H^1} \leq \rho. \tag{A.65}$$

When $\rho < \delta/4$, it follows that

$$1 - m_3^2 \geq 1 - (m_*)_3^2 - 2|m_3 - (m_*)_3| \geq \frac{\delta}{2}. \tag{A.66}$$

As a consequence, the hydrodynamical pair $\mathfrak{v}$ is well-defined. Moreover, since $v = m_3$, we have

$$\|v - v_*\|_{H^1} \leq \rho. \tag{A.67}$$

For the difference $w - w_*$, the identity $w = \langle i\check{m}, \check{m}'\rangle_{\mathbb{C}}/(1-m_3^2)$ provides the decomposition

$$w - w_* = \frac{\langle i(\check{m}-\check{m}_*), \check{m}_*{}'\rangle_{\mathbb{C}}}{1-(m_*)_3^2} + \frac{\langle i\check{m}, \check{m}' - \check{m}_*{}'\rangle_{\mathbb{C}}}{1-(m_*)_3^2} + \frac{m_3^2 - (m_*)_3^2}{(1-m_3^2)(1-(m_*)_3^2)}\langle i\check{m}, \check{m}'\rangle_{\mathbb{C}}. \tag{A.68}$$

In view of (A.65) and (A.66), we can estimate the last two terms in the right-hand side of (A.68) by

$$\Big\|\frac{\langle i\check{m}, \check{m}' - \check{m}_*{}'\rangle_{\mathbb{C}}}{1-(m_*)_3^2}\Big\|_{L^2} + \Big\|\frac{m_3^2 - (m_*)_3^2}{(1-m_3^2)(1-(m_*)_3^2)}\langle i\check{m}, \check{m}'\rangle_{\mathbb{C}}\Big\|_{L^2} \leq \frac{\delta + 4(\|m_*'\|_{L^2} + \rho)}{\delta^2}\rho. \tag{A.69}$$

Concerning the first term, we recall that

$$\big|\check{m}(x) - \check{m}_*(x)\big| \leq \big|\check{m}(0) - \check{m}_*(0)\big| + |x|^{\frac{1}{2}}\|m' - m_*'\|_{L^2} \leq \rho\big(1 + |x|^{\frac{1}{2}}\big),$$

for any $x \in \mathbb{R}$. On the other hand, there exists a radius R such that

$$\int_{(-R,R)^c} \big|m_*'\big|^2 \leq \frac{\delta^2\epsilon^2}{8}.$$

This gives the estimate

$$\int_{\mathbb{R}} \left| \frac{\langle i(\check{m} - \check{m}_*), \check{m}_*' \rangle_{\mathbb{C}}}{1 - (m_*)_3^2} \right|^2 \leq \frac{1}{\delta^2} \bigg(\int_{(-R,R)} \big| m - m_* \big|^2 \big| m_*' \big|^2 + 4 \int_{(-R,R)^c} \big| m_*' \big|^2 \bigg)$$
$$\leq \frac{(1 + R^{\frac{1}{2}})^2 \| m_*' \|_{L^2}^2 \rho^2}{\delta^2} + \frac{\epsilon^2}{2}.$$

Combining with (A.67), (A.68) and (A.69), we can choose the number ρ small enough so that we have (A.64). □

We are now in position to provide the

Proof of Corollary 1. In order to construct the solution m, our strategy is first to consider the corresponding Cauchy problem in the hydrodynamical variables. Theorem 1 provides a solution $\mathfrak{v}$ for this problem. However, the mappings corresponding to $\mathfrak{v}$ in the original variables can only be determined up to a phase factor. The main difficulty in the proof is to establish that we can fix such a phase in order to obtain a (unique) solution to (LL), which satisfies the statements in Corollary 1. We by-pass this difficulty by relying on arguments developed by H. Mohamad in [27] in the context of the Gross-Pitaevskii equation.

We split the proof into six steps.

Step 1. *Given any number $T \in (0, T_{\max})$, there exists a positive number ρ_T such that, given any initial datum $\tilde{m}^0 \in B_{\mathcal{E}}(m^0, \rho_T)$, the corresponding hydrodynamical pair $\tilde{\mathfrak{v}}^0$ is well-defined, and there exists a unique solution $\tilde{\mathfrak{v}} \in \mathcal{C}^0([0,T], \mathcal{NV}(\mathbb{R}))$ to* (HLL) *with initial datum $\mathfrak{v}^0$, which satisfies the statements in Theorem 1.*

Recall that the (HLL) flow provided by Theorem 1 is continuous with respect to the initial datum. In particular, since $T < T_{\max}$, there exists a radius r_T such that, given any initial datum $\tilde{\mathfrak{v}}^0 \in B(\mathfrak{v}^0, r_T)$, the corresponding solution $\tilde{\mathfrak{v}}$ to (HLL) provided by Theorem 1 is well-defined on $[0, T]$. In view of Lemma A.4, there exists a radius ρ_T such that, if $\tilde{m}^0 \in B_{\mathcal{E}}(m^0, \rho_T)$, then the corresponding hydrodynamical pair $\tilde{\mathfrak{v}}^0$ is well-defined, and lies in the ball $B(\mathfrak{v}^0, r_T)$. This is enough to obtain the statements in Step 1.

Fix a number $T \in (0, T_{\max})$, consider initial data m_n^0 in the ball $B(m^0, \rho_T)$, and denote by $\mathfrak{v}_n = (v_n, w_n)$ the solution to (HLL) with initial datum the pair $\mathfrak{v}_n^0$ corresponding to m_n^0. In order to construct the solution m_n to (LL) with initial datum m_n^0, the main difficulty is to define a phase function φ_n such that $m_n := \big((1 - v_n^2)^{1/2} \cos(\varphi_n), (1 - v_n^2)^{1/2} \sin(\varphi_n), v_n \big)$ is solution to (LL) on $\mathbb{R} \times [0, T]$. By definition, the derivative $\partial_x \varphi_n$ must be equal to w_n. As a consequence, it is natural to introduce the primitive

$$\theta_n(x, t) := \int_0^x w_n(y, t) \, dy, \tag{A.70}$$

for any $(x, t) \in \mathbb{R} \times [0, T]$. The function θ_n is then equal to φ_n up to some constant of integration possibly depending on time. We fix this constant so that we eventually obtain a solution to (LL).

At time $t = 0$, the pair $\mathfrak{v}_n^0$ is well-defined so that the map $\check{m}_n^0$ does not vanish. Therefore, we can find a phase number ϕ_n^0 such that

$$\check{m}_n^0(0) = \big| \check{m}_n^0(0) \big| e^{i \phi_n^0}.$$

We next introduce the notation

$$\Phi_n := \partial_x \Big(\frac{\partial_x v_n}{1 - v_n^2} \Big) - v_n \frac{(\partial_x v_n)^2}{(1 - v_n^2)^2} + v_n \big(w_n^2 - 1 \big), \tag{A.71}$$

and we define the map

$$\vartheta_n(x,t) = \int_0^x w_n(y,0)\,dy + \int_0^t \Phi_n(x,s)\,ds. \tag{A.72}$$

Given a smooth function $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \chi = 1$, we consider the function

$$\phi_n(t) = \langle \vartheta_n(\cdot,t) - \theta_n(\cdot,t), \chi \rangle_{\mathcal{D}',\mathcal{D}},$$

for any $t \in [0,T]$, and we set

$$\varphi_n(x,t) = \theta_n(x,t) + \phi_n(t) + \phi_n^0, \tag{A.73}$$

for any $(x,t) \in \mathbb{R}\times[0,T]$. We claim that this construction makes sense, and depends continuously on the $H^1 \times L^2$-norm of the maps $\mathfrak{v}_n$.

Step 2. *The maps*

$$m_n := \Big((1-v_n^2)^{\frac{1}{2}} \cos(\varphi_n), (1-v_n^2)^{\frac{1}{2}} \sin(\varphi_n), v_n \Big), \tag{A.74}$$

are well-defined and continuous on $\mathbb{R} \times [0,T]$. Moreover, given any positive number R, there exists a positive number A, depending only on m_*^0, ρ, T and χ, such that

$$\begin{aligned} \max_{t\in[0,T]} d_{\mathcal{E}}\big(m_n(\cdot,t), m_p(\cdot,t)\big) \leq & A d_{\mathcal{E}}\big(m_n^0, m_p^0\big) + A\big(1+R^{\frac{1}{2}}\big) \max_{t\in[0,T]} \big\| \mathfrak{v}_n(\cdot,t) - \mathfrak{v}_p(\cdot,t) \big\|_{H^1\times L^2} \\ & + A \max_{t\in[0,T]} \bigg(\int_{\mathbb{R}\setminus(-R,R)} \big(\partial_x v_n(\cdot,t)^2 + w_n(\cdot,t)^2 \big) \bigg)^{\frac{1}{2}}, \end{aligned} \tag{A.75}$$

for any $(n,p) \in \mathbb{N}^2$.

Indeed, the pairs $\mathfrak{v}_n$ belong to $\mathcal{C}^0([0,T], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. Moreover, in view of the continuity with respect to the initial datum of (HLL), we can decrease, if necessary, the value of the radius ρ_T such that we can find a positive number δ for which

$$1 - v_n(x,t)^2 \geq \delta, \tag{A.76}$$

for any $n \in \mathbb{N}$ and $(x,t) \in \mathbb{R} \times [0,T]$.

We next derive from (A.70) that the functions θ_n are well-defined and continuous on $\mathbb{R}\times[0,T]$. Concerning the function Φ_n, we can invoke (A.76) and apply the Sobolev embedding theorem to check that $\Phi_n \in \mathcal{C}^0([0,T], H^{-1}(\mathbb{R}))$. As a consequence of (A.72), the functions ϑ_n lie in $\mathcal{C}^1([0,T], \mathcal{C}^0(\mathbb{R}) + H^{-1}(\mathbb{R}))$. Therefore, the numbers $\phi_n(t)$ are well-defined for any $t \in [0,T]$, and there exists a positive number A, depending only on χ, such that they satisfy the estimate

$$\begin{aligned} \big|\phi_n(t_1) - \phi_p(t_2)\big| \leq & A\Big(\|w_n(\cdot,0) - w_p(\cdot,0)\|_{L^2} + \|w_n(\cdot,t_1) - w_p(\cdot,t_2)\|_{L^2} \\ & + T \max_{s\in[0,T]} \|\Phi_n(\cdot,s) - \Phi_p(\cdot,s)\|_{H^{-1}} + |t_1 - t_2| \max_{s\in[0,T]} \|\Phi_n(\cdot,s)\|_{H^{-1}} \Big), \end{aligned}$$

for any $(n,p) \in \mathbb{N}^2$ and $0 \leq t_1 \leq t_2 \leq T$. Combining (A.71) with (A.76), and invoking the conservation of the Landau-Lifshitz energy, we derive the existence of a positive number A, depending only on δ, such that

$$\max_{s\in[0,T]} \|\Phi_n(\cdot,s)\|_{H^{-1}} \leq A\big(1 + E(\mathfrak{v}_n^0)\big),$$

and

$$\max_{s\in[0,T]} \|\Phi_n(\cdot,s)-\Phi_p(\cdot,s)\|_{H^{-1}} \leq A\big(1+E(\mathfrak{v}_n^0)+E(\mathfrak{v}_p^0)\big) \max_{s\in[0,T]} \|\mathfrak{v}_n(\cdot,s)-\mathfrak{v}_p(\cdot,s)\|_{H^1\times L^2}.$$

At this stage, we observe that the energy E is bounded on the ball $B(\mathfrak{v}^0, r_T)$ by a positive number A, depending only on m^0 and ρ_T. Since the pairs $\mathfrak{v}_n^0$ lie in this ball, we have

$$E(\mathfrak{v}_n^0) \leq A, \tag{A.77}$$

for any $n \in \mathbb{N}$. This gives

$$\begin{aligned} &\big|\phi_n(t_1)-\phi_p(t_2)\big| \\ &\quad \leq A\Big(|t_1-t_2| + \|\mathfrak{v}_n(\cdot,t_1)-\mathfrak{v}_p(\cdot,t_2)\|_{H^1\times L^2} + \big(1+T\big) \max_{s\in[0,T]} \|\mathfrak{v}_n(\cdot,s)-\mathfrak{v}_p(\cdot,s)\|_{H^1\times L^2}\Big). \end{aligned} \tag{A.78}$$

In particular, the functions ϕ_n are continuous on $[0,T]$.

As a first consequence of the above considerations, the maps m_n are well-defined and continuous on $\mathbb{R}\times[0,T]$. We now check that they satisfy (A.75). Let $t\in[0,T]$ be fixed, and compute

$$\big|\check{m}_n(0,t)-\check{m}_p(0,t)\big| \leq \Big|(1-v_n^2(0,t))^{\frac{1}{2}} - (1-v_p^2(0,t))^{\frac{1}{2}}\Big| + \Big|e^{i(\phi_n(t)+\phi_n^0-\phi_p(t)-\phi_p^0)}-1\Big|.$$

The first term in the right-hand side can be treated as before, so that

$$\Big|(1-v_n^2(0,t))^{\frac{1}{2}} - (1-v_p^2(0,t))^{\frac{1}{2}}\Big| \leq A\|\mathfrak{v}_n(\cdot,t)-\mathfrak{v}_p(\cdot,t)\|_{H^1\times L^2}.$$

For the second term, we infer from (A.77) and (A.78) that

$$\Big|e^{i(\phi_n(t)+\phi_n^0-\phi_p(t)-\phi_p^0)}-1\Big| \leq A\big(1+T\big) \max_{s\in[0,T]} \|\mathfrak{v}_n(\cdot,s)-\mathfrak{v}_p(\cdot,s)\|_{H^1\times L^2} + \Big|e^{i\phi_n^0}-e^{i\phi_p^0}\Big|. \tag{A.79}$$

On the other hand, we can write

$$\begin{aligned} \Big|e^{i\phi_n^0}-e^{i\phi_p^0}\Big| \leq & \frac{1}{(1-v_n^0(0)^2)^{1/2}}\big|\check{m}_n^0(0)-\check{m}_p^0(0)\big| + \Big|\frac{1}{(1-v_p^0(0)^2)^{1/2}} - \frac{1}{(1-v_n^0(0)^2)^{1/2}}\Big| \\ \leq & A\Big(d_{\mathcal{E}}(m_n^0,m_p^0) + \|\mathfrak{v}_n^0-\mathfrak{v}_p^0\|_{H^1\times L^2}\Big). \end{aligned} \tag{A.80}$$

We conclude that

$$\big|\check{m}_n(0,t)-\check{m}_p(0,t)\big| \leq A\Big((1+T) \max_{s\in[0,T]} \|\mathfrak{v}_n(\cdot,s)-\mathfrak{v}_p(\cdot,s)\|_{H^1\times L^2} + d_{\mathcal{E}}(m_n^0,m_p^0)\Big). \tag{A.81}$$

We now deal with the difference $\partial_x m_n - \partial_x m_p$. Coming back to (A.70), we observe that the equality

$$\partial_x\theta_n(\cdot,t) = w_n(\cdot,t),$$

holds in $L^2(\mathbb{R})$ for any $t\in[0,T]$. In view of (A.73), this provides the identity

$$\partial_x\big(e^{i\varphi_n}\big) = iw_n e^{i\varphi_n},$$

again in $L^2(\mathbb{R})$. Since $v_n \in \mathcal{C}^0([0,T],H^1(\mathbb{R}))$ satisfies (A.76), it follows that

$$\partial_x\check{m}_n = \Big(-\frac{v_n\partial_x v_n}{(1-v_n^2)^{\frac{1}{2}}} + iw_n(1-v_n^2)^{\frac{1}{2}}\Big)e^{i\varphi_n},$$

so that

$$\big\|\partial_x \check{m}_n - \partial_x \check{m}_p\big\|_{L^2} \leq A\bigg(\|\mathfrak{v}_n - \mathfrak{v}_p\|_{H^1\times L^2} + \Big\| \big(|\partial_x v_n|^2 + |w_n|^2\big)^{\frac{1}{2}} \big|e^{i\varphi_n} - e^{i\varphi_p}\big| \Big\|_{L^2} \bigg).$$

At this stage, we introduce a positive number R as in the statement of Step 2, and we bound the last term in the previous inequality by

$$\int_{\mathbb{R}} \big(|\partial_x v_n|^2 + |w_n|^2\big)\big|e^{i\varphi_n} - e^{i\varphi_p}\big|^2 \leq 4 \int_{\mathbb{R}\setminus(-R,R)} \big(|\partial_x v_n|^2 + |w_n|^2\big) + A \max_{[-R,R]} \big|e^{i\varphi_n} - e^{i\varphi_p}\big|^2.$$

We then write

$$\big|e^{i\varphi_n(x,t)} - e^{i\varphi_p(x,t)}\big| \leq \big|e^{i\theta_n(x,t)} - e^{i\theta_p(x,t)}\big| + \Big|e^{i(\phi_n(t)+\phi_n^0-\phi_p(t)-\phi_p^0)} - 1\Big|,$$

so that by (A.70), (A.79) and (A.80),

$$\begin{aligned} \max_{[-R,R]} \big|e^{i\varphi_n} - e^{i\varphi_p}\big| & \\ \leq R^{\frac{1}{2}} \big\|w_n - w_p\big\|_{L^2} &+ A\Big(d_{\mathcal{E}}(m_n^0, m_p^0) + (1+T) \max_{s\in[0,T]} \|\mathfrak{v}_n(\cdot,s) - \mathfrak{v}_p(\cdot,s)\|_{H^1\times L^2}\Big). \end{aligned}$$

As a conclusion, we have

$$\begin{aligned} \big\|\partial_x \check{m}_n(\cdot,t) - \partial_x \check{m}_p(\cdot,t)\big\|_{L^2} \leq & A\bigg(d_{\mathcal{E}}(m_n^0, m_p^0) + \Big(\int_{\mathbb{R}\setminus(-R,R)} \big(\partial_x v_n(\cdot,t)^2 + w_n(\cdot,t)^2\big)\Big)^{\frac{1}{2}} \\ & + (1 + R^{\frac{1}{2}} + T) \max_{s\in[0,T]} \|\mathfrak{v}_n(\cdot,s) - \mathfrak{v}_p(\cdot,s)\|_{H^1\times L^2}\bigg), \end{aligned} \tag{A.82}$$

for any $t \in [0,T]$. Since $[m_n]_3 = v_n$, it is direct to estimate the difference $[m_n]_3 - [m_p]_3$ in $H^1(\mathbb{R})$. Combining with (A.81) and (A.82), we deduce (A.75).

Our goal is now to derive from (A.75) that $(m_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^0([0,T],\mathcal{E}(\mathbb{R}))$. In this direction, we first need a uniform estimate for the last integral in (A.75).

Step 3. *Assume that*

$$\mathfrak{v}_n \to \tilde{\mathfrak{v}} \quad \text{in } \mathcal{C}^0([0,T], H^1(\mathbb{R}) \times L^2(\mathbb{R})), \tag{A.83}$$

as $n \to +\infty$. For any positive number ϵ, there exist a positive number R_0 and an integer n_0, depending only on T and $\tilde{\mathfrak{v}}$, such that

$$\max_{t\in[0,T]} \bigg(\int_{\mathbb{R}\setminus(-R,R)} \big(\partial_x v_n(x,t)^2 + w_n(x,t)^2\big)\, dx \bigg) \leq \epsilon^2, \tag{A.84}$$

for any $R \geq R_0$ and $n \geq n_0$.

In view of (A.83), there exists an integer n_0 such that

$$\int_{\mathbb{R}\setminus(-R,R)} \Big|\partial_x v_n(x,t)^2 + w_n(x,t)^2 - \partial_x \tilde{v}(x,t)^2 - \tilde{w}(x,t)^2\Big|\, dx \leq \max_{t\in[0,T]} \|\mathfrak{v}_n - \tilde{\mathfrak{v}}\|^2_{H^1\times L^2} \leq \frac{\epsilon^2}{9}, \tag{A.85}$$

for any $t \in [0,T]$, $R \in (0,+\infty)$ and $n \geq n_0$. Since $\tilde{\mathfrak{v}}$ is uniformly continuous from $[0,T]$ to $\mathcal{E}(\mathbb{R})$, there exists a positive number τ such that

$$\int_{\mathbb{R}\setminus(-R,R)} \Big|\partial_x \tilde{v}(x,s)^2 + \tilde{w}(x,s)^2 - \partial_x \tilde{v}(x,t)^2 - \tilde{w}(x,t)^2\Big|\, dx \leq \frac{\epsilon^2}{9}, \tag{A.86}$$

for any $R \in (0, +\infty)$ and $(s, t) \in [0, T]$, with $|s - t| < \tau$. Finally, there exist numbers $(t_i)_{1 \leq i \leq \ell}$ such that the intervals $(t_i - \tau, t_i + \tau)$ recover the segment $[0, T]$. We fix R_0 such that

$$\int_{\mathbb{R} \setminus (-R_0, R_0)} \Big(\partial_x \tilde{v}(x, t_i)^2 + \tilde{w}(x, t_i)^2 \Big) \, dx \leq \frac{\epsilon^2}{9},$$

for any $1 \leq i \leq \ell$. Combining with (A.85) and (A.86), we obtain (A.84).

Step 4. *Proof of the existence of a solution m to* (LL)*, which satisfies* (10)*, and statements* (i) *and* (iii) *in Corollary 1.*

Given any $T \in (0, T_{\max})$, we derive from Lemma A.1 the existence of functions $m_n^0 \in B_{\mathcal{E}}(m^0, \rho_T)$, with $\partial_x m_n^0 \in H^\infty(\mathbb{R})$, such that $m_n^0 \to m^0$ in $\mathcal{E}(\mathbb{R})$, as $n \to +\infty$. In view of Step 1, the corresponding hydrodynamical pairs $\mathfrak{v}^0$ and $\mathfrak{v}_n^0$ are well-defined, and the solutions $\mathfrak{v}$ and $\mathfrak{v}_n$ to (HLL) with initial data $\mathfrak{v}^0$ and $\mathfrak{v}_n^0$ belong to $\mathcal{C}^0([0, T], \mathcal{NV}(\mathbb{R}))$. Moreover, they satisfy (A.83) due to (6).

As a consequence, given a positive number ϵ, we deduce from Step 3 the existence of a positive number R_0 and an integer n_0 such that (A.84) holds. In addition, we can assume that

$$d_{\mathcal{E}}(m_n^0, m_p^0) \leq \epsilon, \quad \text{and} \quad \max_{t \in [0,T]} \|\mathfrak{v}_n(\cdot, t) - \mathfrak{v}_p(\cdot, t)\|_{H^1 \times L^2} \leq \frac{\epsilon}{(1 + R_0^{\frac{1}{2}})},$$

when $n, p \geq n_0$. Applying estimate (A.75) with $R = R_0$, we conclude that the maps m_n defined by (A.74) satisfy

$$\max_{t \in [0,T]} d_{\mathcal{E}}\big(m_n(\cdot, t), m_p(\cdot, t)\big) \leq A\epsilon,$$

when $n, p \geq n_0$. Therefore, they form a Cauchy sequence in $\mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R}))$, and there exists a function $m \in \mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R}))$ such that

$$m_n \to m \quad \text{in } \mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R})), \tag{A.87}$$

as $n \to +\infty$. Combining Lemma A.4 with (A.83) (for $\tilde{\mathfrak{v}} \equiv \mathfrak{v}$), we deduce that the hydrodynamical pair corresponding to $m(x, t)$ is equal to $\mathfrak{v}(x, t)$ for any $(x, t) \in \mathbb{R} \times [0, T]$. Moreover, since the choice of T is arbitrary, the map m is actually well-defined in $\mathcal{C}^0([0, T_{\max}), \mathcal{E}(\mathbb{R}))$, and statement (i) in Corollary 1 holds on $\mathbb{R} \times [0, T_{\max})$.

We next check that m is solution to (LL) (in the sense of distributions). When the functions m_n^0 are smooth, it follows from Proposition A.2 that the solutions $\mathfrak{v}_n$ to (HLL) are also smooth. In view of (A.73), the phase functions φ_n are smooth, and their time derivative is given by

$$\partial_t \varphi_n = \Phi_n. \tag{A.88}$$

Going back to (A.74), the maps $\check{m}_n$ are, in turn, smooth, and their time derivative is given by

$$\partial_t \check{m}_n = \Big(- \frac{v_n \partial_t v_n}{(1 - v_n^2)^{\frac{1}{2}}} + i(1 - v_n^2)^{\frac{1}{2}} \partial_t \varphi_n \Big) e^{i\varphi_n}.$$

Since the pair $\mathfrak{v}_n$ is solution to (HLL), we derive from (HLL) and (A.88) that m_n is a smooth solution to (LL). In view of (A.87), the limit function m remains a solution (in the sense of distributions) to (LL) on $[0, T_{\max})$ since the choice of T is arbitrary.

Finally, recall that

$$E\big(m_n(\cdot, t)\big) = E\big(m_n^0\big),$$

for any $n \in \mathbb{N}$ and any $t \in [0, T]$, due to Lemma A.2. The conservation of energy for m then follows from (A.87) and again from the arbitrary nature of T.

In order to complete the proof of Statement (ii), it remains to check the uniqueness of the limit solution m.

Step 5. *Let m be the solution to* (LL) *constructed in Step 4. Assume that there exists a solution $\tilde{m} \in \mathcal{C}^0([0, T_{\max}), \mathcal{E}(\mathbb{R}))$ to* (LL)*, with initial datum m^0, for which there are smooth solutions $\tilde{m}_n$ to* (LL) *such that*

$$\tilde{m}_n \to \tilde{m} \quad \text{in } \mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R})). \tag{A.89}$$

as $n \to +\infty$, for any $T \in (0, T_{\max})$. Then, $m = \tilde{m}$.

Let $T \in (0, T_{\max})$. For n large enough, the initial data $\tilde{m}_n^0$ lie in the ball $B_{\mathcal{E}}(m^0, \rho_T)$ due to (A.89). In particular, the maps $\mathfrak{m}_n$ given by (A.74) are well-defined and smooth on $[0, T]$, and we can argue as in the proof of Step 4 to claim that they solve (LL) for the initial data $\tilde{m}_n^0$. Going back to Proposition A.1, we deduce that $\mathfrak{m}_n = \tilde{m}_n$ on $[0, T]$. In particular, we can apply estimate (A.75) to obtain the bound

$$\begin{aligned}\max_{t \in [0,T]} d_{\mathcal{E}}\big(m_n(\cdot, t), \tilde{m}_p(\cdot, t)\big) \leq & A d_{\mathcal{E}}\big(m_n^0, \tilde{m}_p^0\big) + A\big(1 + R^{\frac{1}{2}}\big) \max_{t \in [0,T]} \big\|\mathfrak{v}_n(\cdot, t) - \tilde{\mathfrak{v}}_p(\cdot, t)\big\|_{H^1 \times L^2} \\ & + A \max_{t \in [0,T]} \bigg(\int_{\mathbb{R} \setminus (-R,R)} \big(\partial_x v_n(\cdot, t)^2 + w_n(\cdot, t)^2\big) \bigg)^{\frac{1}{2}},\end{aligned}$$

for any $(n, p) \in \mathbb{N}^2$. Here, the notation $\tilde{\mathfrak{v}}_p$ refers to the solution to (HLL) with initial datum $\tilde{\mathfrak{v}}_p^0$ corresponding to the function $\tilde{m}_p^0$. In view of Theorem 1, we know that

$$\tilde{\mathfrak{v}}_p \to \mathfrak{v} \quad \text{in } \mathcal{C}^0([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R})),$$

as $p \to +\infty$, where $\mathfrak{v}$ is the solution to (HLL) with initial datum $\mathfrak{v}^0$ corresponding to the function m^0. Using (A.84) and (A.89), we can take the limit $n, p \to +\infty$ in order to obtain the equality $\tilde{m} = m$ on $[0, T]$. Since the choice of T is arbitrary, this is enough to complete the proof of Step 5.

In order to complete the proof of Corollary 1, it remains to give the

Step 6. *Proof of Statement (iv).*

Let $T \in (0, T_{\max})$ be fixed. Assume that $m_n^0 \to m^0$ in $\mathcal{E}(\mathbb{R})$ as $n \to +\infty$. Without loss of generality, we can assume that m_n^0 lies in the ball $B_{\mathcal{E}}(m^0, \rho_T)$, so that we can rely on all the results proved in the previous steps.

Given a fixed integer $n \in \mathbb{N}$, we denote by m_n the solution to (LL) with initial datum m_n^0 provided by the previous steps. In view of Lemma A.1, we can find two sequences $(\tilde{m}_{n,p}^0)_{p \in \mathbb{N}}$ and $(\tilde{m}_p^0)_{p \in \mathbb{N}}$ of smooth functions in the ball $B_{\mathcal{E}}(m^0, \rho_T)$ such that

$$\tilde{m}_{n,p}^0 \to m_n^0, \quad \text{and} \quad \tilde{m}_p^0 \to m^0 \quad \text{in } \mathcal{E}(\mathbb{R}), \tag{A.90}$$

as $p \to +\infty$. Applying Step 4, we know that the corresponding smooth solutions $\tilde{m}_{n,p}$ and $\tilde{m}_p$ to (LL) are well-defined in $\mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R}))$, and satisfy the estimate

$$\begin{aligned}\max_{t \in [0,T]} d_{\mathcal{E}}\big(\tilde{m}_{n,p}(\cdot, t), \tilde{m}_p(\cdot, t)\big) \leq & A d_{\mathcal{E}}\big(\tilde{m}_{n,p}^0, \tilde{m}_p^0\big) + A\big(1 + R^{\frac{1}{2}}\big) \max_{t \in [0,T]} \big\|\tilde{\mathfrak{v}}_{n,p}(\cdot, t) - \tilde{\mathfrak{v}}_p(\cdot, t)\big\|_{H^1 \times L^2} \\ & + A \max_{t \in [0,T]} \bigg(\int_{\mathbb{R} \setminus (-R,R)} \big(\partial_x \tilde{v}_p(\cdot, t)^2 + \tilde{w}_p(\cdot, t)^2\big) \bigg)^{\frac{1}{2}},\end{aligned} \tag{A.91}$$

for any $R \in (0, +\infty)$ and $p \in \mathbb{N}$. Here, the pairs $\tilde{\mathfrak{v}}_{n,p}$ and $\tilde{\mathfrak{v}}_p$ are, as above, the solutions to (HLL) provided by Theorem 1 for the initial data $\tilde{\mathfrak{v}}_{n,p}^0$ and $\tilde{\mathfrak{v}}_p^0$ corresponding to $\tilde{m}_{n,p}^0$ and $\tilde{m}_p^0$. Combining Lemma A.4 and (A.90), we obtain the convergences

$$\tilde{\mathfrak{v}}_{n,p}^0 \to \mathfrak{v}_n^0, \quad \text{and} \quad \tilde{\mathfrak{v}}_p^0 \to \mathfrak{v}^0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

as $p \to +\infty$. Taking into account the continuity with respect to the initial datum of the (HLL) flow, and the construction of the solutions m_n and m in Step 4, this is enough to take the limit $p \to +\infty$ in (A.91) in order to obtain

$$\begin{aligned} \max_{t \in [0,T]} d_{\mathcal{E}}\big(m_n(\cdot,t), m(\cdot,t)\big) \leq & A d_{\mathcal{E}}\big(m_n^0, m^0\big) + A\big(1 + R^{\frac{1}{2}}\big) \max_{t \in [0,T]} \big\|\mathfrak{v}_n(\cdot,t) - \mathfrak{v}(\cdot,t)\big\|_{H^1 \times L^2} \\ & + A \max_{t \in [0,T]} \bigg(\int_{\mathbb{R} \setminus (-R,R)} \big(\partial_x v(\cdot,t)^2 + w(\cdot,t)^2\big) \bigg)^{\frac{1}{2}}. \end{aligned} \tag{A.92}$$

At this stage, given a positive number ϵ, we can argue as in Step 3 to find a positive number R such that

$$\max_{t \in [0,T]} \bigg(\int_{\mathbb{R} \setminus (-R,R)} \big(\partial_x v(\cdot,t)^2 + w(\cdot,t)^2\big) \bigg)^{\frac{1}{2}} \leq \epsilon.$$

Recalling that

$$\mathfrak{v}_n^0 \to \mathfrak{v}^0 \quad \text{in } H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

as $n \to +\infty$, again by Lemma A.4, we derive similarly from (A.92) and the continuity with respect to the initial datum of the (HLL) flow that

$$\limsup_{n \to +\infty} \max_{t \in [0,T]} d_{\mathcal{E}}\big(m_n(\cdot,t), m(\cdot,t)\big) \leq A\epsilon.$$

Since the choice of ϵ is arbitrary, this completes the proofs of Step 6, and as a consequence, of Corollary 1. □

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