THÈSE

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par

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Sujet

Ondes progressives pour les équations de Gross-Pitaevskii

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A mes parents,
Janine et Jean.

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Table des matières

Résumé

Ce mémoire de thèse porte sur les ondes progressives pour l'équation de Gross-Pitaevskii, et les ondes solitaires pour les équations de Kadomtsev-Petviashvili.
L'équation de Gross-Pitaevskii est un modèle pour l'analyse des condensats de Bose-Einstein, de la supraconductivité, de la superfluidité ou de l'optique non linéaire. Les équations de Kadomtsev-Petviashvili décrivent l'évolution d'ondes dispersives, faiblement non linéaires, et des ondes sonores dans les matériaux anti-ferromagnétiques.
On s'intéresse ici aux propriétés d'existence et au comportement asymptotique de ces ondes. On montre la non-existence des ondes progressives supersoniques, non constantes, d'énergie finie, pour l'équation de Gross-Pitaevskii en dimension supérieure ou égale à deux, puis celle des ondes progressives soniques, non constantes, d'énergie finie, en dimension deux. On décrit ensuite le comportement asymptotique des ondes progressives subsoniques, d'énergie finie, pour l'équation de Gross-Pitaevskii, puis celui des ondes solitaires pour les équations de Kadomtsev-Petviashvili en dimension supérieure ou égale à deux.

Mots-clés : Equation de Gross-Pitaevskii ; Equation de Kadomtsev-Petviashvili ; Equations de convolution ; Onde progressive ; Onde solitaire ; Existence de solutions ; Régularité de solutions ; Décroissance à l'infini ; Comportement asymptotique.

Classification AMS : 35A05, 35A08, 35A22, 35B40, 35B65, 35C15, 35C20, 35E05, 35Q40, 35Q51, 35Q53, 35Q55, 42B15, 44A35.

Abstract

This PhD thesis is devoted to the travelling waves in the Gross-Pitaevskii equation, and the solitary waves in the generalised Kadomtsev-Petviashvili equations.
The Gross-Pitaevskii equation is a model for Bose-Einstein condensates, superconductivity, superfluidity or non-linear optics. The generalised Kadomtsev-Petviashvili equations arise in the study of weakly non-linear, dispersive waves, and sound waves in anti-ferromagnetics.
Here, we investigate the existence properties and the asymptotic behaviour of such waves. We first establish the non-existence of non-constant supersonic travelling waves of finite energy in the Gross-Pitaevskii equation in dimension larger than two, and of non-constant sonic travelling waves of finite energy in the Gross-Pitaevskii equation in dimension two. We then describe the asymptotic behaviour of subsonic travelling waves of finite energy in the Gross-Pitaevskii equation, and of solitary waves in the generalised Kadomtsev-Petviashvili equations, in dimension larger than two.

Keywords: Gross-Pitaevskii equation; Kadomtsev-Petviashvili equation; Convolution equations; Travelling wave; Solitary wave; Existence of solutions; Regularity of solutions; Decay at infinity; Asymptotic behaviour.

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Introduction.

1 Motivation physique.

1.1 L'équation de Gross-Pitaevskii.

L'équation de Gross-Pitaevskii s'écrit sous sa forme non dimensionnée

$$i\partial_t u = \Delta u + u(1 - |u|^2). \tag{1}$$

Elle intervient dans de nombreux domaines de la recherche physique contemporaine (supraconductivité, superfluidité de l'hélium II, optique non linéaire...). Elle modélise en particulier la condensation de Bose-Einstein des gaz atomiques ultra-froids. Cet étonnant phénomène se produit à très basse température dans un gaz de bosons sans interactions réciproques : une fraction des particules se condense dans l'état quantique d'énergie minimale (Cf [12] pour de plus amples détails). L'idée d'une telle condensation remonte à une prédiction d'A. Einstein en 1925. Cependant, elle n'a été observée expérimentalement qu'en 1995, ce qui a conduit à un regain d'intérêt pour ce phénomène.

Afin de comprendre les mécanismes sous-jacents à cette condensation, E.P. Gross [28] et L.P. Pitaevskii [45] ont considéré un gaz de N bosons de masse m remplissant un volume V et ont supposé que tous les bosons sont rassemblés dans l'état quantique d'énergie minimale. Ils les ont alors décrits par une fonction d'ondes macroscopique Ψ, puis, en ont déduit l'équation de Gross-Pitaevskii par une méthode analogue à l'approximation de Hartree-Fock en physique atomique :

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\Delta\Psi - \Psi\int_V |\Psi(x', t)|^2 U(x - x')dx' = 0. \tag{2}$$

Dans cette équation, le potentiel U représente les interactions entre particules, qui sont non nulles à une température différente du zéro absolu. A très basse température, ces interactions sont très faibles et à courte portée. Aussi sont-elles le plus souvent modélisées par des potentiels d'interactions U de la forme $U_0\delta_0$.

Pour obtenir l'équation non dimensionnée (1), on introduit le niveau d'énergie moyen par unité de masse des bosons E_b, et l'on pose

$$\tilde{\Psi}(t, x) = e^{\frac{-imE_b t}{\hbar}}\Psi(t, x).$$

L'équation (2) devient alors

$$i\hbar\partial_t\tilde{\Psi} + \frac{\hbar^2}{2m}\Delta\tilde{\Psi} + mE_b\tilde{\Psi} - U_0\tilde{\Psi}|\tilde{\Psi}|^2 = 0.$$

Il suffit ensuite d'opérer les changements d'échelles

$$u(t, x) = \sqrt{\frac{U_0}{mE_b}}\ \overline{\tilde{\Psi}\Big(\frac{mE_b}{\hbar}t, \frac{m\sqrt{2E_b}}{\hbar}x\Big)},$$

pour obtenir l'équation non dimensionnée (1).

L'équation de Gross-Pitaevskii conserve (au moins formellement) deux quantités pertinentes sur le plan physique. La première d'entre elles est l'énergie $E(u)$, usuellement dénommée énergie de Ginzburg-Landau : elle s'exprime sous la forme

$$E(u) = \frac{1}{2}\int_V |\nabla u|^2 + \frac{1}{4}\int_V (1-|u|^2)^2. \tag{3}$$

La seconde est une grandeur vectorielle, le moment $\vec{P}(u)$, qui s'écrit

$$\vec{P}(u) = \frac{1}{2}\int_V i\nabla u.u. \tag{4}$$

Ces deux grandeurs interviennent dans l'étude des ondes progressives pour l'équation de Gross-Pitaevskii, notamment dans la description de leur comportement asymptotique.

Avant d'en venir à l'étude de ces ondes, il faut mentionner une autre forme de l'équation de Gross-Pitaevskii : sa forme hydrodynamique. Si l'on utilise la transformation de Madelung [37]

$$u = \sqrt{\rho}e^{i\theta},$$

et si l'on note

$$v = -2\nabla\theta,$$

on obtient une forme dite hydrodynamique de l'équation (1) qui s'écrit

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho v) = 0 \\ \rho(\partial_t v + v.\nabla v) + \nabla\rho^2 = \rho\nabla(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}). \end{cases} \tag{5}$$

Cette nouvelle forme motive l'introduction de l'équation de Gross-Pitaevskii pour décrire la superfluidité de l'hélium II à très basse température [1]. Le système (5) est en effet semblable aux équations d'Euler pour un fluide irrotationnel de pression $p(\rho) = \rho^2$. Il en diffère cependant par le terme fortement non linéaire de sa seconde équation, que l'on appelle souvent pression quantique. En outre, par l'analogie précédente, le système (5) fournit la vitesse des ondes sonores autour de la solution constante $u = 1$: elle est égale à $c_s = \sqrt{2}$. Cette vitesse joue un rôle crucial dans l'étude des ondes progressives pour l'équation de Gross-Pitaevskii.

1.2 Les ondes progressives pour l'équation de Gross-Pitaevskii.

Les ondes progressives pour l'équation de Gross-Pitaevskii sont les solutions particulières de l'équation (1) qui s'expriment sous la forme

$$u(t,x) = v(x_1 - ct, x_2, \dots, x_N).$$

Elles correspondent à la propagation d'un front d'ondes v suivant la direction x_1 à la vitesse constante c. L'équation vérifiée par le profil v est alors

$$ic\partial_1 v + \Delta v + v(1-|v|^2) = 0. \tag{6}$$

[1] Néanmoins, l'équation de Gross-Pitaevskii n'est pas un très bon modèle pour la superfluidité de l'hélium II : les interactions entre particules à l'intérieur du fluide sont trop importantes pour pouvoir être négligées à des températures différentes du zéro absolu.

On supposera par la suite que la fonction v est définie sur l'espace $\mathbb{R}^N$ (avec $N \geq 1$) [2] et à valeurs dans le corps des complexes $\mathbb{C}$.

Ces ondes progressives jouent un rôle important dans la dynamique associée à l'équation de Gross-Pitaevskii. Aussi les physiciens C.A. Jones, S.J. Putterman et P.H. Roberts [29] [30] les ont-ils minutieusement étudiées d'un point de vue numérique et formel en dimensions deux et trois.

La première question qu'ils ont considérée est bien sûr l'existence d'ondes progressives non constantes. En effet, l'équation (6) présente de nombreuses solutions. Les plus simples sont les solutions constantes, nulles ou de module un. Mais, il existe d'autres solutions comme la fonction $x \mapsto e^{-icx_1}$, et il est assez facile de construire de nouvelles solutions. Par exemple, si l'on connaît une solution v définie sur $\mathbb{R}^N$, v est une solution en toute dimension $M > N$. De même, si l'on translate v, si on la multiplie par un nombre complexe de module un ou si l'on pose

$$\tilde{v}(x) = e^{-icx_1}\overline{v(x)},$$

on obtient de nouvelles solutions en dimension N.

Toutefois, ces solutions sont peu intéressantes sur le plan physique, notamment car leur énergie est infinie. Ceci n'est guère acceptable pour modéliser les condensats de Bose-Einstein (mais, demeure possible dans le cas de l'optique non linéaire). Se pose donc la question de l'existence de solutions non constantes, d'énergie finie de l'équation (6). A partir de calculs numériques et formels, C.A. Jones et P.H. Roberts [30] ont répondu affirmativement à cette question. Selon eux, en dimensions deux et trois, l'équation (6) possède des solutions non constantes, d'énergie finie, si et seulement si la vitesse c est strictement comprise entre 0 et $\sqrt{2}$. En d'autres termes, les seules ondes progressives non constantes, d'énergie finie, sont subsoniques.

En fait, C.A. Jones, S.J. Putterman et P.H. Roberts [29] [30] ont précisé le comportement qualitatif de ces ondes progressives v, qu'ils ont obtenues comme points critiques de l' énergie E pour un moment scalaire $p = P_1$ fixé. Ces ondes forment une branche régulière de solutions pour des vitesses c comprises entre 0 et $\sqrt{2}$, qui vérifient la symétrie naturelle associée à l'équation (6). Elles sont en effet à symétrie axiale autour de l'axe x_1 : elles ne dépendent que de la variable x_1 et de la distance d_1 à l'axe x_1, donnée par la relation

$$\forall x \in \mathbb{R}^N, d_1(x) = |x_\perp| = \sqrt{\sum_{j=2}^{N} x_j^2},\ x_\perp = (x_2, \ldots, x_N).$$

Leur développement formel à l'infini est donné (à une constante multiplicative de module un près) par la formule

$$v(x) \sim 1 + \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2} + \ldots, \tag{7}$$

en dimension deux, tandis qu'en dimension trois, il s'écrit

$$v(x) \sim 1 + \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}} + \ldots. \tag{8}$$

Dans ces formules, le nombre réel α désigne le coefficient dipolaire élastique, qui dépend de l'énergie $E(v)$ et du moment scalaire $p(v)$ à travers les relations

$$2\pi\alpha\sqrt{1 - \frac{c^2}{2}} = cE(v) + 2\left(1 - \frac{c^2}{4}\right)p(v) \tag{9}$$

[2] Néanmoins, on pourrait aussi s'intéresser à ce type de solutions sur d'autres domaines de $\mathbb{R}^N$ (Cf l'article de A. Aftalion et X. Blanc [1] par exemple).

en dimension 2, et

$$4\pi\alpha = \frac{c}{2}E(v) + 2p(v) \tag{10}$$

en dimension 3.

En outre, C.A. Jones, S.J. Putterman et P.H. Roberts ont déterminé le comportement local d'une onde progressive v pour des valeurs de c proches de 0 et de $\sqrt{2}$. En dimension 2, lorsque c est proche de 0, v présente deux vortex de degrés -1 et 1, symétriques par rapport à l'axe x_1 et à une distance équivalente à $\frac{1}{c}$ lorsque c tend vers 0. Les vortex sont des points d'annulation de la fonction v autour desquels elle se comporte comme la fonction

$$\phi^d : z \in \mathbb{C} \mapsto \phi^d(z) = \Big(\frac{z}{|z|}\Big)^d$$

autour de 0. Leur degré est égal à l'exposant $d \in \mathbb{Z}$ de la fonction ϕ^d. Lorsque la vitesse c augmente, les vortex se rapprochent, puis disparaissent au-dessus d'une vitesse critique c_v. Quand c approche la vitesse du son $c_s = \sqrt{2}$, l'onde devient une onde de raréfaction, dont l'amplitude est gouvernée par l'équation de Kadomtsev-Petviashvili. En effet, si l'on note $\eta := 1 - |v|^2$, et si l'on opère le changement d'échelles

$$\forall x \in \mathbb{R}^2, w(x) = \frac{8}{2-c^2}\eta\Big(\frac{1}{\sqrt{2-c^2}}x_1, \frac{\sqrt{2}}{2-c^2}x_2\Big),$$

la fonction w vérifie, lorsque c tend vers $\sqrt{2}$, l'équation des ondes solitaires de vitesse 1 pour l'équation de Kadomtsev-Petviashvili usuelle, qui s'exprime sous la forme

$$\begin{cases} -\partial_1 w + w\partial_1 w + \partial_1^3 w - \partial_2 w_2 = 0, \\ \partial_1 w_2 = \partial_2 w. \end{cases}$$

L'équation de Kadomtsev-Petviashvili usuelle généralise l'équation de Korteweg-de Vries aux dimensions supérieures ou égales à deux. Elle s'écrit sur $\mathbb{R}^N$,

$$\begin{cases} \partial_t u + u\partial_1 u + \partial_1^3 u - \sum\limits_{j=2}^{N} \partial_j u_j = 0, \\ \forall j \in \{2, \dots, N\}, \partial_1 u_j = \partial_j u, \end{cases} \tag{11}$$

et décrit l'évolution d'ondes dispersives, faiblement non linéaires et essentiellement unidirectionnelles dans la direction de propagation. Les ondes solitaires pour cette équation sont les solutions particulières de la forme

$$u(t,x) = v(x_1 - c't, x_2, \dots, x_N).$$

Elles correspondent à la propagation d'un front d'ondes v suivant la direction x_1 à la vitesse constante c'. L'équation vérifiée par le profil v est alors

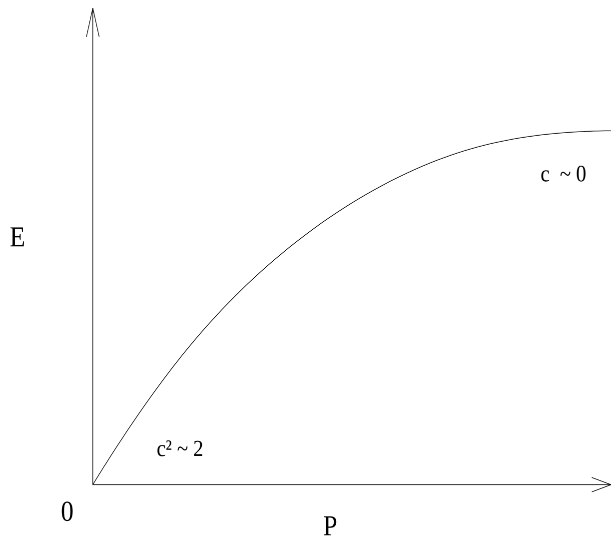
$$\begin{cases} -c'\partial_1 v + v\partial_1 v + \partial_1^3 v - \sum\limits_{j=2}^{N} \partial_j v_j = 0, \\ \forall j \in \{2, \dots, N\}, \partial_1 v_j = \partial_j v. \end{cases} \tag{12}$$

La description des ondes progressives en dimension trois est similaire. Elles présentent aussi des vortex lorsque leur vitesse c est petite. Mais, ils forment un cercle autour de l'axe x_1, dont le diamètre est équivalent à $\frac{1}{c}$ lorsque c tend vers 0. Quand la vitesse c augmente, le diamètre du cercle diminue jusqu'à la disparition des vortex au-dessus d'une vitesse critique c_v. Lorsque la vitesse tend vers $\sqrt{2}$, les ondes progressives deviennent des ondes de raréfaction, dont l'amplitude est gouvernée par l'équation (12). En effet, la fonction w, obtenue après le changement d'échelles

$$\forall x \in \mathbb{R}^3, w(x) = \frac{8}{2-c^2}\eta\Big(\frac{1}{\sqrt{2-c^2}}x_1, \frac{\sqrt{2}}{2-c^2}x_2, \frac{\sqrt{2}}{2-c^2}x_3\Big),$$

vérifie, à la limite $c \to \sqrt{2}$, l'équation (12) pour $c' = 1$.

Enfin, C.A. Jones, S.J. Putterman et P.H. Roberts ont évalué numériquement les valeurs de l'énergie $E(v)$ et du moment scalaire $p(v)$ d'une onde progressive v en fonction de sa vitesse c. En dimension deux, ils ont obtenu le graphe suivant.

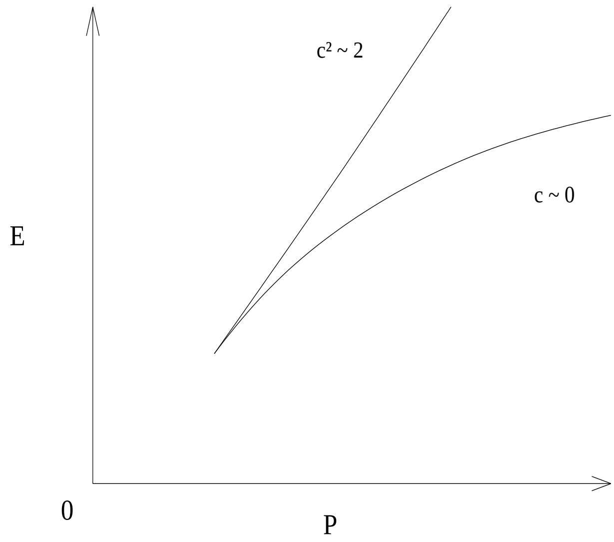


Sur ce graphe, chaque point représente une onde progressive v d'énergie $E(v)$ et de moment scalaire $p(v)$. La vitesse de l'onde correspond à la pente de la tangente à la courbe. En effet, v est (au moins formellement) un point critique de l'énergie E pour un moment scalaire p fixé. Sa vitesse est le multiplicateur de Lagrange associé à ce point critique, ce qui conduit à la relation formelle

$$c(v) = \frac{\partial E}{\partial p}\Big(v\Big). \tag{13}$$

La pente de la tangente à la courbe ci-dessus désigne donc bien la vitesse de l'onde considérée. Il s'ensuit que l'énergie et le moment scalaire de l'onde progressive tendent vers $+\infty$ lorsque sa vitesse c tend vers 0, tandis que ces deux grandeurs tendent vers 0 lorsque c tend vers $\sqrt{2}$.

En dimension trois, la situation n'est guère différente.



Sur ce graphe, chaque point correspond toujours à une onde progressive v d'énergie $E(v)$, de moment scalaire $p(v)$ et de vitesse c égale à la pente de la tangente à la courbe. L'énergie et le moment scalaire tendent toujours vers $+\infty$ lorsque c tend vers 0. Néanmoins, ces quantités tendent aussi vers $+\infty$ lorsque c tend vers $\sqrt{2}$. La courbe énergie-moment scalaire dessinée ci-dessus présente donc un point de rebroussement pour une vitesse c_r strictement comprise entre c_v et $\sqrt{2}$.

Ces estimations d'énergie jouent un rôle dans la stabilité orbitale des ondes progressives : une onde progressive qui minimise l'énergie E à moment scalaire p fixé est vraisemblablement stable. Aussi semble-t-il que les ondes progressives obtenues par C.A. Jones, S.J. Putterman et P.H. Roberts soient stables en dimension deux, de même que celles qui appartiennent à la branche inférieure du graphe ci-dessus, en dimension trois. Quant à la stabilité de celles de la branche supérieure, elle semble sujette à caution car ces ondes ne minimisent pas l'énergie E à moment scalaire p fixé.

2 Contexte mathématique.

Le problème de Cauchy associé à l'équation de Gross-Pitaevskii (ainsi que la dynamique attachée) soulève de multiples difficultés mathématiques. F. Béthuel et J.C. Saut [4] l'ont résolu en dimension supérieure ou égale à deux à l'aide d'arguments de T. Kato [32, 33].

Théorème ([4]). *Supposons que N soit supérieur ou égal à 2, et que u_0 soit une fonction de $1 + H^1(\mathbb{R}^N)$. L'équation (1) a alors une unique solution de donnée initiale u_0 dans l'espace $C^0(1+H^1(\mathbb{R}^N))$. De plus, cette solution conserve l'énergie E définie par la relation (3).*

Ce mémoire de thèse ne portera pas sur ce problème, mais sur les nombreuses conjectures formulées par C.A. Jones, S.J. Putterman et P.H. Roberts sur l'existence et le comportement qualitatif des ondes progressives pour l'équation de Gross-Pitaevskii. Elles sont la source de nombreux travaux mathématiques qui les ont pour la plupart confirmées. On peut ainsi illustrer leur bien-fondé dans le cas élémentaire de la dimension un.

2.1 Cas de la dimension un.

L'équation (6) est entièrement intégrable en dimension un. Son intégration conduit au théorème suivant (Cf [25] [39]).

Théorème 1 ([25]). *Supposons que $N = 1$ et $c > 0$, et considérons une solution v d'énergie finie de l'équation (6). Alors,*
– *si $c \geq \sqrt{2}$, v est une constante de module un.*
– *si $0 < c < \sqrt{2}$, à multiplication par une constante de module un et translation près, v est soit identiquement égale à la constante 1, soit à la fonction*

$$v_c(x) = \sqrt{1 - \frac{2-c^2}{2\mathrm{ch}^2\left(\frac{\sqrt{2-c^2}}{2}x\right)}}\exp\left(i \arctan\left(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2-c^2}}\right) - i \arctan\left(\frac{c}{\sqrt{2-c^2}}\right)\right).$$

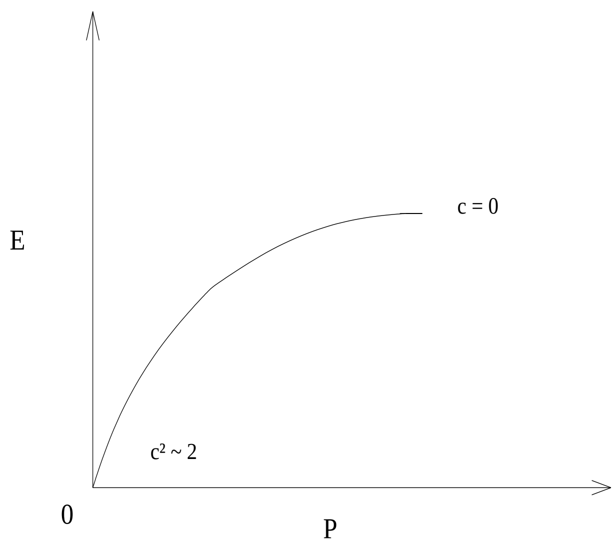
Conformément aux conjectures de C.A. Jones, S.J. Putterman et P.H. Roberts, les seules ondes progressives v_c non constantes, d'énergie finie, sont subsoniques en dimension un. Elles forment (à multiplication par une constante de module un et translation près) une famille régulière de solutions pour des vitesses strictement comprises entre 0 et $\sqrt{2}$. Leur énergie est égale à

$$E(v_c) = \frac{1}{3}(2-c^2)^{\frac{3}{2}}. \tag{14}$$

Leur moment scalaire vaut

$$p(v_c) = \frac{\pi}{2} - \arctan\left(\frac{c}{\sqrt{2-c^2}}\right) - \frac{c}{2}\sqrt{2-c^2}. \tag{15}$$

En particulier, la fonction $c \mapsto p(v_c)$ est strictement décroissante sur l'intervalle $]0, \sqrt{2}[$. L'énergie $E(v_c)$ s'exprime donc en fonction de $p(v_c)$, ce qui conduit au tracé du graphe suivant.



Ce graphe ressemble à celui de C.A. Jones, S.J. Putterman et P.H. Roberts en dimension deux. La pente de la courbe donne la vitesse c de la solution v_c considérée : la relation formelle (13) se justifie ici rigoureusement grâce aux formules (14) et (15). Comme en dimension deux, l'énergie et le moment scalaire de v_c tendent vers 0 lorsque c tend vers $\sqrt{2}$. Néanmoins, ces deux quantités ne tendent pas vers $+\infty$ lorsque c tend vers 0. En dimension un, les ondes progressives présentent en effet au moins deux différences avec celles des dimensions supérieures. D'une part, elles ne comportent pas de vortex lorsque leur vitesse est petite : c'est la raison pour laquelle l'énergie et le moment scalaire restent bornés lorsque c tend vers 0. D'autre part, leur décroissance à l'infini est exponentielle au lieu d'être algébrique. De fait, en dimension un, l'équation (6) est associée au noyau K de transformée de Fourier

$$\forall \xi \in \mathbb{R}, \widehat{K}(\xi) = \frac{1}{\xi^2 + 2 - c^2},$$

qui impose une décroissance exponentielle à l'infini.

2.2 Ondes progressives pour l'équation de Gross-Pitaevskii.

L'étude mathématique de ces ondes est plus délicate en dimension supérieure ou égale à deux. L'équation (6) n'est plus intégrable. Se pose donc la question de l'existence de solutions non constantes, d'énergie finie. En dimension deux, F. Béthuel et J.C. Saut [4, 5] ont répondu à cette question par les deux théorèmes suivants.

Théorème ([4]). *Soit $N = 2$. Il existe une constante $c_0 > 0$ telle que l'équation* (6) *a une solution v_c non constante, d'énergie finie, pour chaque valeur de $c \in]0, c_0[$. De plus, il existe des constantes Λ_0 et Λ_1 telles que l'énergie de cette solution vérifie*

$$\forall c \in]0, c_0[, 2\pi|\ln(c)| + \Lambda_0 \leq E(v_c) \leq 2\pi|\ln(c)| + \Lambda_1. \tag{16}$$

Théorème ([5]). *Soit $N = 2$. Il existe une suite $(c_n)_{n\in\mathbb{N}}$ de réels compris dans l'intervalle $]0, \sqrt{2}[$, qui tend vers $\sqrt{2}$, et telle que l'équation* (6) *a une solution v_{c_n} non constante, d'énergie finie et de vitesse c_n pour tout entier n.*

Ces deux théorèmes découlent d'une approche variationnelle de l'équation (6). F. Béthuel et J.C. Saut obtiennent les solutions v_c en appliquant le lemme du col (ou une version améliorée due à N. Ghoussoub et D. Preiss [21]) à la fonctionnelle F_c, définie pour tout $v \in 1 + H^1(\mathbb{R}^N)$ par

$$F_c(v) = E_c(v) - p(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4c^2}\int_{\mathbb{R}^N} (1 - |v|^2)^2 - \frac{1}{2}\int_{\mathbb{R}^N} i\partial_1 v.(v-1). \tag{17}$$

Leur approche fournit de plus des informations qualitatives sur les solutions v_c (comme l'estimation (16) de leur énergie). Par exemple, lorsque c est suffisamment petit, la solution v_c présente deux vortex de degrés -1 et 1, dont la distance est équivalente à $\frac{1}{c}$ lorsque c tend vers 0.

En dimension supérieure ou égale à trois, F. Béthuel, G. Orlandi et D. Smets [7] ont également établi l'existence d'ondes progressives non constantes, d'énergie finie, pour des vitesses petites.

Théorème ([7]). *Soit $N \geq 3$. Il existe une suite $(c_n)_{n\in\mathbb{N}}$ de réels compris dans l'intervalle $]0, \sqrt{2}[$, qui tend vers 0, et telle que l'équation* (6) *a une solution v_{c_n} non constante, d'énergie finie et de vitesse c_n pour tout entier n. De plus, l'énergie et le moment scalaire de v_{c_n} vérifient*

$$E(v_{c_n}) \underset{n\to+\infty}{\sim} \pi|\mathbb{S}^{N-2}|(N-2)^{N-2}c_n^{2-N}|\ln(c_n)|^{N-1}, \tag{18}$$

$$p(v_{c_n}) \underset{n\to+\infty}{\sim} \frac{2\pi}{N-1}|\mathbb{S}^{N-2}|(N-2)^{N-1}c_n^{1-N}|\ln(c_n)|^{N-1}. \tag{19}$$

Ce théorème résulte aussi d'un argument variationnel. Il s'agit ici de la minimisation de l'énergie de Ginzburg-Landau E_c, définie par

$$E_c(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4c^2}\int_{\mathbb{R}^N} (1-|v|^2)^2, \tag{20}$$

sous la contrainte que le moment scalaire $p(v)$ soit fixé. Comme en dimension deux, cette analyse fournit des propriétés qualitatives des solutions : les estimations (18) et (19) de leur énergie et de leur moment scalaire, ainsi que la preuve de l'existence de vortex. Ces derniers constituent un anneau dont le diamètre est équivalent à $\frac{1}{c}$ lorsque c tend vers 0. De plus, ce raisonnement par minimisation sous contrainte permet d'envisager la stabilité orbitale des ondes obtenues (Cf le théorème 6 ci-dessous).

Néanmoins, cette approche ne fournit pas un intervalle complet de solutions pour des vitesses proches de 0. D. Chiron [9] a donc complété ce théorème en prouvant l'existence de solutions pour un intervalle de la forme $]0, c_0[$.

Théorème ([9]). *Soit $N \geq 3$. Il existe une constante $c_0 > 0$ telle que l'équation* (6) *a une solution v_c non constante, d'énergie finie, pour chaque valeur de c dans l'intervalle $]0, c_0[$. De plus, l'énergie et le moment scalaire de v_c vérifient les relations* (18) *et* (19).

La preuve de D. Chiron reprend l'argumentation de F. Béthuel, G. Orlandi et D. Smets [7]. Cependant, sa démarche repose sur le lemme du col pour la fonctionnelle F_c définie par la relation (17). Il obtient ainsi un intervalle complet de solutions comme F. Béthuel et J.C. Saut [4, 5] en dimension deux.

Ces quatre théorèmes forment l'ensemble des résultats d'existence d'ondes progressives non constantes, d'énergie finie, pour l'équation de Gross-Pitaevskii. L'existence de ces ondes pour toute valeur de la vitesse dans l'intervalle $]0, \sqrt{2}[$ reste un problème ouvert en dimension supérieure ou égale à deux.

La preuve de la non-existence de ces ondes est par contre au coeur de ce mémoire (Cf les théorèmes 2 et 3). A l'aide des identités de Pohozaev, F. Béthuel et J.C. Saut [4] ont d'abord montré leur non-existence lorsque leur vitesse est nulle en dimension supérieure ou égale à deux.

Théorème ([4]). *Soit $N \geq 2$. Toute solution de l'équation* (6) *d'énergie finie et de vitesse $c = 0$ est une constante de module un.*

Par ailleurs, F. Béthuel et J.C. Saut [4, 5] d'une part, A. Farina [18] d'autre part, ont justifié de manière rigoureuse certaines propriétés qualitatives énoncées par C.A. Jones, S.J. Putterman et P.H. Roberts [29, 30]. F. Béthuel et J.C. Saut [4, 5] ont prouvé l'existence d'une limite à l'infini des ondes progressives subsoniques d'énergie finie en dimension deux.

Théorème ([4, 5]). *Supposons que $N = 2$ et $0 < c < \sqrt{2}$, et considérons une solution v de l'équation* (6) *d'énergie finie. Il existe alors une constante λ_∞ de module un telle que*

$$v(x) \underset{|x|\to+\infty}{\longrightarrow} \lambda_\infty.$$

Quant à A. Farina [18], il a calculé une borne universelle sur leur module grâce à une version raffinée du principe du maximum.

Théorème ([18]). *Soit $N \geq 1$ et $c \in \mathbb{R}$. Si v_c est une solution sur $\mathbb{R}^N$ de l'équation* (6), *alors,*

$$\forall x \in \mathbb{R}^N, |v_c(x)| \leq \sqrt{1 + \frac{c^2}{4}}.$$

Enfin, deux résultats récents se rattachent également à l'étude des ondes progressives pour l'équation de Gross-Pitaevskii. Le premier, dû à D. Chiron [10], énonce l'existence d'ondes progressives non constantes (mais d'énergie infinie) pour des vitesses petites en dimension trois.

Théorème ([10]). *Soit $N = 3$ et $L > 0$. Il existe un réel $\varepsilon_0(L)$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[$, l'équation (6) a une solution non constante v_ε, périodique pour la variable x_1 (de période $\frac{2\pi}{\varepsilon}$), dont la vitesse $c(\varepsilon)$ et le moment scalaire $p(v_\varepsilon)$ vérifient*

$$c(\varepsilon) \underset{\varepsilon \to 0}{\sim} \frac{\varepsilon |\ln(\varepsilon)|}{\sqrt{1+L^2}},$$
$$p(v_\varepsilon) = \frac{2\pi^2 L^2}{\varepsilon^2}.$$

De plus, la fonction $x \mapsto v_\varepsilon(\frac{x}{\varepsilon})$ présente des vortex qui se concentrent suivant une hélice d'axe x_1, de rayon L et de longueur $2\pi\sqrt{1+L^2}$ lorsque ε tend vers 0.

Ce théorème découle d'une approche variationnelle par minimisation sous contrainte de l'énergie de Ginzburg-Landau donnée par la relation (20). Il illustre de manière rigoureuse un phénomène observé depuis longtemps sur le plan expérimental ou numérique (Cf [2, 42]) : l'existence d'ondes progressives dont les vortex se concentrent le long d'une hélice.

Le second résultat, dû à A. Aftalion et X. Blanc [1], stipule l'existence d'ondes progressives de vitesses petites pour un problème avec obstacle en dimension deux : ce problème consiste à résoudre l'équation (6) sur le domaine $\Omega := \mathbb{R}^2 \setminus \overline{B(0,1)}$ avec une condition de Dirichlet sur le bord de la boule $B(0,1)$.

Théorème ([1]). *Soit $N = 2$. Il existe une constante $c_0 > 0$ telle que pour toute valeur de $c \in]0, c_0[$, l'équation (6) a une solution v_c, définie sur le domaine Ω, qui ne s'annule pas sur ce domaine et qui vérifie la condition de Dirichlet,*

$$v_c = 0 \text{ on } \mathbb{S}^1.$$

Ce théorème, déduit d'arguments variationnels, permet de résoudre un problème analogue à l'équation (6) en dimension trois.

Théorème ([1]). *Il existe une constante $0 < c_1 \leq c_0$ telle que pour toute valeur de $c \in]0, c_1[$, l'équation*

$$\forall x = (x_1, x_2, x_3) \in \Omega \times]0,1[, ic\partial_1 u(x) + \Delta u(x) + u(x)(x_3 - |u(x)|^2) = 0,$$

munie des conditions aux limites

$$\begin{cases} \forall (x_1, x_2) \in \Omega, u(x_1, x_2, 0) = 0, \\ \forall x \in \mathbb{S}^{N-1} \times]0,1[, u(x) = 0, \\ \forall (x_1, x_2) \in \Omega, u(x_1, x_2, 1) = v_c(x_1, x_2). \end{cases}$$

a une solution u_c qui ne s'annule pas sur l'ensemble $\Omega \times]0,1[$.

2.3 Ondes solitaires pour les équations de Kadomtsev-Petviashvili.

Les équations de Kadomtsev-Petviashvili s'écrivent

$$\begin{cases} \partial_t u + u^p \partial_1 u + \partial_1^3 u - \sum_{j=2}^{N} \partial_j u_j = 0, \\ \forall j \in \{2, \ldots, N\}, \partial_1 u_j = \partial_j u, \end{cases} \tag{21}$$

pour une fonction u définie sur $\mathbb{R}^N$ (avec $N \geq 2$) et à valeurs réelles. Le réel p est un nombre rationnel de la forme $p = \frac{m}{n}$, où n est impair, et m et n sont premiers entre eux. La fonction $u \mapsto u^p$ est définie par la convention usuelle

$$\forall u \in \mathbb{R}, u^p = Sign(u)^m |u|^p.$$

L'équation de Kadomtsev-Petviashvili usuelle (souvent notée KP I) correspond au cas $p = 1$.

L'équation KP I décrit l'évolution d'ondes dispersives, faiblement non linéaires et essentiellement unidirectionnelles dans la direction de propagation x_1 (Cf l'article de B.B. Kadomtsev et de V.I. Petviashvili [31]). Pour p égal à 2, l'équation (21) modélise l'évolution d'ondes sonores dans les matériaux anti-ferromagnétiques (Cf l'article de G.E. Falkovitch et de S.K. Turitsyn [19]).

Le problème de Cauchy associé à l'équation (21) a motivé un grand nombre de travaux mathématiques (Cf [17, 20, 46, 47, 50, 52, 53]) que l'on peut synthétiser par le théorème que voici.

Théorème. *Soit $N \in \{2, 3\}$ et $s \geq 3$. Considérons l'espace X_s défini par*

$$X_s := \Big\{ f \in H^s(\mathbb{R}^N), \widehat{\Big(\frac{\widehat{f}}{\xi_1} \Big)} \in H^s(\mathbb{R}^N) \Big\}.$$

Supposons que $u(0)$ appartienne à l'espace X_s et qu'il existe des fonctions $u_j(0) \in L^2(\mathbb{R}^N)$ $(2 \leq j \leq N)$ telles que

$$\partial_j^2 u(0) = \partial_1^2 u_j(0).$$

Il existe alors un réel strictement positif T tel que l'équation (21) a une unique solution u, de donnée initiale $u(0)$, qui vérifie

$$u \in C^0([-T, T], H^s(\mathbb{R}^N)) \cap C^1([-T, T], H^{s-3}(\mathbb{R}^N)),$$

et, pour tout $j \in \{2, \ldots, N\}$,

$$u_j \in C^0([-T, T], H^{s-1}(\mathbb{R}^N)).$$

De plus, la norme L^2 de la fonction u

$$I(u) = \int_{\mathbb{R}^N} u^2(x) dx,$$

et son énergie

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \Big(\partial_1 u(x)^2 + \sum_{j=2}^N u_j(x)^2 \Big) dx - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^N} u(x)^{p+2} dx,$$

sont définies pour tout $t \in [-T, T]$ et sont indépendantes de t.

Ce mémoire de thèse ne portera pas sur ce problème de Cauchy, mais sur un type particulier de solutions de l'équation (21) : les ondes solitaires.

Ces ondes sont les solutions u de l'équation (21) de la forme

$$u(t, x) = v(x_1 - ct, x_\perp), \ x_\perp = (x_2, \ldots, x_N),$$

dont le profil v appartient à l'espace Y, défini comme l'adhérence de l'espace $\partial_1 C_0^\infty(\mathbb{R}^N)$ pour la norme

$$\forall \phi \in C_0^\infty(\mathbb{R}^N), \|\partial_1 \phi\|_Y = \Big(\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \|\partial_{1,1}^2 \phi\|_{L^2(\mathbb{R}^N)}^2 \Big)^{\frac{1}{2}}.$$

Ce sont (au moins formellement) des points critiques sur Y de l'action S, définie par

$$\forall v \in Y, S(v) = E(v) + \frac{c}{2} I(v). \tag{22}$$

Le paramètre $c > 0$ désigne la vitesse de l'onde, qui se déplace dans la direction de propagation x_1. L'équation pour le profil v s'écrit

$$\begin{cases} -c\partial_1 v + v^p \partial_1 v + \partial_1^3 v - \sum\limits_{j=2}^{N} \partial_j v_j = 0, \\ \forall j \in \{2, \ldots, N\}, \partial_1 v_j = \partial_j v. \end{cases} \tag{23}$$

En particulier, si v vérifie l'équation (23), la fonction $\tilde{v}$ donnée par le changement d'échelles,

$$\forall x \in \mathbb{R}^N, \tilde{v}(x_1, x_\perp) = c^{-\frac{1}{p}} v\Big(\frac{x_1}{\sqrt{c}}, \frac{x_\perp}{c}\Big), \tag{24}$$

est une onde solitaire de vitesse 1. Afin de simplifier les notations et les énoncés qui suivent, on supposera par la suite que

$$c = 1$$

grâce au changement d'échelles (24). En raison de cette hypothèse supplémentaire, le profil v vérifie l'équation

$$-\Delta v + \partial_1^4 v + \frac{1}{p+1} \partial_1^2 (v^{p+1}) = 0, \tag{25}$$

sur laquelle repose l'analyse du comportement asymptotique des ondes solitaires.

Ces ondes jouent un rôle prépondérant dans la dynamique des équations de Kadomtsev-Petviashvili. De plus, comme l'ont souligné C.A. Jones, S.J Putterman et P.H. Roberts, elles décrivent le comportement des ondes progressives pour l'équation de Gross-Pitaevskii à la limite $c \to \sqrt{2}$. Ainsi, A. de Bouard et J.C. Saut [13, 14, 15] (Cf aussi [46]) les ont attentivement étudiées en dimensions deux et trois. Ils ont d'abord établi l'existence d'ondes solitaires non constantes en dimensions deux et trois.

Théorème ([13]). *Supposons que N soit égal à 2 ou à 3, et que c soit égal à 1. L'équation* (23) *a des solutions non constantes dans l'espace Y si et seulement si*

$$0 < p < \frac{4}{2N-3}. \tag{26}$$

La non-existence des ondes solitaires non constantes pour $p \geq \frac{4}{2N-3}$ provient des identités de Pohozaev. Quant à la preuve de leur existence, lorsque p vérifie la relation (26), elle découle une nouvelle fois d'un argument variationnel : la minimisation de la norme de Y sous la contrainte que la fonctionnelle

$$\begin{array}{ccc} Y & \to & \mathbb{R} \\ v & \mapsto & \int_{\mathbb{R}^N} v^{p+2}(y) dy \end{array}$$

soit fixée. Ce raisonnement exploite le principe de concentration-compacité de P.L. Lions [34] et le théorème d'injection pour les espaces de Sobolev anisotropes (Cf l'ouvrage de

O.V. Besov, V.P. Il'in et S.M. Nikolskii [3]). Il ne dépend de la dimension considérée qu'à travers ce théorème d'injection. Aussi peut-il s'étendre sans difficulté aux dimensions supérieures ou égales à 4 (ce qui est réalisé dans [27] pour l'aspect non-existence).

En outre, A. de Bouard et J.C. Saut [13, 14, 15] ont déterminé certaines propriétés qualitatives des ondes solitaires, comme leur régularité.

Théorème ([13, 14]). *Supposons que N soit égal à 2 ou à 3, que c soit égal à 1 et que $0 < p < \frac{4}{2N-3}$. Les solutions de l'équation (23) dans l'espace Y sont continues et tendent vers 0 à l'infini. De plus, si p est entier, elles appartiennent à l'espace*

$$H^{\infty}(\mathbb{R}^N) := \bigcap_{n\in\mathbb{N}} H^n(\mathbb{R}^N).$$

Ils ont également précisé leur décroissance à l'infini.

Théorème ([14]). *Supposons que c soit égal à 1 et que $0 < p < \frac{4}{2N-3}$. Si N est égal à 2, alors, les solutions v de l'équation (23) dans l'espace Y vérifient la propriété*

$$|.|^2 v \in L^{\infty}(\mathbb{R}^2).$$

Si N est égal à 3, elles satisfont

$$\forall \delta \in [0, \frac{3}{2}[, |.|^{\delta} v \in L^2(\mathbb{R}^3).$$

Cette analyse asymptotique s'appuie sur un argument sur lequel on reviendra longuement par la suite. Cependant, il faut d'ores et déjà souligner le caractère optimal de ce théorème en dimension deux. En effet, l'équation (23) possède une solution explicite lorsque N est égal à 2 et p, à 1. Cette solution "bosse" s'écrit

$$\forall (x_1, x_2) \in \mathbb{R}^2, v_c(x_1, x_2) = 24c \frac{3 - cx_1^2 + c^2 x_2^2}{(3 + cx_1^2 + c^2 x_2^2)^2}.$$

En raison de l'existence de cette solution, on ne peut espérer un taux de décroissance supérieur à celui énoncé par le théorème ci-dessus, lequel est donc optimal.

Finalement, A. de Bouard et J.C. Saut [14] ont décrit les propriétés des états fondamentaux. Les états fondamentaux (dont l'ensemble sera noté $\mathcal{E}_F$) sont les ondes solitaires qui minimisent l'action S donnée par la relation (22). Ils sont à symétrie axiale autour de l'axe x_1 en dimensions deux et trois.

Théorème ([14]). *Supposons que N soit égal à 2 ou à 3, que c soit égal à 1 et que $0 < p < \frac{4}{2N-3}$. Un état fondamental v de l'équation (23) ne dépend (à translation près) que de la variable x_1 et de la distance d_1 à l'axe x_1, donnée par*

$$\forall x \in \mathbb{R}^N, d_1(x) = |x_{\perp}| = \sqrt{\sum_{j=2}^{N} x_j^2}.$$

La preuve de ce théorème, qui découle d'un argument d'O. Lopes [36], permet d'amorcer l'analyse de la dynamique des équations de Kadomtsev-Petviashvili. Elle conduit en effet à la stabilité orbitale des états fondamentaux en dimension deux lorsque p est strictement inférieur à $\frac{4}{3}$.

Théorème ([15]). *Supposons que* $N = 2$, $1 \leq p < \frac{4}{3}$ *et* $s \geq 3$, *et considérons un état fondamental* v *de l'équation* (23). *Il existe alors pour tout réel* $\varepsilon > 0$, *un réel* $\delta > 0$ *tel que, quelle que soit la fonction* $u_0 \in X_s$ *qui vérifie*

$$\|u_0 - v\|_Y \leq \delta,$$

la solution u *de l'équation* (21), *de donnée initiale* u_0, *satisfait*

$$\sup_{t \geq 0} \Big(\inf_{w \in \mathcal{E}_F} \|u(t) - w\|_Y \Big) \leq \varepsilon.$$

La stabilité dépend fortement de l'exposant p considéré. A. de Bouard et J.C. Saut [15] ont ainsi établi l'instabilité orbitale des ondes solitaires à symétrie axiale lorsque p est strictement supérieur à $\frac{4}{3}$.

Théorème ([15]). *Soit* $N = 2$ *et* $\frac{4}{3} < p < 4$. *Les ondes solitaires solutions de l'équation* (23), *à symétrie axiale autour de l'axe* x_1, *sont orbitalement instables.*

3 Principaux résultats.

Ce mémoire de thèse contient trois types de résultats :
- la non-existence des ondes progressives non constantes, d'énergie finie et de vitesse supersonique ou sonique, pour l'équation de Gross-Pitaevskii,
- la description asymptotique des ondes progressives d'énergie finie et de vitesse subsonique pour l'équation de Gross-Pitaevskii,
- la description asymptotique des ondes solitaires pour les équations de Kadomtsev-Petviashvili.

3.1 Non-existence des ondes progressives de vitesse sonique ou supersonique pour l'équation de Gross-Pitaevskii.

En dimension deux et trois, C.A. Jones, S.J. Putterman et P.H. Roberts [29, 30] ont conjecturé la non-existence des ondes progressives non constantes, d'énergie finie, pour l'équation de Gross-Pitaevskii lorsque leur vitesse est supérieure ou égale à $\sqrt{2}$. On corrobore la validité de cette conjecture dans les deux cas suivants.

Théorème 2 ([23]). *Soit* $N \geq 2$. *Une onde progressive d'énergie finie et de vitesse* $c > \sqrt{2}$ *pour l'équation de Gross-Pitaevskii est constante.*

Théorème 3 ([25]). *Soit* $N = 2$. *Une onde progressive d'énergie finie et de vitesse* $c = \sqrt{2}$ *pour l'équation de Gross-Pitaevskii est constante.*

Ne demeure donc que le cas où la vitesse c est égale à $\sqrt{2}$ et la dimension est supérieure ou égale à trois. On tâchera d'expliquer pour quelle raison ce dernier cas diffère des précédents (Cf le paragraphe 4.2.1).

3.2 Comportement asymptotique des ondes progressives pour l'équation de Gross-Pitaevskii.

En dimension deux et trois, C.A. Jones, S.J. Putterman et P.H. Roberts [29, 30] ont également calculé les développements asymptotiques formels donnés par les formules (7), (8), (9) et (10) pour les ondes progressives subsoniques pour l'équation de Gross-Pitaevskii. Afin d'établir rigoureusement ces formules, on détermine dans un premier temps la limite à l'infini de ces ondes en dimension supérieure ou égale à trois.

Théorème 4 ([22]). *Supposons que $N \geq 3$ et $0 < c < \sqrt{2}$, et considérons une onde progressive v d'énergie finie et de vitesse c pour l'équation de Gross-Pitaevskii. Il existe alors un nombre complexe λ_∞ de module un tel que*

$$v(x) \underset{|x| \to +\infty}{\to} \lambda_\infty.$$

Remarques. 1. L'équation (6), vérifiée par la fonction v, est invariante par multiplication par un nombre complexe de module un. Quitte à considérer la fonction $\frac{v}{\lambda_\infty}$, on supposera dans la suite que λ_∞ est égal à 1.

2. F. Béthuel et J.C. Saut [4, 5] ont démontré ce théorème en dimension deux.

En outre, ce théorème reste valable pour des ondes progressives d'énergie finie et de vitesse $c = \sqrt{2}$ en dimension supérieure ou égale à trois (ce qui peut constituer une première étape vers leur non-existence (Cf [25] pour de plus amples détails)).

Théorème 5 ([25]). *Soit $N \geq 3$. Si v est une onde progressive d'énergie finie et de vitesse $c = \sqrt{2}$ pour l'équation de Gross-Pitaevskii, alors, il existe un nombre complexe λ_∞ de module un tel que*

$$v(x) \underset{|x| \to +\infty}{\to} \lambda_\infty.$$

On calcule ensuite le taux de décroissance à l'infini des ondes progressives subsoniques d'énergie finie en dimension supérieure ou égale à deux.

Théorème 6 ([24]). *Supposons que $N \geq 2$ et $0 < c < \sqrt{2}$, et considérons une onde progressive v d'énergie finie et de vitesse c pour l'équation de Gross-Pitaevskii. Il existe alors un nombre réel A tel que*

$$\forall x \in \mathbb{R}^N, |v(x) - 1| \leq \frac{A}{1 + |x|^N}.$$

Le théorème 6 fournit un corollaire important. Selon ce théorème, la fonction $v - 1$ appartient à tous les espaces $L^p(\mathbb{R}^N)$ pour

$$\frac{N}{N-1} < p \leq +\infty.$$

Si $N \geq 3$, elle appartient donc à l'espace $1 + H^1(\mathbb{R}^N)$, dans lequel l'équation de Gross-Pitaevskii est globalement bien posée (Cf [4]). Aussi est-il possible désormais d'étudier la stabilité orbitale de v dans cet espace (ce qui constitue une première étape vers la description de la dynamique de l'équation de Gross-Pitaevskii). Par exemple, en dimension supérieure ou égale à trois, F. Béthuel, G. Orlandi et D. Smets [7] ont construit des ondes progressives qui minimisent l'énergie E sur l'espace $1 + H^1(\mathbb{R}^N)$ pour un moment scalaire p fixé. Pour déterminer la stabilité orbitale de ces ondes, il suffit grâce au théorème 6 de prouver un peu de compacité pour ce problème.

Enfin, on confirme la validité des conjectures (7), (8), (9) et (10).

Théorème 7 ([26, 27]). *Considérons une onde progressive d'énergie finie et de vitesse* $0 < c < \sqrt{2}$ *pour l'équation de Gross-Pitaevskii en dimension* $N \geq 2$. *Il existe une fonction* v_∞ *définie sur la sphère* $\mathbb{S}^{N-1}$ *et à valeurs réelles telle que*

$$|x|^{N-1}(v(x)-1) - iv_\infty\left(\frac{x}{|x|}\right) \underset{|x|\to+\infty}{\to} 0.$$

De plus, il existe des constantes α, β_2, ..., β_N *telles que la fonction* v_∞ *est égale à*

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + \sum_{j=2}^{N} \beta_j \frac{\sigma_j}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}. \quad (27)$$

Les constantes α *et* β_j *sont données par les relations*

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\left(1 - \frac{c^2}{2}\right)^{\frac{N-3}{2}}\left(\frac{4-N}{2}cE(v) + \left(2 + \frac{N-3}{2}c^2\right)p(v)\right), \quad (28)$$

$$\beta_j = \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}}\left(1 - \frac{c^2}{2}\right)^{\frac{N-1}{2}} P_j(v). \quad (29)$$

Remarque. La définition du moment $\vec{P}(v)$ présente une difficulté. L'intégrale qui apparaît dans la définition (4) n'est pas toujours convergente pour des ondes progressives subsoniques. Afin de formuler les équations (28) et (29) rigoureusement, il faut définir le moment par

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla v.(v-1),$$

et le moment scalaire dans la direction x_1 par

$$p(v) = P_1(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\partial_1 v.(v-1).$$

Toutes ces intégrales sont bien définies lorsque v est une onde progressive subsonique (Cf [24]).

Le théorème 7 est plus précis que les conjectures (7), (8), (9) et (10), qui ne concernent que le cas d'ondes progressives à symétrie axiale autour de l'axe x_1. Au contraire, le théorème 7 décrit le comportement asymptotique de n'importe quelle onde progressive.

De plus, le théorème 7 est optimal. Grâce aux résultats de F. Béthuel et J.C. Saut [4, 5] en dimension deux, et de F. Béthuel, G. Orlandi et D. Smets [7] en dimension trois, on sait qu'il existe en toute dimension supérieure ou égale à deux, des ondes progressives non constantes, d'énergie finie, à symétrie axiale autour de l'axe x_1. La constante α associée à de telles ondes est non nulle, les constantes β_j étant nulles (Cf [26]). Il s'ensuit qu'une onde progressive de ce type a exactement le comportement asymptotique donné par le théorème 7, lequel est donc optimal. Néanmoins, on ne sait pas s'il existe des ondes progressives v qui correspondent à chacun des comportements asymptotiques décrits par le théorème 7. En particulier, on ne connaît pas à ce jour d'ondes progressives pour lesquelles au moins une des constantes β_j est non nulle.

3.3 Comportement asymptotique des ondes solitaires pour les équations de Kadomtsev-Petviashvili.

Ce dernier résultat concerne le comportement asymptotique des ondes solitaires pour les équations de Kadomtsev-Petviashvili. Il complète ceux de A. de Bouard et J.C. Saut [14].

Théorème 8. *Supposons que* $N \geq 2$ *et* $0 < p < \frac{4}{2N-3}$, *et considérons une solution de vitesse égale à* 1 *de l'équation* (23). *La fonction* $x \mapsto (1 + |x|^N)v(x)$ *est alors bornée sur* $\mathbb{R}^N$. *De plus, si on note* v_∞ *la fonction définie sur la sphère* $\mathbb{S}^{N-1}$ *par*

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(p+1)}(1 - N\sigma_1^2) \int_{\mathbb{R}^N} v(x)^{p+1} dx, \tag{30}$$

alors, on a la convergence

$$\forall \sigma \in \mathbb{S}^{N-1}, R^N v(R\sigma) \underset{R \to +\infty}{\to} v_\infty(\sigma). \tag{31}$$

En outre, cette convergence est uniforme (c'est-à-dire qu'elle a lieu dans $L^\infty(\mathbb{S}^{N-1})$*) si* $\frac{1}{N} \leq p < \frac{4}{2N-3}$. *Enfin, si* p *est égal à* 1, *la fonction* v_∞ *s'exprime sous la forme*

$$\begin{aligned} \forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) &= \frac{(7-2N)\Gamma(\frac{N}{2})}{2(2N-5)\pi^{\frac{N}{2}}}(1 - N\sigma_1^2)E(v) \\ &= \frac{(7-2N)\Gamma(\frac{N}{2})}{4\pi^{\frac{N}{2}}}(1 - N\sigma_1^2)S(v). \end{aligned} \tag{32}$$

Comme le théorème 7, ce théorème est optimal si le numérateur m de p est impair. En effet, si m est impair, l'intégrale $\int_{\mathbb{R}^N} v(x)^{p+1} dx$ associée à une onde solitaire non nulle ne peut être nulle. Le théorème 8 décrit donc le comportement asymptotique exact de toutes les ondes solitaires non nulles. Au contraire, cette intégrale peut être nulle si m est pair. En particulier, lorsque m est pair, L. Paumond [43] a montré l'existence d'ondes solitaires pour une équation analogue à celle de Kadomtsev-Petviashvili, qui vérifient cette condition d'intégrale nulle : cette équation s'écrit sur $\mathbb{R}^5$,

$$\begin{cases} \partial_t u + u^p \partial_1 u + \partial_1^7 u - \sum\limits_{j=2}^{5} \partial_j u_j = 0, \\ \forall j \in \{2, \ldots, 5\}, \partial_1 u_j = \partial_j u. \end{cases}$$

De telles ondes existent vraisemblablement pour les équations de Kadomtsev-Petviashvili. Aussi est-il possible que le théorème 8 ne soit pas optimal lorsque m est pair.

4 Principales techniques employées.

Les théorèmes précédents reposent sur un ensemble de techniques que l'on va maintenant décrire plus abondamment.

4.1 Équations de convolution.

Que ce soit pour l'étude de propriétés de non-existence ou pour l'analyse de certains comportements asymptotiques, on commence par transformer les équations aux dérivées partielles considérées en équations de convolution. Cette transformation fait apparaître explicitement les noyaux associés à ces équations, qui fournissent la plupart des propriétés des solutions. Par exemple, le comportement asymptotique des noyaux d'une équation surlinéaire dicte le comportement asymptotique des solutions de cette équation, comme on le vérifiera ci-dessous.

Dans le cas d'une onde progressive v pour l'équation de Gross-Pitaevskii, on transforme l'équation (6) en un système d'équations de convolution. Cette transformation, réalisée par F. Béthuel et J.C. Saut [5], repose sur l'introduction de nouvelles variables. Grâce à la régularité de v et à la convergence de son module $\rho := |v|$ vers 1 à l'infini, [3] on construit un relèvement régulier θ de la fonction v sur un voisinage $B(0, R_0)^c$ de l'infini : il s'agit d'une fonction $\theta \in C^\infty(B(0, R_0)^c)$ qui vérifie

$$v = \rho e^{i\theta}$$

sur l'ouvert $B(0, R_0)^c$. Comme la fonction θ n'est pas définie sur l'espace $\mathbb{R}^N$ tout entier, on introduit [4] une fonction plateau $\psi \in C^\infty(\mathbb{R}^N, [0, 1])$ telle que

$$\begin{cases} \psi = 0 \text{ sur } \mathrm{B}(0, 2\mathrm{R}_0), \\ \psi = 1 \text{ sur } \mathrm{B}(0, 3\mathrm{R}_0)^{\mathrm{c}}, \end{cases}$$

puis on calcule le système d'équations suivant pour les nouvelles variables $\eta := 1 - \rho^2$ et $\nabla(\psi\theta)$:

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}), \tag{33}$$

et

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{34}$$

où les fonctions F et G sont définies par

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta), \tag{35}$$

et

$$G = i\nabla v.v + \nabla(\psi\theta). \tag{36}$$

Les équations (33) et (34) conduisent enfin aux équations de convolution :

$$\eta = K_0 * F + 2c\sum_{j=1}^{N} K_j * G_j, \tag{37}$$

où K_0 et K_j sont les noyaux de transformée de Fourier,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{38}$$

respectivement

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{39}$$

et

$$\forall j \in \{1, \ldots, N\}, \partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{40}$$

où $L_{j,k}$ et $R_{j,k}$ sont les noyaux de transformée de Fourier,

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{41}$$

[3]En dimension deux, il faut invoquer l'appartenance du gradient de v à l'espace $L^2(\mathbb{R}^N)$ pour justifier la nullité du degré de la fonction $\frac{v}{|v|}$ à l'infini.

[4]Malgré le caractère arbitraire des choix du réel R_0 et de la fonction ψ, les propriétés de v considérées par la suite ne dépendent pas de ces choix.

respectivement

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}. \tag{42}$$

On peut accomplir la même transformation pour les ondes solitaires pour les équations de Kadomtsev-Petviashvili. Dans ce cas, l'équation (25) conduit directement à l'équation de convolution

$$v = \frac{1}{p+1} H_0 * v^{p+1}, \tag{43}$$

où H_0 est le noyau de transformée de Fourier,

$$\widehat{H_0}(\xi) = \frac{\xi_1^2}{|\xi|^2 + \xi_1^4}. \tag{44}$$

Les équations (37), (40) et (43) sont toutes de la forme

$$g = K * f. \tag{45}$$

Les noyaux K sont donnés explicitement par leur transformée de Fourier, qui est une fraction rationnelle. Les fonctions f dépendent de manière surlinéaire des variables g étudiées (à savoir η et $\nabla(\psi\theta)$ pour l'équation de Gross-Pitaevskii, et v pour celles de Kadomtsev-Petviashvili) : ceci signifie qu'il existe des réels $A \geq 0$ et $p > 1$ tels que

$$|f| \leq A|g|^p. \tag{46}$$

Ainsi, l'étude qualitative des solutions des équations (6) et (23) se ramène à celle des solutions g des équations de la forme (45)-(46). En particulier, il s'agit maintenant de préciser le comportement qualitatif des noyaux K, avant d'utiliser le caractère surlinéaire des équations pour en déduire les propriétés attendues.

4.2 Étude des noyaux de convolution.

Les noyaux associés aux formules (38), (39), (41), (42) et (44) sont caractérisés par la forme de leur transformée de Fourier, qui est une fraction rationnelle

$$\widehat{K} = \frac{P}{Q}. \tag{47}$$

Leurs propriétés dépendent fortement des singularités de la fonction $\widehat{K}$, à savoir de son comportement au voisinage des points d'annulation du polynôme Q, et au voisinage de l'infini. Elles diffèrent sensiblement selon que la fonction $\widehat{K}$ (ou l'une de ses dérivées partielles) est intégrable au voisinage de ses singularités ou non.

L'objectif de cette partie n'est pas de détailler de manière exhaustive ces différences, ce qui est fort délicat. On se contentera plutôt d'illustrer sur des exemples élémentaires des méthodes générales qui fournissent les informations qualitatives sur les noyaux K_0, K_j, $L_{j,k}$, $R_{j,k}$ et H_0 qui sont nécessaires à la preuve des théorèmes 2, 3, 6, 7 et 8.

4.2.1 Équations de convolution et singularités non intégrables.

La première méthode concerne les noyaux K dont la transformée de Fourier présente des singularités locales non intégrables. Avec les notations de la formule (47), ceci signifie que $\widehat{K}$ n'appartient pas à certains espaces $L^p(V)$, où V est un voisinage d'un point de l'ensemble d'annulation de Q,

$$\mathcal{Z}(Q) := \{\xi \in \mathbb{R}^N, Q(\xi) = 0\}.$$

Cette méthode fournit des informations sur les fonctions f et g solutions de l'équation (45) associée au noyau K grâce aux singularités locales de $\widehat{K}$. Elle se résume par le lemme suivant.

Lemme 1. *Soit $\xi_0 \in \mathcal{Z}(Q)$ et V, un voisinage de ξ_0. Supposons qu'il existe un entier $p \geq 2$ telle que la fonction $\widehat{K}$ n'appartienne pas à $L^p(V)$, et que les fonctions f et g appartiennent à $L^1(\mathbb{R}^N)$, respectivement $L^{p'}(\mathbb{R}^N)$ pour $p' = \frac{p}{p-1}$. Alors, la fonction $\widehat{f}$ s'annule au point ξ_0.*

Remarque. Le lemme 1 s'applique plus généralement à tous les noyaux K de la forme (47), avec des fonctions P et Q quelconques.

Démonstration. D'après les hypothèses du lemme 1, les transformées de Fourier des fonctions f et g sont respectivement dans les espaces $C^0(\mathbb{R}^N)$ et $L^p(\mathbb{R}^N)$. L'équation (45) devient ainsi pour presque tout $\xi \in \mathbb{R}^N$,

$$\widehat{g}(\xi) = \widehat{K}(\xi)\widehat{f}(\xi).$$

Supposons alors que

$$\widehat{f}(\xi_0) \neq 0.$$

La fonction $\widehat{f}$ est continue en ξ_0 : il existe donc un voisinage V de ξ_0 tel que

$$\forall \xi \in V, |\widehat{f}(\xi)| \geq A > 0.$$

Il s'ensuit que

$$\forall \xi \in V, |\widehat{g}(\xi)| \geq A|\widehat{K}(\xi)|.$$

Néanmoins, la fonction $\widehat{g}$ appartient à $L^p(V)$, alors que la fonction $\widehat{K}$ n'est pas dans cet espace. On aboutit ainsi à une contradiction qui prouve que

$$\widehat{f}(\xi_0) = 0.$$

□

A travers l'annulation de $\widehat{f}$, le lemme 1 fournit de nouvelles relations intégrales sur la non-linéarité f, qui amènent à leur tour des informations qualitatives sur la fonction g. On peut illustrer cette affirmation par l'exemple des ondes progressives supersoniques d'énergie finie pour l'équation de Gross-Pitaevskii. Le lemme 1 est en effet valable pour ces ondes. Comme leur énergie est finie, les fonctions η et $\nabla(\psi\theta)$ appartiennent à $L^2(\mathbb{R}^N)$. D'après les formules (35) et (36), les fonctions F et G sont donc dans l'espace $L^1(\mathbb{R}^N)$. Par ailleurs, les noyaux K_0 et K_j, dont les dénominateurs sont égaux à

$$\forall \xi \in \mathbb{R}^N, Q_0(\xi) := |\xi|^4 + 2|\xi|^2 - c^2\xi_1^2,$$

n'appartiennent pas aux espaces $L^2(V)$, où V est un voisinage d'un point quelconque de l'ensemble

$$Z(Q_0) := \{\xi \in \mathbb{R}^N, |\xi|^4 + 2|\xi|^2 - c^2\xi_1^2 = 0\}.$$

Il découle donc du lemme 1 et de l'équation (37) que la fonction

$$\xi \mapsto |\xi|^2 \widehat{F}(\xi) + 2c \sum_{j=1}^{N} \xi_1 \xi_j \widehat{G_j}(\xi)$$

s'annule sur l'ensemble $Z(Q_0)$, ce qui conduit à la nouvelle relation intégrale,

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = 2c\left(1 - \frac{2}{c^2}\right)p(v). \tag{48}$$

L'équation (48), qui relie l'énergie et le moment scalaire de l'onde, est au coeur de la preuve du théorème 2. Elle est en effet incompatible avec d'autres identités intégrales plus classiques (les identités de Pohozaev), sauf si l'onde progressive est constante.

De même, le théorème 3 résulte de l'argument du lemme 1. Par cet argument, les ondes progressives soniques, d'énergie finie, vérifient aussi l'équation (48) en dimension deux, ce qui prouve qu'elles sont constantes. [5]

En définitive, c'est le caractère non intégrable des singularités locales des fonctions $\widehat{K_0}$ et $\widehat{K_j}$ qui conduit à la non-existence des ondes progressives dans les deux cas précédents.

Par ailleurs, les relations intégrales fournies par le lemme 1 ont bien d'autres applications. Par exemple, A. de Bouard et J.C. Saut [14] les ont utilisées afin d'établir le caractère optimal de leur théorème sur la décroissance asymptotique des ondes solitaires pour les équations de Kadomtsev-Petviashvili en dimension deux.

4.2.2 Singularités intégrables et comportement local et asymptotique des noyaux.

L'équation de convolution (45) relie le comportement asymptotique de la fonction g à celui du noyau K. Si l'on connaît la décroissance algébrique du noyau K, c'est-à-dire son appartenance à un espace de la forme

$$M_\alpha^\infty(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{C}, \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} := \sup\{|x|^\alpha |u(x)|, x \in \mathbb{R}^N\} < +\infty\}$$

pour un indice $\alpha > 0$, on en déduit dans des cas simples la décroissance algébrique de la fonction g.

Lemme 2. *Soit $f \in C_c^0(\mathbb{R}^N)$ et $K \in C^0(\mathbb{R}^N) \cap M_\alpha^\infty(\mathbb{R}^N)$, où α est un réel strictement positif. Alors, la fonction g, solution de l'équation de convolution* (45)*, appartient à l'espace $M_\alpha^\infty(\mathbb{R}^N)$.*

Démonstration. Considérons un nombre réel R tel que le support de f soit inclus dans la boule $B(0, R-1)$. Comme la fonction g est continue sur $\mathbb{R}^N$, l'équation (45) s'écrit

$$\forall x \in B(0,R)^c, g(x) = \int_{B(0,R-1)} K(x-y)f(y)dy,$$

[5] Au contraire, l'argument du lemme 1 ne s'applique plus aux noyaux K_0 et K_j lorsque $c = \sqrt{2}$ et $N \geq 3$: ils ne sont plus suffisamment singuliers au voisinage de l'origine. On ne sait donc pas comment établir la formule (48) dans ce cas, ce qui empêche de prouver la non-existence des ondes progressives soniques, non constantes, d'énergie finie, en dimension supérieure ou égale à trois par cet argument.

ce qui donne

$$\begin{aligned}\forall x \in B(0,R)^c, |x|^\alpha |g(x)| \leq & |x|^\alpha \int_{B(0,R-1)} |K(x-y)||f(y)|dy \\ \leq & A|x|^\alpha \int_{B(0,R-1)} \frac{|f(y)|}{|x-y|^\alpha} dy \\ \leq & A\frac{|x|^\alpha}{(|x|-R+1)^\alpha} \leq A.\end{aligned}$$

Puisque g est continue, elle appartient ainsi à l'espace $M_\alpha^\infty(\mathbb{R}^N)$. □

Cet exemple élémentaire illustre la facilité avec laquelle le comportement asymptotique d'une fonction g, donnée par l'équation de convolution (45), se déduit de celui du noyau K. Néanmoins, cet exemple requiert la connaissance de plusieurs propriétés du noyau : sa continuité et sa décroissance algébrique. Dans un cadre plus général, la description asymptotique de la fonction g nécessite l'analyse d'au moins deux aspects du noyau K :
- sa décroissance et sa convergence à l'infini (afin de déterminer le comportement asymptotique de la fonction g),
- l'absence ou la présence de singularités locales pour ce noyau (afin de donner un sens rigoureux à l'équation de convolution (45), ce qui n'est pas toujours évident).

Plusieurs méthodes permettent d'atteindre cet objectif pour les noyaux dont la transformée de Fourier présente des singularités intégrables. Cette notion désigne les fonctions u pour lesquelles il existe un recouvrement fini de $\mathbb{R}^N$ par des ouverts réguliers $\Omega_1, \dots, \Omega_n$ et des multi-indices $\alpha_1, \dots, \alpha_n$ tels que

$$\forall 1 \leq i \leq N, \partial^{\alpha_i} u \in L^1(\Omega_i).$$

Ces méthodes reposent sur des formules intégrales, classiques en analyse harmonique, que l'on va maintenant exposer.

Décroissance et convergence à l'infini des noyaux dont la transformée de Fourier présente des singularités intégrables.

On considère un noyau K dont la transformée de Fourier $\widehat{K}$ présente des singularités intégrables, et on cherche à déterminer sa décroissance algébrique à l'infini, puis, si possible, un équivalent simple de ce noyau à l'infini. On dit ici que K présente une décroissance algébrique à l'infini de taux $\alpha > 0$ si K appartient à l'espace

$$M_\alpha^\infty(B(0,R)^c) := \Big\{ u : B(0,R)^c \to \mathbb{C}, \|u\|_{M_\alpha^\infty(B(0,R)^c)} := \sup_{x \in B(0,R)^c} (|x|^\alpha |u(x)|) < +\infty \Big\}$$

pour un réel R donné. Dans ce cas, la recherche d'un équivalent de K à l'infini, à savoir d'une limite non nulle pour la fonction $x \mapsto |x|^\alpha K(x)$ lorsque $|x|$ tend vers $+\infty$, permet de conclure que la décroissance algébrique obtenue au préalable est bien optimale.

Il s'agit donc de relier la décroissance algébrique du noyau K à l'intégrabilité de sa transformée de Fourier (ou de l'une de ses dérivées). Ce lien repose d'abord sur l'inégalité L^1-L^∞ pour la transformée de Fourier : si u est une fonction de $L^1(\mathbb{R}^N)$, alors, $\widehat{u}$ est une fonction de $L^\infty(\mathbb{R}^N)$, qui vérifie

$$\|\widehat{u}\|_{L^\infty(\mathbb{R}^N)} \leq \|u\|_{L^1(\mathbb{R}^N)}.$$

Il résulte ensuite de la formule de la dérivée d'une transformée de Fourier : si u appartient à l'espace de Schwartz $\mathcal{S}(\mathbb{R}^N)$, et si α est un multi-indice, alors, la dérivée partielle d'ordre

α de u a pour transformée de Fourier,

$$\forall \xi \in \mathbb{R}^N, \widehat{\partial^\alpha u}(\xi) = i^\alpha \xi^\alpha \widehat{u}(\xi).$$

Il découle de ces deux arguments que, si la dérivée d'ordre α de $\widehat{K}$ appartient à $L^1(\mathbb{R}^N)$, la fonction $x \mapsto x^\alpha K(x)$ appartient à $L^\infty(\mathbb{R}^N)$. En définitive, montrer de la décroissance algébrique pour K revient à montrer de l'intégrabilité pour une dérivée de sa transformée de Fourier. Néanmoins, ce raisonnement comporte au moins deux difficultés. D'une part, il impose des décroissances algébriques entières du fait d'ordres de dérivation entiers. D'autre part, il ne s'applique qu'à l'espace $\mathbb{R}^N$ tout entier. Pour remédier à ces deux difficultés, deux approches semblent possibles.

La première repose sur l'introduction des espaces de Sobolev $W^{s,1}(\mathbb{R}^N)$ pour un réel $s > 0$. Ces espaces sont définis par

$$W^{s,1}(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N), \forall |\alpha| \leq s, \partial^\alpha u \in L^1(\mathbb{R}^N)\},$$

lorsque s est entier, et par

$$W^{s,1}(\mathbb{R}^N) := \left\{u \in W^{\sigma,1}(\mathbb{R}^N), \forall |\alpha| = \sigma, \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\partial^\alpha u(z) - \partial^\alpha u(y)|}{|z-y|^{N+r}} dydz < +\infty\right\},$$

lorsque s s'écrit sous la forme $s = \sigma + r$ avec $\sigma \in \mathbb{N}$ et $r \in]0,1[$. Ils donnent un sens aux dérivées d'ordre non entier d'une fonction : appartenir à l'espace $W^{s,1}(\mathbb{R}^N)$ signifie en quelque sorte avoir des dérivées dans $L^1(\mathbb{R}^N)$ jusqu'à l'ordre s. L'inégalité L^1-L^∞ s'étend ainsi à cet espace.

Lemme 3 ([25]). *Soit $0 < s < 1$ et $\widehat{u} \in W^{s,1}(\mathbb{R}^N)$. La fonction $x \mapsto |x|^s u(x)$ appartient à l'espace $C_0^0(\mathbb{R}^N)$. De plus, il existe une constante A_N ne dépendant que de la dimension N telle que*

$$\| |.|^s u\|_{L^\infty(\mathbb{R}^N)} \leq A_N \|\widehat{u}\|_{W^{s,1}(\mathbb{R}^N)}. \tag{49}$$

Remarque. Le lemme 3 est prouvé dans l'article [24]. De plus, dans cet article, l'inégalité (49) est étendue à quelques espaces plus généraux que $W^{s,1}(\mathbb{R}^N)$.

Le lemme 3 permet par exemple de montrer que les noyaux K_0, K_j et $L_{j,k}$ sont dans tous les espaces $M_\alpha^\infty(\mathbb{R}^N)$ pour $N-2 < \alpha < N$ (Cf [24] pour de plus amples détails). Néanmoins, il a deux défauts inhérents. Il ne s'applique qu'à l'espace $\mathbb{R}^N$ tout entier, ce qui empêche de tenir compte de singularités locales ou de la non-isotropie des noyaux. De plus, il ne peut donner la décroissance optimale des noyaux considérés, ce qui est plus problématique encore. De fait, le lemme 3 affirme que si $\widehat{K}$ est dans l'espace $W^{s,1}(\mathbb{R}^N)$, la fonction $x \mapsto |x|^s K(x)$ tend vers 0 à l'infini. Le taux de décroissance de K peut donc être strictement supérieur à s. C'est le cas pour les noyaux K_0, K_j et $L_{j,k}$ qui appartiennent à l'espace $M_N^\infty(\mathbb{R}^N)$ (ce que ne donne pas le lemme 3).

Une seconde technique corrige ces défauts et donne les décroissances optimales des noyaux associés aux équations de Gross-Pitaevskii, et de Kadomtsev-Petviashvili. Elle repose sur des formules intégrales bien connues des experts. On se donne un multi-indice $\alpha \in \mathbb{N}^N$ et un noyau K dont la transformée de Fourier présente des singularités intégrables, mais, demeure régulière en dehors de ces singularités. On écrit formellement la formule de Fourier inverse pour la fonction $x \mapsto x^\alpha K(x)$,

$$x^\alpha K(x) = \frac{i^{|\alpha|}}{(2\pi)^N} \int_{\mathbb{R}^N} \partial^\alpha \widehat{K}(\xi) e^{ix.\xi} d\xi,$$

puis, on découpe l'intégrale du second membre conformément au recouvrement associé à $\widehat{K}$,

$$x^\alpha K(x) = \frac{i^{|\alpha|}}{(2\pi)^N} \sum_{i=1}^{n} \int_{\Omega_i} \partial^\alpha \widehat{K}(\xi) e^{ix.\xi} d\xi.$$

Enfin, on intègre par parties les différentes intégrales du second membre pour se ramener aux dérivées d'ordre α_i, ce qui donne

$$x^\alpha K(x) = \frac{1}{(2\pi)^N} \sum_{j=1}^{n} \left(\frac{i^{|\alpha_j|}}{x^{\alpha_j - \alpha}} \int_{\Omega_j} \partial^{\alpha_j} \widehat{K}(\xi) e^{ix.\xi} d\xi + \text{ Termes de bords sur } \partial\Omega_j \right). \quad (50)$$

A ce stade, cette formule est bien sûr formelle. Néanmoins, toutes les intégrales de son membre de droite sont bien définies. Aussi est-il possible de lui donner un sens rigoureux dans la plupart des cas. Par exemple, le lemme suivant justifie la formule (50) dans des conditions qui s'appliquent pour les noyaux K_0, K_j et $L_{j,k}$ (Cf [24]).

Lemme 4 ([24, 27]). *Soit u une distribution tempérée sur $\mathbb{R}^N$ dont la transformée de Fourier appartient à $C^\infty(\mathbb{R}^N \setminus \{0\})$. Supposons qu'il existe des entiers $1 \leq j \leq N$ et $p \in \mathbb{N}^*$ tels que*
(i) $\partial_j^p \widehat{u} \in L^1(B(0,1)^c)$,
(ii) $\partial_j^{p-1} \widehat{u} \in L^1(B(0,1))$,
(iii) $|.|\partial_j^p \widehat{u} \in L^1(B(0,1))$.
La fonction $x \mapsto x_j^p u(x)$ est alors continue sur $\mathbb{R}^N$ et vérifie pour presque tout réel $\lambda > 0$,

$$\forall x \in \mathbb{R}^N, x_j^p u(x) = \frac{i^p}{(2\pi)^N} \Bigg(\int_{B(0,\lambda)^c} \partial_j^p \widehat{u}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p-1} \widehat{u}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^p \widehat{u}(\xi)(e^{ix.\xi} - 1) d\xi \Bigg). \quad (51)$$

Remarque. Le lemme 4 est démontré dans les articles [24] et [27]. Dans ces articles, sont mentionnées d'autres formules analogues à la formule (50).

Les formules (50) et (51) donnent la décroissance algébrique du noyau K : il suffit de majorer leur second membre indépendamment de x grâce à l'intégrabilité des fonctions $\partial^{\alpha_j} \widehat{K}$ pour déterminer cette décroissance. Ces formules offrent l'avantage de s'adapter aux singularités de la transformée de Fourier de K (grâce à un recouvrement adapté), et à son éventuelle non-isotropie. En effet, en modifiant la valeur du multi-indice α, on modifie les directions d'étude de la décroissance. En pratique, le principal avantage de ces formules provient de leur dépendance vis-à-vis de paramètres (les exposants α_j, les ouverts Ω_j...) que l'on peut adapter aux noyaux étudiés. Par exemple, la formule (51) introduit un paramètre λ qui module la taille des ouverts du recouvrement. Dans l'article [27], cette formule donne à la fois la décroissance algébrique du noyau H_0 et ses singularités locales, grâce à des choix appropriés de la valeur de λ.

Ces formules conduisent de plus au calcul explicite d'équivalents simples à l'infini des noyaux K. Afin d'illustrer cette affirmation, considérons l'exemple du noyau H_0, traité dans [27]. Par un lemme analogue au lemme 4, ce noyau vérifie une formule similaire à la formule (50) : si $j \in \{1, \ldots, N\}$, $\lambda > 0$ et $x \in \Omega_j = \{x \in \mathbb{R}^N, x_j \neq 0\}$, alors,

$$x_j^N H_0(x) = \frac{i^N}{(2\pi)^N} \Bigg((-ix_j)^{N-m_j} \int_{B(0,\lambda)^c} \partial_j^{m_j} \widehat{H_0}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \sum_{k=N}^{m_j-1} (-ix_j)^{N-k-1} \int_{S(0,\lambda)} \xi_j \partial_j^k \widehat{H_0}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{N-1} \widehat{H_0}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{H_0}(\xi)(e^{ix.\xi} - 1) d\xi \Bigg), \tag{52}$$

pour $m_1 = 2N - 2$ et $m_j = N$ si $j \geq 2$ (Cf [27]). On cherche alors à calculer la limite de la fonction $x \mapsto x_j^N H_0(x)$ lorsque $|x|$ tend vers $+\infty$. En fait, cette limite dépend de la direction d'étude $\frac{x}{|x|}$. On pose donc $x = R\sigma$ pour $R > 0$ et $\sigma \in \mathbb{S}^{N-1}$, et on s'intéresse à la limite lorsque R tend vers $+\infty$ de la fonction $R \mapsto R^N H_0(R\sigma)$. Pour un choix de λ égal à $\frac{1}{R}$, la formule (52) donne après le changement de variable $u = R\xi$,

$$R^N H_0(R\sigma) = \frac{i^N}{(2\pi\sigma_j)^N} \Bigg((-i\sigma_j)^{N-m_j} \int_{B(0,1)^c} R^{-m_j} \partial_j^{m_j} \widehat{H_0}\Big(\frac{u}{R}\Big) e^{i\sigma.u} du + \sum_{k=N}^{m_j-1} (-i\sigma_j)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j R^{-k} \partial_j^k \widehat{H_0}\Big(\frac{u}{R}\Big) e^{i\sigma.u} du + \int_{\mathbb{S}^{N-1}} u_j R^{1-N} \partial_j^{N-1} \widehat{H_0}\Big(\frac{u}{R}\Big) du + \int_{B(0,1)} R^{-N} \partial_j^N \widehat{H_0}\Big(\frac{u}{R}\Big)(e^{i\sigma.u} - 1) du \Bigg). \tag{53}$$

Pour déterminer la limite de cette fonction, on applique le théorème de convergence dominée aux intégrales du second membre de la formule (53), ce qui donne

$$R^N H_0(R\sigma) \underset{R \to +\infty}{\to} \frac{i^N}{(2\pi\sigma_j)^N} \Bigg(\frac{i}{\sigma_j} \int_{B(0,1)^c} \partial_j^{N+1} \widehat{R_{1,1}}(\xi) e^{i\sigma.\xi} d\xi + \frac{i}{\sigma_j} \int_{\mathbb{S}^{N-1}} \xi_j \partial_j^N \widehat{R_{1,1}}(\xi) e^{i\sigma.\xi} d\xi + \int_{\mathbb{S}^{N-1}} \xi_j \partial_j^{N-1} \widehat{R_{1,1}}(\xi) d\xi + \int_{B(0,1)} \partial_j^N \widehat{R_{1,1}}(\xi)(e^{i\sigma.\xi} - 1) d\xi \Bigg), \tag{54}$$

où $R_{1,1}$ est le noyau défini par la formule (42). En fait, les noyaux $R_{j,k}$ sont liés aux opérateurs de Riesz pour lesquels on a des formules explicites. Le noyau $R_{j,k}$ est ainsi donné par

$$R_{j,k}(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \Bigg(PV\Big(\frac{\delta_{j,k}|x|^2 - Nx_j x_k}{|x|^{N+2}} 1_{B(0,1)} \Big) + \frac{\delta_{j,k}|x|^2 - Nx_j x_k}{|x|^{N+2}} 1_{B(0,1)^c} \Bigg), \tag{55}$$

où $PV\Big(\frac{\delta_{j,k}|x|^2 - Nx_j x_k}{|x|^{N+2}} 1_{B(0,1)} \Big)$ est la valeur principale à l'origine de la fonction

$$x \mapsto \frac{\delta_{j,k}|x|^2 - Nx_j x_k}{|x|^{N+2}} 1_{B(0,1)}(x),$$

qui est définie par

$$\Big\langle PV\Big(\frac{\delta_{j,k}|x|^2 - Nx_j x_k}{|x|^{N+2}} 1_{B(0,1)} \Big), \phi \Big\rangle = \int_{B(0,1)} \frac{\delta_{j,k}|x|^2 - Nx_j x_k}{|x|^{N+2}} (\phi(x) - \phi(0)) dx,$$

pour toute fonction $\phi \in C_0^\infty(\mathbb{R}^N)$. Les formules (54) et (55) conduisent finalement à l'équivalent suivant du noyau H_0,

$$\forall \sigma \in \mathbb{S}^{N-1}, H_0(R\sigma) \underset{R\to+\infty}{\sim} \frac{\Gamma(\frac{N}{2})(1-N\sigma_1^2)}{2\pi^{\frac{N}{2}} R^N}. \tag{56}$$

En conclusion, les formules (50), (51) et (52) fournissent des équivalents explicites des noyaux à l'infini, ce qui permet de caractériser leur décroissance algébrique optimale.

Singularité locale des noyaux dont la transformée de Fourier présente des singularités intégrables.

Le noyau K contient souvent des singularités locales, notamment au voisinage de l'origine. Par la formule (55), le noyau $R_{j,k}$ est ainsi égal à une valeur principale au voisinage de l'origine. Ceci soulève une difficulté pour définir rigoureusement l'équation de convolution (45). En effet, il n'est pas possible de convoler deux fonctions K et f qui ne sont pas suffisamment intégrables.

Pour un noyau K dont la transformée de Fourier a des singularités intégrables, on peut parfois résoudre cette difficulté grâce aux égalités de la forme (50). Elles permettent en effet l'analyse des singularités locales à l'origine de K. Plus précisément, elles décrivent l'explosion algébrique au voisinage de l'origine du noyau K, c'est-à-dire s'il appartient à l'espace

$$M_\alpha^\infty(B(0,R)) := \Big\{u : B(0,R) \to \mathbb{C}, \|u\|_{M_\alpha^\infty(B(0,R))} := \sup_{x\in B(0,R)} (|x|^\alpha |u(x)|) < +\infty\Big\}$$

pour des réels $R > 0$ et $\alpha > 0$ donnés. Il suffit pour cela de borner le second membre de la formule (50) indépendamment de x lorsque x est suffisamment petit.

Lorsque l'on connaît l'explosion algébrique du noyau K, on peut donner un sens rigoureux à l'équation de convolution (45), quitte à modifier certains de ces termes. Illustrons cette affirmation sur l'exemple des dérivées du noyau H_0 (Cf [27]). On note pour tout $k \in \{1, \ldots, N\}$, H_k, le noyau dont la transformée de Fourier est égale à

$$\widehat{H_k}(\xi) = \frac{\xi_k \xi_1^2}{|\xi|^2 + \xi_1^4}. \tag{57}$$

Ce noyau est associé aux dérivées du noyau H_0 par la formule

$$H_k = -i\partial_k H_0.$$

Par un lemme analogue au lemme 4, le noyau H_k vérifie une formule similaire à la formule (51), de laquelle on déduit l'explosion algébrique à l'origine du noyau H_k :

$$\forall x \in B(0,1), (x_1^2 + |x_\perp|)^{N-\frac{1+\delta_{k,1}}{2}} |H_k(x)| \leq A. \tag{58}$$

Cette explosion à l'origine n'est pas isotrope, ce qui provient de la non-isotropie dans la direction x_1 de la formule (57). En outre, la formule (58) laisse présager que le noyau H_k n'est pas dans l'espace $L^1(B(0,1))$, ce qui implique que l'équation (45) associée à H_k n'a pas de sens par exemple pour certaines fonctions continues à support compact. Néanmoins, par la formule (58), la fonction $x \mapsto x_j H_k(x)$ appartient à $L^1(B(0,1))$, ce qui conduit au lemme suivant.

Lemme 5 ([27]). *Soit une fonction $f \in C^0(\mathbb{R}^N)$ telle que*
(i) $f \in L^\infty(\mathbb{R}^N) \cap M_{N(p+1)}^\infty(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N$,
et soit $g = H_0 * f$. *La fonction* g *est de classe* C^1 *sur* $\mathbb{R}^N$. *De plus, sa dérivée partielle* $\partial_k g$ *s'écrit*

$$\begin{aligned}\forall x \in \mathbb{R}^N, \partial_k g(x) = i \int_{B(0,1)^c} H_k(y) f(x-y) dy + i \int_{B(0,1)} H_k(y)(f(x-y) - f(x)) dy \\ + \left(\int_{\mathbb{S}^{N-1}} H_0(y) y_k dy \right) f(x).\end{aligned} \tag{59}$$

L'analyse de l'explosion algébrique à l'origine d'un noyau K donne un sens à une équation qui n'est pas valable en général. Il suffit de restreindre l'espace des fonctions f avec lesquelles on convole le noyau K, en leur imposant un peu plus de régularité. Cette approche intégrale émane de la théorie des distributions. Dans l'exemple précédent, le noyau H_k est égal en tant que distribution à

$$H_k = H_k 1_{B(0,1)^c} + PV(H_k 1_{B(0,1)}) - i \left(\int_{\mathbb{S}^{N-1}} K_0(y) y_k dy \right) \delta_0,$$

où $PV(H_k 1_{B(0,1)})$ désigne la valeur principale à l'origine de la fonction H_k,

$$\forall \phi \in C_0^\infty(B(0,1)), < PV(H_k 1_{B(0,1)}), \phi >= \int_{B(0,1)} H_k(x)(\phi(x) - \phi(0)) dx.$$

L'analyse de l'explosion algébrique à l'origine d'un noyau K permet de mieux le décrire en tant que distribution, puis de donner un sens à l'équation de convolution (45) (quitte à restreindre l'espace des fonctions f considérées).

En conclusion, la formule (50) constitue un outil remarquable pour décrire les noyaux de convolution dont la transformée de Fourier présente des singularités intégrables. Elle donne leur décroissance algébrique optimale, fournit des équivalents simples à l'infini de ces noyaux, ainsi que la description de leur explosion locale à l'origine. En fait, elle procure toutes les propriétés nécessaires au calcul de la décroissance à l'infini des solutions d'équations de convolution par l'argument du lemme 2.

4.3 Détermination itérative de la décroissance à l'infini.

Il s'agit maintenant de calculer la décroissance algébrique optimale d'une fonction g qui vérifie une équation de convolution surlinéaire de la forme (45)-(46). Ce calcul découle d'une méthode affinée dans une série d'articles de J.L. Bona et Yi A. Li [8], A. de Bouard et J.C. Saut [14] et M. Maris [40, 41]. Par un argument itératif, elle exploite le caractère surlinéaire de la non-linéarité f pour prouver que la décroissance algébrique de la fonction g est au moins égale à celle du noyau K (qui est connue grâce aux arguments du paragraphe précédent). Le lemme suivant illustre cette méthode pour un modèle simplifié.

Lemme 6. *Soit* $\alpha_K > 0$. *On considère une fonction continue* $g \in L^1(\mathbb{R}^N)$ *et un noyau continu* $K \in L^1(\mathbb{R}^N) \cap M^\infty_{\alpha_K}(\mathbb{R}^N)$, *solutions d'une équation de convolution surlinéaire de la forme* (45)-(46). *On suppose de plus qu'il existe un réel* $\alpha > 0$ *tel que*

$$g \in M^\infty_\alpha(\mathbb{R}^N). \tag{60}$$

Alors, la fonction g *appartient à* $M^\infty_{\alpha_K}(\mathbb{R}^N)$.

Démonstration. Soit $x \in \mathbb{R}^N$ et $\beta > 0$. L'équation (45) entraîne

$$\begin{aligned} |x|^\beta |g(x)| \leq & |x|^\beta \int_{\mathbb{R}^N} |K(x-y)||f(y)|dy \\ \leq & A\Big(\int_{\mathbb{R}^N} |x-y|^\beta |K(x-y)||f(y)|dy + \int_{\mathbb{R}^N} |K(x-y)||y|^\beta |f(y)|dy \Big), \end{aligned}$$

ce qui devient par l'inégalité (46),

$$|x|^\beta |g(x)| \leq A\Big(\int_{\mathbb{R}^N} |x-y|^\beta |K(x-y)||g(y)|^p dy + \int_{\mathbb{R}^N} |K(x-y)||y|^\beta |g(y)|^p dy \Big),$$

puis, par l'inégalité de Hölder,

$$|x|^\beta |g(x)| \leq A\Big(\|K\|_{M^\infty_\beta(\mathbb{R}^N)} \|g\|^p_{L^p(\mathbb{R}^N)} + \|K\|_{L^1(\mathbb{R}^N)} \|g\|^p_{M^\infty_{\frac{\beta}{p}}(\mathbb{R}^N)} \Big).$$

En outre, la fonction g est continue et appartient à $L^1(\mathbb{R}^N)$. Par l'hypothèse (60), elle appartient donc à tous les espaces $L^q(\mathbb{R}^N)$ pour $q \geq 1$. De même, le noyau K est continu sur $\mathbb{R}^N$ et appartient à l'espace $M^\infty_{\alpha_K}(\mathbb{R}^N)$, donc, il est dans tous les espaces $M^\infty_\beta(\mathbb{R}^N)$ pour $0 \leq \beta \leq \alpha_K$. Lorsque $0 \leq \beta \leq \alpha_K$, il s'ensuit que

$$\|g\|_{M^\infty_\beta(\mathbb{R}^N)} \leq A\Big(1 + \|g\|^p_{M^\infty_{\frac{\beta}{p}}(\mathbb{R}^N)}\Big).$$

Comme la fonction g appartient aux espaces $M^\infty_\beta(\mathbb{R}^N)$ pour $0 \leq \beta \leq \alpha$, on en déduit qu'elle appartient aux espaces $M^\infty_\beta(\mathbb{R}^N)$ pour $0 \leq \beta \leq \min\{p\alpha, \alpha_K\}$, puis en itérant, pour $0 \leq \beta \leq \min\{p^2\alpha, \alpha_K\}$... En définitive, comme la suite $(p^n\alpha)_{n\in\mathbb{N}}$ tend vers $+\infty$, la fonction g appartient à tous les espaces $M^\infty_\beta(\mathbb{R}^N)$ pour $0 \leq \beta \leq \alpha_K$. □

Remarque. Le lemme 6 n'utilise que l'inégalité de Hölder L^1-L^∞, mais l'on peut bien sûr utiliser l'inégalité plus générale L^p-$L^{p'}$.

Le lemme 6 expose une situation fort simplifiée, mais, il met en valeur les éléments décisifs pour obtenir la décroissance algébrique d'une fonction solution d'une équation de convolution surlinéaire :
- l'intégrabilité et la décroissance algébrique du noyau K,
- l'intégrabilité de la non-linéarité f,
- un peu de décroissance algébrique pour la fonction g.

Quitte à utiliser des formules intégrales comme celles du lemme 5 dans les cas les plus singuliers, l'analyse menée dans le paragraphe précédent fournit l'intégrabilité et la décroissance requises pour le noyau K.

L'intégrabilité de la non-linéarité f s'obtient par deux arguments distincts. D'une part, des méthodes de bootstrap elliptique donnent plus de régularité pour la fonction g. Par exemple, pour les ondes progressives pour l'équation de Gross-Pitaevskii, les solutions faibles sont en fait des solutions classiques. Cette affirmation résulte du fait que leur énergie est finie, ce qui permet d'amorcer un bootstrap elliptique. Cependant, cet argument, qui prouve que les ondes progressives sont dans les espaces $L^p(\mathbb{R}^N)$ pour p suffisamment grands, ne permet pas de savoir si elles sont dans les espaces $L^p(\mathbb{R}^N)$ pour p petit (ce qui est plus intéressant au voisinage de l'infini). Cette difficulté se résout par des arguments classiques en analyse harmonique, qui tirent profit du caractère surlinéaire de l'équation de convolution (45)-(46). Puisque la fonction g est dans l'espace $L^q(\mathbb{R}^N)$

pour q assez grand, la non-linéarité f est dans l'espace $L^{\frac{q}{p}}(\mathbb{R}^N)$. En outre, la transformée de Fourier du noyau K constitue souvent un multiplicateur de Fourier dans les espaces $L^q(\mathbb{R}^N)$. Par un théorème de P.I. Lizorkin (Cf [35]), les noyaux K_0, K_j, $L_{j,k}$ et H_0 sont ainsi des multiplicateurs de Fourier dans tous les espaces $L^q(\mathbb{R}^N)$ pour $1 < q < +\infty$. Aussi la fonction g est-elle finalement dans l'espace $L^{\frac{q}{p}}(\mathbb{R}^N)$. Ce raisonnement itéré autant que nécessaire fournit l'intégrabilité L^q de la fonction g pour q petit. Par la formule (46), on en déduit l'intégrabilité de la non-linéarité f.

Enfin, il faut déterminer un peu de décroissance algébrique pour la fonction g pour initialiser l'argument itératif du lemme 6 : cette décroissance survient souvent sous forme intégrale. En intégrant par parties l'équation aux dérivées partielles d'origine, on montre qu'une intégrale de la forme

$$\int_{\mathbb{R}^N} |x|^\alpha |f(x)|^r dx$$

est finie pour des exposants $r \geq 1$ et $\alpha > 0$ bien choisis. Cette décroissance intégrale est très faible, mais elle se substitue sans difficulté à l'assertion (60) grâce aux inégalités de Hölder L^p-$L^{p'}$. Par exemple, pour les ondes progressives ou solitaires dans les équations de Gross-Pitaevskii, de Kadomtsev-Petviashvili, de Benney-Luke et de Benjamin-Ono (Cf [14, 24, 27, 40, 41]), elle suffit à initier l'argument itératif du lemme 6 pour calculer la décroissance algébrique de ces ondes.

Après avoir déterminé l'intégrabilité et la décroissance algébrique de K, l'intégrabilité de f et un peu de décroissance pour g, l'argument du lemme 6 fournit la décroissance optimale de g. La principale conclusion que l'on peut tirer de cet argument est que, dans le cas d'équations de convolution surlinéaires, la décroissance des solutions est au moins égale à celle des noyaux. Ainsi, les décroissances des ondes progressives pour l'équation de Gross-Pitaevskii, et des ondes solitaires pour les équations de Kadomtsev-Petviashvili, qui sont données par les théorèmes 6 et 8, sont identiques à celles des noyaux K_0, K_j, $L_{j,k}$ et H_0 correspondants (Cf [24, 26, 27]). En quelque sorte, les noyaux imposent leur décroissance aux solutions.

4.4 Détermination de l'asymptote à l'infini.

Pour compléter la description asymptotique de la fonction g, on détermine un équivalent à l'infini de g. Ceci permet en particulier de prouver le caractère optimal de la décroissance algébrique obtenue au préalable. Les théorèmes 7 et 8 fournissent ainsi des équivalents simples à l'infini des ondes progressives subsoniques pour l'équation de Gross-Pitaevskii, et des ondes solitaires pour les équations de Kadomtsev-Petviashvili.

Une nouvelle fois, le calcul de cet équivalent s'appuie sur l'équation de convolution (45). En effet, après avoir déterminé au paragraphe précédent le taux m de la décroissance algébrique de g, il s'agit désormais de calculer la limite à l'infini de la fonction $x \mapsto |x|^m g(x)$. Comme pour le noyau K, cette limite dépend souvent de la direction d'étude $\frac{x}{|x|}$. On pose donc $x = R\sigma$ où $R > 0$ et $\sigma \in \mathbb{S}^{N-1}$, et on se ramène au calcul de la limite, lorsque R tend vers $+\infty$, des fonctions $g_R : \sigma \mapsto R^m g(R\sigma)$, égales à

$$\forall \sigma \in \mathbb{S}^{N-1}, g_R(\sigma) = \int_{\mathbb{R}^N} R^m K(R\sigma - y) f(y) dy. \tag{61}$$

On invoque alors le théorème de convergence dominée. En effet, l'analyse menée au paragraphe 4.2.2 fournit un équivalent à l'infini du noyau K de la forme

$$\forall (\sigma, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N, K(R\sigma - y) \underset{R \to +\infty}{\sim} \frac{K_\infty(\sigma)}{R^m},$$

où K_∞ est une fonction définie sur $\mathbb{S}^{N-1}$ (Cf par exemple la formule (56)). On en déduit la convergence de l'intégrant du terme de droite de l'équation (61),

$$\forall(\sigma, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N, R^m K(R\sigma - y) f(y) \underset{R \to +\infty}{\to} K_\infty(\sigma) f(y).$$

On domine alors cet intégrant grâce à l'intégrabilité et à la décroissance algébrique du noyau K et de la fonction g précisées dans les paragraphes 4.2.2 et 4.3. Il s'ensuit par le théorème de convergence dominée que les fonctions g_R ont une limite ponctuelle g_∞ lorsque R tend vers $+\infty$,

$$\forall \sigma \in \mathbb{S}^{N-1}, g_R(\sigma) \underset{R \to +\infty}{\to} g_\infty(\sigma) = K_\infty(\sigma) \int_{\mathbb{R}^N} f(y) dy. \tag{62}$$

Néanmoins, cette convergence ponctuelle n'est guère satisfaisante : la notion classique de limite à l'infini correspond plutôt à la convergence uniforme de la famille de fonctions $(g_R)_{R>0}$ vers la fonction g_∞. Il s'agit donc de montrer que la famille de fonctions $(g_R)_{R>0}$ converge vers la fonction g_∞ dans $L^\infty(\mathbb{S}^{N-1})$ lorsque R tend vers $+\infty$. Puisqu'on connaît la limite ponctuelle de la famille $(g_R)_{R>0}$, le théorème d'Ascoli-Arzela fournit cette uniformité. Cependant, ce théorème requiert de la compacité pour la famille $(g_R)_{R>0}$. On l'obtient en montrant que la famille $(\nabla^{\mathbb{S}^{N-1}} g_R)_{R>0}$ est uniformément bornée sur $\mathbb{S}^{N-1}$. [6] Cette affirmation provient également de l'équation (45). De fait, le gradient de la fonction g vérifie au moins formellement l'équation de convolution

$$\nabla g = \nabla K * f.$$

Quitte à utiliser des formules intégrales comme celles du lemme 6, on peut donner un sens rigoureux à cette équation. Grâce aux méthodes des paragraphes 4.2.2 et 4.3, on en déduit le taux m' de décroissance algébrique de ∇g. Grâce à l'inégalité

$$\forall (R, \sigma) \in \mathbb{R}_+^* \times \mathbb{S}^{N-1}, |\nabla^{\mathbb{S}^{N-1}} g_R(\sigma)| \leq R^{m+1} |\nabla g(R\sigma)|,$$

il s'ensuit que la famille $(\nabla^{\mathbb{S}^{N-1}} g_R)_{R>0}$ est uniformément bornée dès que

$$m' \geq m + 1.$$

Le théorème d'Ascoli-Arzela prouve alors la convergence uniforme à l'infini de la famille $(g_R)_{R>0}$.

Remarque. En général, le noyau ∇K impose le taux m', qui vaut alors le plus souvent

$$m' = m + 1.$$

Par exemple, les ondes progressives pour l'équation de Gross-Pitaevskii vérifient cette allégation (Cf le théorème 7). Il arrive cependant que la non-linéarité f impose un taux m' strictement inférieur à $m+1$. Ce cas de figure se produit pour les ondes solitaires pour les équations de Kadomtsev-Petviashvili (Cf le théorème 8) : l'argument précédent ne fournit plus l'uniformité de la convergence. Néanmoins, la convergence ponctuelle demeure valable.

Pour conclure l'analyse asymptotique de g, on peut chercher à expliciter la valeur de g_∞. La formule (62) fournit parfois cette valeur : il suffit en effet de connaître explicitement celle de la fonction K_∞ par des formules de la forme (56). Par exemple, ceci est possible pour

[6] Si u est une fonction régulière sur $\mathbb{S}^{N-1}$, la notation $\nabla^{\mathbb{S}^{N-1}} u$ désigne le gradient de u sur la sphère $\mathbb{S}^{N-1}$.

tous les noyaux associés aux équations de Gross-Pitaevskii et de Kadomtsev-Petviashvili (Cf [27]).

Néanmoins, une autre approche est envisageable. A l'origine, la fonction g vérifie une équation aux dérivées partielles. Quitte à étudier le comportement asymptotique de certaines dérivées partielles de g, on peut calculer la limite à l'infini de cette équation, ce qui donne une équation aux dérivées partielles linéaire pour la fonction g_∞ sur la sphère $\mathbb{S}^{N-1}$. Il suffit alors d'intégrer cette équation pour obtenir une valeur explicite de g_∞. Cette méthode fournit par exemple un équivalent des ondes progressives à symétrie axiale autour de l'axe x_1 pour l'équation de Gross-Pitaevskii (Cf [26]). Cependant, elle produit parfois des solutions parasites, solutions de l'équation aux dérivées partielles vérifiée par la fonction g_∞, sans donner d'équivalents pour g. C'est la raison pour laquelle il est préférable de calculer explicitement la valeur de la limite K_∞ (si possible).

5 Perspectives.

L'étude des ondes progressives pour l'équation de Gross-Pitaevskii et des ondes solitaires pour les équations de Kadomtsev-Petviashvili amène de nombreuses questions toujours sans réponses. Néanmoins, certaines de ces questions ne semblent pas totalement hors de portée sur un plan mathématique.

Pour ce qui concerne les ondes progressives pour l'équation de Gross-Pitaevskii, se pose d'abord le problème de leur existence pour toute valeur de c comprise entre 0 et $\sqrt{2}$. Cette question semble fort délicate. Néanmoins, deux problèmes semblables paraissent plus accessibles. Le premier consiste à chercher des solutions avec vortex analogues à celles de F. Béthuel et J.C. Saut [4, 5] et F. Béthuel, G. Orlandi et D. Smets [7] pour le problème avec obstacle étudié par A. Aftalion et X. Blanc [1] en dimension deux. Le second vise à obtenir des ondes progressives v, d'énergie finie, pour l'équation de Gross-Pitaevskii, qui vérifient la propriété

$$\exists j \in \{2, \ldots, N\}, P_j(v) \neq 0.$$

En effet, la description asymptotique des ondes progressives pour l'équation de Gross-Pitaevskii semble indiquer l'existence de telles solutions, ce qui n'est confirmé par aucun résultat numérique ou mathématique (Cf le théorème 7).

L'existence (au moins locale) d'une branche d'ondes progressives pour l'équation de Gross-Pitaevskii constitue un second développement intéressant. Cette existence paraît envisageable par des méthodes d'inversion locale pour des vitesses petites, ou pour des vitesses proches de $\sqrt{2}$. Dans ce cas, on peut concevoir une telle méthode à partir des ondes solitaires pour les équations de Kadomtsev-Petviashvili. Une application de ce développement serait de retrouver l'allure générale des courbes énergie-moment scalaire obtenues par C.A. Jones, S.J. Putterman et P.H. Roberts en dimensions deux et trois. En particulier, la présence d'un point de rebroussement en dimension trois reste une question ouverte.

Dans le même ordre d'idée, on peut espérer prouver rigoureusement le lien entre les ondes progressives pour l'équation de Gross-Pitaevskii et les ondes solitaires pour les équations de Kadomtsev-Petviashvili. Les articles de C.A. Jones, S.J. Putterman et P.H. Roberts [29, 30] fournissent en effet un argument formel qui décrit la limite des ondes progressives pour l'équation de Gross-Pitaevskii lorsque leur vitesse tend vers $\sqrt{2}$.

Une dernière question concerne le rôle des ondes progressives dans la dynamique de l'équation de Gross-Pitaevskii : est-il possible de vérifier la stabilité orbitale des ondes obtenues pour des vitesses petites par F. Béthuel et J.C. Saut [4, 5], et F. Béthuel, G.

Orlandi et D. Smets [7] ? Pour les ondes obtenues par minimisation sous contrainte par F. Béthuel, G. Orlandi et D. Smets [7], la difficulté semble essentiellement technique (une question de compacité de l'ensemble d'étude) et paraît donc plus accessible.

Pour les ondes solitaires pour les équations de Kadomtsev-Petviashvili, se pose la question de leur unicité (à translation près), au moins dans le cas où p est égal à 1. Une approche possible repose sur leur développement asymptotique : il s'agirait de prouver qu'une onde qui décroît plus vite que ce qui est attendu est nécessairement constante. Néanmoins, il semble qu'il faille utiliser un développement au second ordre des ondes solitaires (afin d'éliminer la difficulté posée par les translations).

Ceci constitue l'une des perspectives les plus prometteuses des techniques décrites dans cette introduction. Après avoir obtenu un développement asymptotique à tout ordre d'une onde solitaire (ou au moins à un ordre suffisant), il s'agirait de montrer qu'il existe au plus une solution associée à ce développement. Ce n'est évidemment pas possible pour des fonctions quelconques. Mais, les ondes solitaires pour les équations de Kadomtsev-Petviashvili sont analytiques lorsque p est entier, ce qui impose une rigidité très forte. Ainsi, est-il envisageable d'utiliser leur comportement asymptotique pour étudier leur unicité.

Cette approche s'applique aussi aux ondes progressives pour les équations de Gross-Pitaevskii. En particulier, elle pourrait fournir la non-existence des ondes progressives non constantes, d'énergie finie et de vitesse $\sqrt{2}$, en dimension supérieure ou égale à trois (Cf [25]). Mais, elle pourrait également conduire à l'existence d'ondes progressives subsoniques : si l'on connaît le développement asymptotique à tout ordre d'une onde, et le caractère analytique de cette onde, il suffit d'envisager un prolongement analytique pour obtenir une solution globale. En conclusion, la description asymptotique de ces ondes pourrait permettre de déterminer leur unicité ou leur existence grâce à leur analyticité.

6 Plan de la thèse.

Ce mémoire de thèse comporte six chapitres. Le premier est formé par un article paru dans la revue Communications in Mathematical Physics sous le titre "A non-existence result for supersonic travelling waves in the Gross-Pitaevskii equation". Il porte sur la preuve du théorème 2 qui énonce la non-existence des ondes progressives supersoniques, non constantes, d'énergie finie, pour l'équation de Gross-Pitaevskii.

Le second chapitre est constitué par un article accepté par la revue Differential and Integral Equations sous le titre "Limit at infinity and non-existence results for sonic travelling waves in the Gross-Pitaevskii equation". Il décrit les ondes progressives d'énergie finie et de vitesse $c = \sqrt{2}$ pour l'équation de Gross-Pitaevskii. En particulier, il donne les preuves du théorème 3 sur leur non-existence en dimension deux, et du théorème 5 sur leur comportement asymptotique en dimension supérieure ou égale à trois. De plus, la preuve du théorème 1 sur le cas de la dimension un est mentionnée en annexe de cet article.

Le troisième chapitre consiste en une note parue aux Comptes Rendus Mathématiques sous le titre "Limit at infinity for travelling waves in the Gross-Pitaevskii equation". Cette note fournit la preuve du théorème 4 qui affirme l'existence d'une limite à l'infini pour les ondes progressives subsoniques pour l'équation de Gross-Pitaevskii.

Le quatrième chapitre présente un article accepté par la revue Annales de l'Institut Henri Poincaré (Analyse Non Linéaire) sous le titre "Decay for travelling waves in the Gross-

Pitaevskii equation". Il décrit la décroissance des ondes progressives subsoniques pour l'équation de Gross-Pitaevskii formulée par le théorème 6.

Quant au cinquième chapitre, il se compose d'un article soumis sous le titre "Asymptotics for the travelling waves in the Gross-Pitaevskii equation". Cet article porte sur l'étude du comportement asymptotique des ondes progressives subsoniques pour l'équation de Gross-Pitaevskii. Il contient la preuve du théorème 7 en dimension deux, et dans le cas des ondes à symétrie axiale autour de l'axe x_1.

Enfin, le dernier chapitre présente un projet d'article intitulé "Asymptotics for solitary waves in the generalised Kadomtsev-Petviashvili equations", qui apporte la preuve du théorème 8 sur le comportement asymptotique des ondes solitaires pour les équations de Kadomtsev-Petviashvili. De plus, la preuve du théorème 7 amorcée dans le cinquième chapitre est achevée en annexe de ce chapitre.

Chapitre I

A non-existence result for supersonic travelling waves in the Gross-Pitaevskii equation.

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Abstract.
We prove the non-existence of non-constant travelling waves of finite energy and of speed $c > \sqrt{2}$ in the Gross-Pitaevskii equation in dimension $N \geq 2$.

Introduction.

In this paper, we will focus on the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2). \tag{1}$$

One of the motivations for this equation is the analysis of Bose-Einstein condensation, which describes the behaviour of interacting bosons near absolute zero. When condensation occurs, equation (1) can be used as a model for the Bose condensate (see [12] for more details). In particular, this model is relevant to describe Bose-condensed gases. The model is also sometimes proposed to describe the superfluid state of *Helium II*, though in this case the interactions between particles are important and cannot be neglected at temperature different from zero.

In order to describe this condensation, E.P. Gross [28] and L.P. Pitaevskii [45] considered a set of N bosons of mass m that fill a volume V: they then assumed almost all bosons are Bose-condensed in the fundamental state of energy. Therefore, they can be described by a macroscopic wave function Ψ. They then deduced the Gross-Pitaevskii equation satisfied by the function Ψ from a Hartree-Fock approach:

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\Delta\Psi - \Psi\int_V |\Psi(x',t)|^2 U(x-x')dx' = 0.$$

Here, $U(x-x')$ denotes the interaction between the bosons at positions x and x': this interaction being of very short range, it is often approached by $U_0\,\delta(x-x')$. Thus, denoting E_b, the average energy level per unit mass of a boson, and,

$$u(t,x) = e^{\frac{-imE_b t}{\hbar}}\Psi(t,x),$$

they computed the equation

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u + mE_b u - U_0 u|u|^2 = 0.$$

They finally rescaled the equation by taking the mean density $\rho_0 = \sqrt{\frac{mE_b}{U_0}}$ as unity, $\frac{\hbar}{\sqrt{2m^2E_b}}$ as unit length, and $\frac{\hbar}{mE_b}$ as unit time, in order to obtain the dimensionless equation

$$i\partial_t u + \Delta u + u(1-|u|^2) = 0.$$

At this point, we can write the hydrodynamic form of this equation by using the Madelung transform [37],

$$u = \sqrt{\rho}e^{i\theta},$$

which is only meaningful where ρ does not vanish. Denoting $v = 2\nabla\theta$, we deduce the equations

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v.\nabla v) + \nabla\rho^2 = \rho\nabla(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}). \end{cases}$$

Those equations are similar to Euler's equations for an irrotational ideal fluid with pressure $p(\rho) = \rho^2$: the term of the right member is then considered as a quantic pressure term. Here, we can remark that the sound speed is $c_s = \sqrt{2}$.

In this article, we will consider equation (1) in the space $\mathbb{R}^N$ for every integer $N \geq 2$: we can notice that this equation is associated to the energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(1-|u|^2)^2 = \int_{\mathbb{R}^N} e(u).$$

We will study the travelling waves of finite energy and of speed $c \geq 0$ for this equation, i.e. the solutions u which are of the form

$$u(t,x) = v(x_1 - ct, \ldots, x_N).$$

The simplified equation for v, which we will consider now, is

$$ic\partial_1 v + \Delta v + v(1-|v|^2) = 0. \tag{2}$$

C.A. Jones, S.J. Putterman and P.H. Roberts [29] [30] first considered formally and numerically those particular solutions because they suppose they play an important role in the long time dynamics of general solutions. They conjectured that non-constant travelling waves only exist when their speed c is in the interval $]0,\sqrt{2}[$, i.e. they all are subsonic. They then noticed the apparition of vortices for those solutions when c tends to 0 in dimension two (two parallel oppositely directed vortices) and in dimension three (a vortex ring). They also gave for each value of c their asymptotic development at infinity in dimension two

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1-\frac{c^2}{2})x_2^2},$$

and in dimension three

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}},$$

where the constant α is the stretched dipole coefficient.

F. Béthuel and J.C. Saut [4, 5] first studied mathematically those travelling waves. They proved their existence in dimension two when c is small and the apparition of vortices in this case. They also gave a mathematical proof for their limit at infinity.
In dimension $N \geq 3$, F. Béthuel, G. Orlandi and D. Smets [7] showed their existence when c is small and the apparition of a vortex ring.
In every dimension, A. Farina [18] proved a universal bound for their modulus.
Finally, we proved their uniform convergence to a constant of modulus one in dimension $N \geq 3$ [22], and also studied their decay at infinity in dimension $N \geq 2$ [24].

In this paper, we will complete all those results by the following theorem.

Theorem 1. *In dimension $N \geq 2$, a solution of equation (2) of finite energy and speed $c > \sqrt{2}$ is constant.*

This paper will be organised around the proof of Theorem 1. In the first step, we will write the equation satisfied by

$$\eta := 1 - |v|^2.$$

Then, we will derive a new integral identity when $c > \sqrt{2}$: this is the crucial step of the proof of Theorem 1. Finally, we will write the Pohozaev identities in order to prove that the energy $E(v)$ vanishes and that the travelling wave v is constant.

1 Equation satisfied by η.

In this part, we will write the equation satisfied by the variable η for every $c \geq 0$. In particular, the results in this section (i.e. Propositions 1, 2 and 3) are valid for every $c \geq 0$. We first recall two useful propositions yet mentioned in [22, 24] and based on arguments taken from F. Béthuel and J.C. Saut [4, 5].

Proposition 1. *For every $c \geq 0$, if v is a solution of equation (2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy, then, v is smooth, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty]$.*

Thus, a travelling wave is a smooth function and a classical solution of equation (2), which will simplify the following discussion.

Proposition 2. *The modulus ρ of v satisfies*

$$\rho(x) \underset{|x|\to+\infty}{\to} 1.$$

Proof. Indeed, the function η^2 is uniformly continuous because v is bounded and lipschitzian by Proposition 1. As $\int_{\mathbb{R}^N} \eta^2$ is finite, η converges uniformly to 0 at infinity, which completes the proof of this proposition. □

Thus, the function ρ does not vanish at infinity, and we can define a smooth function θ on a neighbourhood of infinity such that v can be written

$$v = \rho e^{i\theta}.$$

Denoting ψ, a smooth function from $\mathbb{R}^N$ to $[0,1]$ such that $\psi = 0$ on a neighbourhood of $Z = \{x \in \mathbb{R}^N, \rho(x) = 0\}$, and $\psi = 1$ on a neighbourhood of infinity, and denoting $v = v_1 + iv_2$, we can write the equation satisfied by the function η.

Proposition 3. *For every $c \geq 0$, the function η satisfies the equation*

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F + 2c\partial_1 \mathrm{div}(\mathrm{G}), \tag{3}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta)$$

and

$$G = v_1\nabla v_2 - v_2\nabla v_1 - \nabla(\psi\theta).$$

Proof. By equation (2), we have

$$\Delta v_1 - c\partial_1 v_2 + v_1(1 - |v|^2) = 0, \tag{4}$$

$$\Delta v_2 + c\partial_1 v_1 + v_2(1 - |v|^2) = 0. \tag{5}$$

We then compute

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -2\Delta|\nabla v|^2 - 2\Delta(v.\Delta v) - 2\Delta\eta + c^2\partial_{1,1}^2\eta,$$

and by equations (4) and (5), we have on one hand,

$$v.\Delta v = v_1\Delta v_1 + v_2\Delta v_2 = c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - |v|^2\eta,$$

and, on the other hand,

$$c\partial_1\eta = -2c(v_1\partial_1 v_1 + v_2\partial_1 v_2) = 2(\Delta v_2 v_1 - \Delta v_1 v_2) = 2\mathrm{div}(\nabla v_2 v_1 - \nabla v_1 v_2).$$

Therefore, we finally get

$$\begin{aligned}\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta &= -2\Delta|\nabla v|^2 - 2\Delta\eta^2 - 2c\Delta(v_1\partial_1 v_2 - v_2\partial_1 v_1)\\ &+ 2c\partial_1\mathrm{div}(v_1\nabla v_2 - v_2\nabla v_1)\\ &= -\Delta(2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta))\\ &+ 2c\partial_1\mathrm{div}(v_1\nabla v_2 - v_2\nabla v_1 - \nabla(\psi\theta))\\ &= -\Delta F + 2c\partial_1\mathrm{div}(G),\end{aligned}$$

which is the desired equality. □

2 A new integral relation.

We have

Proposition 4. *If $c > \sqrt{2}$, the travelling wave v satisfies the integral equation*

$$\int_{\mathbb{R}^N}(|\nabla v|^2 + \eta^2) = c\left(\frac{2}{c^2} - 1\right)\int_{\mathbb{R}^N}(v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\psi\theta)). \tag{6}$$

Remark. This is the only point where we use the assumption $c > \sqrt{2}$.

For the proof, we use

Lemma 1. *F and G belong to the space $W^{2,1}(\mathbb{R}^N)$.*

Proof. Indeed, G is smooth and satisfies at infinity

$$G = (\rho^2 - 1)\nabla\theta.$$

By Proposition 1, the functions η and ∇v belong to $H^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$. Since

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2,$$

and since ρ uniformly converges to 1 at infinity by Proposition 2, the function $\nabla\theta$ belongs to $H^2 \cap W^{2,\infty}$ on a neighbourhood of infinity. Thus, the function G belongs to the space $W^{2,1}(\mathbb{R}^N)$. Since

$$F = 2(|\nabla v|^2 + \eta^2) + 2cG_1,$$

the function F also belongs to this space, which completes the proof of Lemma 1. □

Proof of Proposition 4. By Proposition 1, the function η belongs to $H^4(\mathbb{R}^N)$, so, by taking the Fourier transformation of equation (3), we can write for almost every $\xi \in \mathbb{R}^N$,

$$(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)\widehat{\eta}(\xi) = |\xi|^2\widehat{F}(\xi) - 2c\sum_{j=1}^{N} \xi_1\xi_j\widehat{G_j}(\xi) := H(\xi). \tag{7}$$

Consider the set

$$\Gamma = \{\xi \in \mathbb{R}^N, |\xi|^4 + 2|\xi|^2 - c^2\xi_1^2 = 0\}.$$

This set is reduced to $\{0\}$ when $c \leq \sqrt{2}$, but, when $c > \sqrt{2}$, it is a smooth hypersurface of codimension 1 except at $\{0\}$: in dimension 2, it has the geometry of a bretzel, and in higher dimensions, it has the geometry of two spheres linked at some point. Indeed, Γ is a surface of revolution around axis x_1: in spherical coordinates $\xi = (r\cos(\alpha), r\sin(\alpha)\cos(\beta), \ldots)$, it is described by the equation

$$r^2 = c^2\cos^2(\alpha) - 2.$$

In particular, we notice that there are two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of points of $\Gamma \setminus \{0\}$ which tend to 0 when n tends to $+\infty$ and which satisfy

$$\frac{x_n}{|x_n|} \underset{n\to+\infty}{\to} \left(\frac{\sqrt{2}}{c}, \sqrt{1 - \frac{2}{c^2}}, 0, \ldots\right), \text{ and, } \frac{y_n}{|y_n|} \underset{n\to+\infty}{\to} \left(\frac{\sqrt{2}}{c}, -\sqrt{1 - \frac{2}{c^2}}, 0, \ldots\right). \tag{8}$$

Coming back to the study of equation (7), we claim that

Lemma 2. *The function H defined by equation* (7) *is continuous on $\mathbb{R}^N$ and satisfies*

$$H = 0 \text{ on } \Gamma.$$

Proof of Lemma 2. The first assertion follows from Lemma 1. Indeed, since the functions F and G belong to the space $W^{2,1}(\mathbb{R}^N)$, the functions $\xi \mapsto |\xi|^2\widehat{F}(\xi)$ and $\xi \mapsto \xi_1\xi_j\widehat{G_j}(\xi)$ are continuous on $\mathbb{R}^N$, and therefore, the function H is continuous on $\mathbb{R}^N$ too.
In order to prove the second assertion, we argue by contradiction and assume there is some point $\xi_0 \in \Gamma \setminus \{0, (\sqrt{c^2 - 2}, 0, \ldots, 0)\}$ such that

$$H(\xi_0) \neq 0.$$

Since the function H is continuous on $\mathbb{R}^N$, there is some neighbourhood V of the point ξ_0 and some positive real number A such that

$$\forall \xi \in V, |H(\xi)| \geq A.$$

Hence, by equation (7), we have for almost every $\xi \in V \setminus \Gamma$,

$$|\widehat{\eta}(\xi)|^2 \geq \frac{A^2}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2}.$$

Integrating this relation and using spherical coordinates, we get

$$\begin{aligned}
\int_{V\setminus\Gamma} |\widehat{\eta}(\xi)|^2 d\xi &\geq A^2 \int_{V\setminus\Gamma} \frac{d\xi}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2} \\
&\geq A_N \int_{V\setminus\Gamma\cap\mathbb{R}\times\mathbb{R}_+} \frac{s^{N-2} ds d\xi_1}{((s^2+\xi_1^2)^2 + 2s^2 + (2-c^2)\xi_1^2)^2} \\
&\geq A_N \int_{V\setminus\Gamma\cap\mathbb{R}_+\times[0,\pi]} \frac{r^{N-1}\sin^{N-2}(\alpha) dr d\alpha}{r^4(r^2+2-c^2\cos^2(\alpha))^2.}
\end{aligned}$$

Thus, denoting

$$\xi_0 = (r_0 \cos(\alpha_0), r_0 \sin(\alpha_0)\cos(\beta_0), \ldots),$$

there is some real number $\varepsilon > 0$ such that

$$\int_{V\setminus\Gamma} |\widehat{\eta}(\xi)|^2 d\xi \geq A_N \int_{r_0-\varepsilon}^{r_0+\varepsilon} \int_{\alpha_0-\varepsilon}^{\alpha_0+\varepsilon} \frac{r^{N-1}\sin^{N-2}(\alpha) dr d\alpha}{r^4(r^2+2-c^2\cos^2(\alpha))^2} := A_N I(\alpha_0, r_0, \varepsilon).$$

Since $\xi_0 \in \Gamma \setminus \{0, (\sqrt{c^2-2}, 0, \ldots, 0)\}$, r_0 is different from 0 and α_0 is different from 0 and $\frac{\pi}{2}$, so, we can compute for ε sufficiently small,

$$I(\alpha_0, r_0, \varepsilon) \geq A(\alpha_0, r_0, \varepsilon) \int_{r_0-\varepsilon}^{r_0+\varepsilon} \int_{\alpha_0-\varepsilon}^{\alpha_0+\varepsilon} \frac{dr d\alpha}{(r^2+2-c^2\cos^2(\alpha))^2}.$$

By doing the change of variable $r = \sqrt{c^2\cos^2(\beta) - 2}$, we know that there is some real number $\delta > 0$ such that

$$I(\alpha_0, r_0, \varepsilon) \geq A(\alpha_0, r_0, \varepsilon) \int_{\alpha_0-\delta}^{\alpha_0+\delta} \int_{\alpha_0-\delta}^{\alpha_0+\delta} \frac{d\beta d\alpha}{(c^2\cos^2(\beta) - c^2\cos^2(\alpha))^2},$$

and finally, by denoting $a = \alpha - \alpha_0$ and $b = \beta - \alpha_0$, we get

$$\begin{aligned}
I(\alpha_0, r_0, \varepsilon) &\geq A(\alpha_0, r_0, \varepsilon, c) \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{da db}{(\cos^2(b+\alpha_0) - \cos^2(a+\alpha_0))^2} \\
&\geq A(\alpha_0, r_0, \varepsilon, c) \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{da db}{(\cos(2b+2\alpha_0) - \cos(2a+2\alpha_0))^2} \\
&\geq A(\alpha_0, r_0, \varepsilon, c) \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{da db}{(\sin(b-a))^2}.
\end{aligned}$$

Since the function $(a,b) \mapsto \frac{1}{(\sin(b-a))^2}$ is not integrable at the origin in $\mathbb{R}^2$, the integral $I(\alpha_0, r_0, \varepsilon)$ is not finite and we can conclude that

$$\int_{V\setminus\Gamma} |\widehat{\eta}(\xi)|^2 d\xi = +\infty.$$

Since the energy of the function v is finite, so is the integral $\int_{\mathbb{R}^N} \eta^2$, and by Plancherel theorem, we deduce

$$\int_{\mathbb{R}^N} |\widehat{\eta}(\xi)|^2 d\xi < +\infty,$$

which leads to a contradiction and proves that H is identically equal to 0 on the set $\Gamma \setminus \{0, (\sqrt{c^2-2}, 0, \dots, 0)\}$. The second assertion of Lemma 2 then follows from the continuity of the function H. □

End of the proof of Proposition 4. By Lemma 2, we now know that

$$\forall n \in \mathbb{N}, H(x_n) = 0,$$

which gives by dividing by $|x_n|^2$,

$$\forall n \in \mathbb{N}, \widehat{F}(x_n) = 2c \sum_{j=1}^{N} \frac{(x_n)_1}{|x_n|} \frac{(x_n)_j}{|x_n|} \widehat{G_j}(x_n).$$

By continuity of the functions $\widehat{F}$ and $\widehat{G_j}$, we can take the limit as $x_n \to 0$ of this expression and obtain by assertion (8)

$$\widehat{F}(0) = \frac{4}{c}\widehat{G_1}(0) + 2\sqrt{2}\sqrt{1-\frac{2}{c^2}}\widehat{G_2}(0).$$

Likewise, we know that

$$\forall n \in \mathbb{N}, H(y_n) = 0,$$

which gives by the same method,

$$\widehat{F}(0) = \frac{4}{c}\widehat{G_1}(0) - 2\sqrt{2}\sqrt{1-\frac{2}{c^2}}\widehat{G_2}(0).$$

Finally, we have

$$\widehat{F}(0) = \frac{4}{c}\widehat{G_1}(0),$$

so that,

$$\int_{\mathbb{R}^N} F(x)dx = \frac{4}{c}\int_{\mathbb{R}^N} G_1(x)dx.$$

The conclusion then follows from the expression of the functions F and G. □

3 Pohozaev identities.

We now prove for sake of completeness two well-known identities based on the use of Pohozaev multipliers (see [4] [7] [29] [30] for more details). Those estimates do not use the fact that $c > \sqrt{2}$.

Proposition 5. *Let $c \geq 0$. A finite energy solution v to equation (2) satisfies the two identities*

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2, \tag{9}$$

$$\forall 2 \leq j \leq N, E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + \frac{c}{2}\int_{\mathbb{R}^N} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)). \tag{10}$$

Proof. We first fix some real number $R > 0$ and we multiply equation (2) by Pohozaev multiplier $x_1\partial_1 v$ on the ball $B(0,R)$ to obtain

$$\int_{B(0,R)} (\Delta v.x_1\partial_1 v + x_1\partial_1 v.v(1-|v|^2)) = 0. \tag{11}$$

Integrating by parts, we compute

$$\int_{B(0,R)} \Delta v.x_1\partial_1 v = \int_{B(0,R)} \frac{|\nabla v|^2}{2} - \int_{B(0,R)} |\partial_1 v|^2 + \int_{S(0,R)} x_1\partial_1 v.\partial_\nu v - \int_{S(0,R)} \nu_1 x_1 \frac{|\nabla v|^2}{2},$$

and

$$\int_{B(0,R)} x_1\partial_1 v.v(1-|v|^2) = \int_{B(0,R)} \frac{(1-|v|^2)^2}{4} - \int_{S(0,R)} x_1\nu_1 \frac{(1-|v|^2)^2}{4}.$$

By equation (11), we then get

$$\int_{B(0,R)} e(v) = \int_{B(0,R)} |\partial_1 v|^2 - \int_{S(0,R)} x_1\partial_1 v.\partial_\nu v + \int_{S(0,R)} \nu_1 x_1 e(v). \tag{12}$$

On one hand, by Proposition 1, we know that

$$\int_{B(0,R)} e(v) - \int_{B(0,R)} |\partial_1 v|^2 \underset{R\to+\infty}{\to} E(v) - \int_{\mathbb{R}^N} |\partial_1 v|^2.$$

On the other hand, we have

$$\Big| \int_{S(0,R)} \Big(x_1\partial_1 v.\partial_\nu v - \nu_1 x_1 e(v)\Big)\Big| \leq AR \int_{S(0,R)} e(v).$$

Since the integral $\int_{\mathbb{R}_+}(\int_{S(0,R)} e(v))dR$ is finite, there are some positive real numbers R_n such that $R_n \underset{n\to+\infty}{\to} +\infty$ and

$$\forall n \in \mathbb{N}, R_n \int_{S(0,R_n)} e(v) \leq \frac{1}{\ln(R_n)},$$

which gives

$$\int_{S(0,R_n)} (x_1\partial_1 v.\partial_\nu v - \nu_1 x_1 e(v)) \underset{n\to+\infty}{\to} 0,$$

and finally, by equation (12),

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2.$$

In order to prove the second identity, we multiply equation (2) by Pohozaev multiplier $x_j\partial_j v$ on the ball $B(0,R)$ to find

$$\int_{B(0,R)} (\Delta v.x_j\partial_j v + ic\partial_1 v.x_j\partial_j v + x_j\partial_j v.v(1-|v|^2)) = 0. \tag{13}$$

Integrating by parts, we compute

$$\int_{B(0,R)} \Delta v.x_j\partial_j v = \int_{B(0,R)} \frac{|\nabla v|^2}{2} - \int_{B(0,R)} |\partial_j v|^2 + \int_{S(0,R)} x_j\partial_j v.\partial_\nu v - \int_{S(0,R)} \nu_j x_j \frac{|\nabla v|^2}{2},$$

and

$$\int_{B(0,R)} x_j\partial_j v.v(1-|v|^2) = \int_{B(0,R)} \frac{(1-|v|^2)^2}{4} - \int_{S(0,R)} x_j\nu_j\frac{(1-|v|^2)^2}{4}.$$

If R is sufficiently large such that $\psi = 1$ on $S(0,R)$, we also compute

$$\int_{B(0,R)} i\partial_1 v.x_j\partial_j v = \frac{1}{2}\Big(\int_{S(0,R)} x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta) - \int_{B(0,R)}(v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta))\Big),$$

which leads to

$$\int_{B(0,R)} e(v) = \int_{B(0,R)} |\partial_j v|^2 + \frac{c}{2}\int_{B(0,R)}(v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)) + \int_{S(0,R)}\Big(x_j\nu_j e(v) - x_j\partial_j v.\partial_\nu v - x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta)\Big). \tag{14}$$

On one hand, by Proposition 1, we know that

$$\int_{B(0,R)} e(v) - \int_{B(0,R)} |\partial_j v|^2 - \frac{c}{2}\int_{B(0,R)}(v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)) \underset{R\to+\infty}{\to} E(v) - \int_{\mathbb{R}^N}|\partial_j v|^2 - \frac{c}{2}\int_{\mathbb{R}^N}(v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)).$$

On the other hand, we have

$$\Big|\int_{S(0,R)}(x_j\nu_j e(v) - x_j\partial_j v.\partial_\nu v - x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta))\Big| \leq AR\int_{S(0,R)} e(v).$$

By using the sequence of positive real numbers R_n constructed to prove equality (9), we get

$$\int_{S(0,R_n)}(x_j\nu_j e(v) - x_j\partial_j v.\partial_\nu v - x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta)) \underset{n\to+\infty}{\to} 0,$$

and finally, by equation (14),

$$E(v) = \int_{\mathbb{R}^N}|\partial_j v|^2 + \frac{c}{2}\int_{\mathbb{R}^N}(v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)).$$

□

4 Conclusion.

We now complete the proof of Theorem 1. By Proposition 5, we have

$$E(v) = \int_{\mathbb{R}^N}|\partial_1 v|^2,$$

which gives by denoting $\nabla_\perp v = (\partial_2 v, \ldots, \partial_N v)$,

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla_\perp v|^2 + \frac{1}{4}\int_{\mathbb{R}^N}\eta^2 = \frac{1}{2}\int_{\mathbb{R}^N}|\partial_1 v|^2 = \frac{E(v)}{2},$$

and

$$\int_{\mathbb{R}^N} \eta^2 = 2E(v) - 2\int_{\mathbb{R}^N} |\nabla_\perp v|^2. \tag{15}$$

We then compute

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = 3E(v) - \int_{\mathbb{R}^N} |\nabla_\perp v|^2, \tag{16}$$

and, by Proposition 5,

$$c\int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\psi\theta)) = \frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_\perp v|^2 - 2E(v). \tag{17}$$

Proposition 4 gives

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = c\left(\frac{2}{c^2} - 1\right)\int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\psi\theta)),$$

which leads by equations (16) and (17) to

$$(c^2+4)(N-1)E(v) = ((N-3)c^2+4)\int_{\mathbb{R}^N} |\nabla_\perp v|^2. \tag{18}$$

If $N = 2$, we get

$$(c^2+4)E(v) = (4-c^2)\int_{\mathbb{R}^N} |\nabla_\perp v|^2,$$

which gives by equation (15),

$$\frac{c^2+4}{2}\int_{\mathbb{R}^N} \eta^2 = -2c^2\int_{\mathbb{R}^N} |\nabla_\perp v|^2 = 0.$$

Finally, we have involving equation (15) once more

$$E(v) = 0.$$

If $N \geq 3$, since by equation (15),

$$\int_{\mathbb{R}^N} |\nabla_\perp v|^2 \leq E(v),$$

equation (18) gives

$$(2c^2 + 4N - 8)E(v) \leq 0,$$

and finally, $E(v)$ is also equal to 0 in this case.

In conclusion, since $E(v) = 0$, the function ∇v vanishes on $\mathbb{R}^N$ and v is a constant (of modulus one since η also vanishes on $\mathbb{R}^N$).

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Chapitre II

Limit at infinity and non-existence results for sonic travelling waves in the Gross-Pitaevskii equation.

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Abstract.
We study the limit at infinity of sonic travelling waves of finite energy in the Gross-Pitaevskii equation in dimension $N \geq 2$ and prove the non-existence of non-constant sonic travelling waves of finite energy in dimension two.

Introduction.

In this article, we focus on the travelling waves of speed $c > 0$ in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2), \tag{1}$$

which are of the form

$$u(t, x) = v(x_1 - ct, \dots, x_N).$$

The equation for v, which we will study now, is

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{2}$$

The Gross-Pitaevskii equation is a physical model for Bose-Einstein condensation. It is formally associated to the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |v|^2)^2 = \int_{\mathbb{R}^N} e(v), \tag{3}$$

and to the momentum

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla v.v. \tag{4}$$

Equation (1) presents an hydrodynamic form. Indeed, if we make use of Madelung transform [37]

$$v = \sqrt{\rho} e^{i\theta}$$

(which is only meaningful where ρ does not vanish), and if we denote

$$\mathrm{v} = 2\nabla\theta,$$

we compute

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathrm{v}) = 0, \\ \rho(\partial_t \mathrm{v} + \mathrm{v}.\nabla \mathrm{v}) + \nabla \rho^2 = \rho \nabla(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}). \end{cases} \tag{5}$$

Equations (5) look like Euler equations for an irrotational ideal fluid with pressure $p(\rho) = \rho^2$ (see [6, 7] for more details). In particular, the sound speed of this fluid near the constant solution $u = 1$ is

$$c_s = \sqrt{2}.$$

The non-constant travelling waves of finite energy play a great role in the long time dynamics of general solutions. This motivates their study by C.A. Jones, S.J. Putterman and P.H. Roberts [29, 30]. In particular, they conjectured that they can only exist when

$$0 < c < \sqrt{2},$$

i.e. they are subsonic. F. Béthuel and J.C. Saut [4] first studied mathematically this conjecture. In dimension $N \geq 2$, they proved that all the travelling waves of finite energy and of speed $c = 0$ are constant. On the other hand, we proved in [23] the non-existence of non-constant travelling waves of finite energy and of speed $c > \sqrt{2}$ in dimension $N \geq 2$. Thus, the non-existence conjecture of C.A. Jones, S.J. Putterman and P.H. Roberts remains an open problem only in the case $c = \sqrt{2}$. That is the reason why we focus here on the sonic travelling waves of finite energy, i.e. we assume

$$c = \sqrt{2}.$$

In particular, we will prove their conjecture in dimension two.

Theorem 1. *In dimension two, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $c = \sqrt{2}$ is constant.*

Remarks. 1. Theorem 1 holds also in dimension one, but its proof is fairly elementary. Indeed, equation (2) is entirely integrable in dimension one. If $c \geq \sqrt{2}$, the solutions v of equation (2) are constant functions of modulus one. Instead, if $0 < c < \sqrt{2}$, up to a multiplication by a constant of modulus one and a translation, the solutions v of equation (2) are equal either to the constant function 1 or to the function

$$v(x) = \sqrt{1 - \frac{2 - c^2}{2\mathrm{ch}^2\Big(\frac{\sqrt{2-c^2}}{2} x\Big)}} \exp\Bigg(i \arctan\Bigg(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2 - c^2}}\Bigg) - i \arctan\Bigg(\frac{c}{\sqrt{2 - c^2}}\Bigg)\Bigg).$$

We refer to the appendix for more details (see also the article of M. Maris [39]).

2. In dimension two, F. Béthuel and J.C. Saut [4, 5] showed the existence of travelling waves of finite energy when c is small and for a sequence of values of c tending to $\sqrt{2}$.

In dimension $N \geq 3$, Theorem 1 is still open. We believe that a positive answer to the non-existence of non-constant sonic travelling waves of finite energy would be an important step towards another fundamental conjecture: the non-existence of non-constant travelling waves of small energy [1]. Indeed, if the speed $c = \sqrt{2}$ is excluded, we may use the rescaling given by the parameter $\varepsilon = \sqrt{2 - c^2}$ to prove that the travelling waves for the Gross-Pitaevskii equation converge towards the solitary waves for the Kadomtsev-Petviashvili equation when ε tends to 0 (see the articles of A. de Bouard and J.C. Saut [13, 14] for more details on the solitary waves for the Kadomtsev-Petviashvili equation). In particular, in dimension $N \geq 3$, the energy of a non-constant travelling wave for the Gross-Pitaevskii equation would tend to $+\infty$ when ε tends to 0, which would presumably prevent the existence of non-constant travelling waves of small energy.

In order to prove the non-existence of non-constant sonic travelling waves of finite energy in dimension $N \geq 3$, one fruitful argument seems to study their behaviour at infinity (see the conclusion for more details). In particular, we can already state their convergence at infinity towards a constant of modulus one.

Theorem 2. *Let $N \geq 3$ and v a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $c = \sqrt{2}$. There exists a constant λ_∞ of modulus one such that*

$$v(x) \underset{|x| \to +\infty}{\to} \lambda_\infty.$$

Remarks. 1. In dimension two, F. Béthuel and J.C. Saut [4] gave a mathematical evidence of the limit at infinity of subsonic travelling waves of finite energy. We complemented their work in dimension $N \geq 3$ [22].

2. C.A. Jones, S.J. Putterman and P.H. Roberts [29, 30] derived a formal asymptotic expansion of subsonic travelling waves which are axisymmetric around axis x_1. In dimension two, they computed

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2}, \tag{6}$$

while in dimension three, they obtained

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}}. \tag{7}$$

Here, the constant α is the stretched dipole coefficient linked to the energy $E(v)$ and to the scalar momentum $p(v) = P_1(v)$ by the formulae

$$2\pi\alpha\sqrt{1 - \frac{c^2}{2}} = cE(v) + 2\left(1 - \frac{c^2}{4}\right)p(v) \tag{8}$$

in dimension two and

$$4\pi\alpha = \frac{c}{2}E(v) + 2p(v) \tag{9}$$

in dimension three. In [24, 26], we derived rigorously conjectures (6), (7), (8) and (9). However, the study of the asymptotic behaviour of sonic travelling waves is much more involved than in the subsonic case. Indeed, in the subsonic case, it relies on a lemma (Lemma 10 of [24]) which is not valid anymore for $c = \sqrt{2}$.

3. In dimension $N \geq 3$, F. Béthuel, G. Orlandi and D. Smets [7] showed the existence of travelling waves of finite energy when c is small. A. Farina [18] proved a universal bound for their modulus.

[1] In particular, if this is true, a scattering theory for small energy solutions to equation (1) is possible, although presumably difficult.

Our paper is organised around the proofs of Theorems 1 and 2. In the first part, we recall some preliminary results yet mentioned in [22, 23, 24]. In particular, we derived some convolution equations from equation (2). They are the basic ingredient of the proofs.
The second part is devoted to the proof of Theorem 1. It relies on the same argument as in [23]: the singularity at the origin of the Fourier transforms of the kernels which appear in the convolution equations of the first part.
Finally, the last part deals with the proof of Theorem 2. It follows from the use of the convolution kernels as Fourier multipliers and from Proposition 2 of [22].

1 Some convolution equations.

In this part, we write some convolution equations which are the key ingredient of all the proofs of this article. In order to state them, we first recall two useful propositions yet mentioned in [22, 23, 24] and based on arguments taken from F. Béthuel and J.C. Saut [4, 5].

Proposition 1 ([24]). *Let $c > 0$ and $N \geq 2$. Consider a solution v of equation (2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy. Then, v is of class C^∞ and bounded on $\mathbb{R}^N$. Moreover, its gradient ∇v and the function $\eta := 1 - \rho^2$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty]$.*

Remark. By Proposition 1, any weak solution of finite energy of (2) is a classical solution.

We deduce from Proposition 1 a first lemma which gives the convergence of the modulus of a travelling wave at infinity.

Lemma 1 ([22, 23, 24]). *Let $c > 0$ and $N \geq 2$. Consider a solution v of equation (2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy. The modulus ρ of v uniformly converges to 1 at infinity.*

In particular, there is some real number R_0 such that

$$\rho \geq \frac{1}{2} \text{ on } B(0, R_0)^c.$$

Thus, using a standard degree argument in dimension two, we can construct a lifting θ of v on $B(0, R_0)^c$, that is a function in $C^\infty(B(0, R_0)^c, \mathbb{R})$ such that

$$v = \rho e^{i\theta}.$$

We next compute new equations for the new functions η and θ. However, since θ is not well-defined on $\mathbb{R}^N$, we must introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N, [0, 1])$ such that

$$\begin{cases} \psi = 0 \text{ on } \mathrm{B}(0, 2\mathrm{R}_0), \\ \psi = 1 \text{ on } \mathrm{B}(0, 3\mathrm{R}_0)^{\mathrm{c}}. \end{cases}$$

All the results in the following will be independent of the choice of R_0 and ψ. Finally, we deduce

Proposition 2 ([24]). *Let $c > 0$ and $N \geq 2$. Consider a solution v of equation (2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy. Then, the functions η and $\psi\theta$ satisfy*

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}) \tag{10}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{11}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta) \tag{12}$$

and

$$G = i\nabla v.v + \nabla(\psi\theta). \tag{13}$$

Remark. The functions F and G are related to the density of energy and of momentum. In order to clarify this claim, we must remove a difficulty in the definition of $\vec{P}(v)$. Indeed, the integral which appears in definition (4) is not always convergent for a travelling wave of finite energy. In order to give a rigorous definition of the momentum $\vec{P}(v)$, we state

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N}(i\nabla v.v + \nabla(\psi\theta)). \tag{14}$$

This new definition is rather suitable (see for instance [23, 24, 26]). In particular, it is now straightforward to link the functions F and G to the density of energy and of momentum.

Finally, equations (10) and (11) lead to the desired convolution equations

$$\eta = K_0 * F + 2\sqrt{2}\sum_{j=1}^{N} K_j * G_j \tag{15}$$

and for every $j \in \{1, \ldots, N\}$,

$$\partial_j(\psi\theta) = \frac{1}{\sqrt{2}}K_j * F + 2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k \tag{16}$$

where K_0, K_j, $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transform

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi_\perp|^2}, \tag{17}$$

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi_\perp|^2}, \tag{18}$$

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi_\perp|^2)}, \tag{19}$$

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}. \tag{20}$$

Remarks. 1. Here, we denoted $\xi_\perp$ the variable given by

$$\forall \xi \in \mathbb{R}^N, \xi_\perp = (\xi_2, \ldots, \xi_N).$$

In particular, the value of $|\xi_\perp|^2$ is

$$|\xi_\perp|^2 = \sum_{j=2}^{N} \xi_j^2.$$

2. We only wrote equations (15) and (16) in the sonic case $c = \sqrt{2}$. However, we can compute similar equations for other values of c.

Now, thanks to equations (15) and (16), we turn to the proofs of Theorems 1 and 2.

2 Non-existence of non-constant travelling waves of finite energy in dimension two.

The proof of Theorem 1 relies on the form of the Fourier transforms of the kernels K_0 and K_j. They are singular at the origin, in particular in direction ξ_1. In dimension two, we deduce from this singularity a new integral relation (formula (21) just below), which provides the non-existence of non-constant sonic travelling waves of finite energy.

Proposition 3. *Let $N = 2$. Any sonic travelling wave v of finite energy satisfies the integral equation*

$$\int_{\mathbb{R}^2} (|\nabla v|^2 + \eta^2) = 0. \tag{21}$$

Remark. Actually, we recover formula (6) of [23]

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = 2c\Big(1 - \frac{2}{c^2}\Big)p(v),$$

in the specific case $c = \sqrt{2}$ and $N = 2$. As in the present paper, it was the key ingredient of the non-existence of non-constant supersonic travelling waves of finite energy.

Theorem 1 is a direct consequence of Proposition 3.

Proof of Theorem 1. By equation (21), the gradient of v vanishes on $\mathbb{R}^2$. Therefore, v is constant on $\mathbb{R}^2$. Moreover, it is a constant of modulus one since the function η also vanishes on $\mathbb{R}^2$ by equation (21). □

Now, it remains to prove Proposition 3. In order to explain the difficulty which appears in dimension $N \geq 3$, we keep in our analysis the dimension $N \geq 2$ arbitrary and only specify the case of dimension two at the very end.

Proof of Proposition 3. By Proposition 1, the functions η, F and G respectively belong to $H^4(\mathbb{R}^N)$, $W^{2,1}(\mathbb{R}^N)$ and $W^{2,1}(\mathbb{R}^N)$. Therefore, we can write for almost every $\xi \in \mathbb{R}^N$ by taking the Fourier transform of equation (15),

$$\widehat{\eta}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi_\perp|^2}\widehat{F}(\xi) + 2\sqrt{2}\sum_{j=1}^{N} \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi_\perp|^2}\widehat{G_j}(\xi). \tag{22}$$

The strategy of the proof now relies on the finiteness of the energy. Indeed, since the energy is finite, the function η belongs to $L^2(\mathbb{R}^N)$. By Plancherel's theorem, the function $\widehat{\eta}$ is also in $L^2(\mathbb{R}^N)$. On the other hand, equation (22) gives an expression of the function $\widehat{\eta}$. We are going to integrate its square modulus on a suitable subset of $\mathbb{R}^N$ and prove that this integral cannot be finite unless equality (21) holds. The choice of the set of integration is motivated by the singularity at the origin of the Fourier transforms of the kernels K_0 and K_j. Indeed, by formulae (17) and (18), they are both more singular in case $\xi_\perp$ vanishes. That is the reason why we are going to integrate the function $|\widehat{\eta}|^2$ on the set

$$\Omega = \{\xi \in \mathbb{R}^N, 0 \leq \xi_1 \leq 1, |\xi_\perp| \leq \xi_1^2\}.$$

Indeed, it follows from equation (22) that

$$\int_\Omega |\widehat{\eta}(\xi)|^2 d\xi = \int_0^1 \int_{|\xi_\perp| \leq \xi_1^2} \frac{\big||\xi|^2\widehat{F}(\xi_1,\xi_\perp) + 2\sqrt{2}(\xi_1^2\widehat{G_1}(\xi_1,\xi_\perp) + \xi_1\xi_\perp.\widehat{G_\perp}(\xi_1,\xi_\perp))\big|^2}{(|\xi|^4 + 2|\xi_\perp|^2)^2} d\xi_\perp d\xi_1.$$

Consider then the function H defined by

$$\forall \xi_1 \in]0,1], H(\xi_1) = \int_{|y|\leq \xi_1^2} \frac{\left|(|y|^2+\xi_1^2)\widehat{F}(\xi_1,y) + 2\sqrt{2}(\xi_1^2\widehat{G_1}(\xi_1,y) + \xi_1 y.\widehat{G_\perp}(\xi_1,y))\right|^2}{((|y|^2+\xi_1^2)^2+2|y|^2)^2} dy,$$

so that

$$\int_\Omega |\widehat{\eta}(\xi)|^2 d\xi = \int_0^1 H(\xi_1) d\xi_1. \tag{23}$$

We claim that

Claim 1. *If* $\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0) \neq 0$, *then*

$$H(\xi_1) \underset{\xi_1\to 0}{\sim} \left(|\mathbb{S}^{N-2}||\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0)|^2 \int_{-\delta_{N,2}}^1 \frac{s^{N-2}}{(1+2s^2)^2} ds\right) \xi_1^{2(N-3)}.$$

Indeed, the function H satisfies for every $\xi = (\xi_1, r\sigma)$,

$$\begin{aligned} H(\xi_1) =& \int_{-\delta_{N,2}\xi_1^2}^{\xi_1^2} \int_{\mathbb{S}^{N-2}} \frac{\left|(r^2+\xi_1^2)\widehat{F}(\xi) + 2\sqrt{2}(\xi_1^2\widehat{G_1}(\xi) + \xi_1 r\sigma.\widehat{G_\perp}(\xi))\right|^2}{((\xi_1^2+r^2)^2+2r^2)^2} r^{N-2} d\sigma dr \\ :=& \xi_1^{2(N-3)} I(\xi_1), \end{aligned}$$

where, denoting $\xi' = (\xi_1, \xi_1^2 s\sigma)$, we let

$$I(\xi_1) = \int_{-\delta_{N,2}}^1 \int_{\mathbb{S}^{N-2}} \frac{\left|(s^2\xi_1^2+1)\widehat{F}(\xi') + 2\sqrt{2}\widehat{G_1}(\xi') + 2\sqrt{2}\xi_1 s\sigma.\widehat{G_\perp}(\xi')\right|^2}{((\xi_1^2 s^2+1)^2+2s^2)^2} s^{N-2} d\sigma ds.$$

Moreover, by Proposition 1, the functions F and G belong to $L^1(\mathbb{R}^N)$, so their Fourier transforms are continuous on $\mathbb{R}^N$. Therefore, the dominated convergence theorem yields

$$I(\xi_1) \underset{\xi_1\to 0}{\to} |\mathbb{S}^{N-2}||\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0)|^2 \int_{-\delta_{N,2}}^1 \frac{s^{N-2}}{(1+2s^2)^2} ds.$$

In particular, if $\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0) \neq 0$, it gives

$$H(\xi_1) \underset{\xi_1\to 0}{\sim} \left(|\mathbb{S}^{N-2}||\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0)|^2 \int_{-\delta_{N,2}}^1 \frac{s^{N-2}}{(1+2s^2)^2} ds\right) \xi_1^{2(N-3)},$$

which is the desired result.

We next argue by contradiction and assume that

$$\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0) \neq 0. \tag{24}$$

If assertion (24) were true, then, by Claim 1,

$$H(\xi_1) \underset{\xi_1\to 0}{\sim} A\xi_1^{2(N-3)}.$$

However, in dimension two, the function $\xi_1 \mapsto \frac{1}{\xi_1^2}$ is not integrable near 0. By formula (23), it yields

$$\int_\Omega |\widehat{\eta}(\xi)|^2 d\xi = +\infty.$$

Thus, it gives a contradiction with the fact that the function $\widehat{\eta}$ is in $L^2(\mathbb{R}^2)$. Therefore, assumption (24) does not hold and we find

$$\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0) = 0.$$

However, by formulae (12) and (13),

$$\begin{aligned}\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0) &= 2\int_{\mathbb{R}^2}(|\nabla v|^2 + \eta^2 - \sqrt{2}i\partial_1 v.v - \sqrt{2}\partial_1(\psi\theta)) + \sqrt{8}\int_{\mathbb{R}^2}(i\partial_1 v.v + \partial_1(\psi\theta)) \\ &= 2\int_{\mathbb{R}^2}(|\nabla v|^2 + \eta^2),\end{aligned}$$

which gives

$$\int_{\mathbb{R}^2}(|\nabla v|^2 + \eta^2) = 0.$$

□

Remark. The argument fails in dimension $N \geq 3$ since the function $\xi_1 \mapsto \xi_1^{2(N-3)}$ is then integrable near 0.

3 Limit at infinity in dimension $N \geq 3$.

Theorem 2 follows from two arguments.
• The first one is to improve the L^p-integrability of the functions η and $\nabla(\psi\theta)$, and of their derivatives.

Proposition 4. *Consider $\alpha \in \mathbb{N}^N$ such that $|\alpha| \geq 2$. Then, we claim*

(i) $(\eta, \nabla(\psi\theta)) \in L^p(\mathbb{R}^N)$ *for every* $p > \frac{2N-1}{2N-3}$,

(ii) $(\nabla\eta, d^2(\psi\theta)) \in L^p(\mathbb{R}^N)$ *for every* $p > \frac{2N-1}{2N-2}$,

(iii) $(\partial^\alpha\eta, \partial^\alpha\nabla(\psi\theta)) \in L^p(\mathbb{R}^N)$ *for every* $p > 1$.

Proposition 4 follows from Lizorkin's theorem [35].

Theorem ([35]). *Let $0 \leq \beta < 1$ and $\widehat{K}$ a bounded function in $C^N(\mathbb{R}^N \setminus \{0\})$. Assume*

$$\prod_{j=1}^N(\xi_j^{k_j+\beta})\partial_1^{k_1}\dots\partial_N^{k_N}\widehat{K}(\xi) \in L^\infty(\mathbb{R}^N)$$

as soon as $(k_1, \dots, k_N) \in \{0,1\}^N$ satisfies

$$0 \leq \sum_{j=1}^N k_j \leq N.$$

Then, $\widehat{K}$ is a multiplier from $L^p(\mathbb{R}^N)$ to $L^{\frac{p}{1-\beta p}}(\mathbb{R}^N)$ for every $1 < p < \frac{1}{\beta}$.

By Lizorkin's theorem, the kernels K_0, K_j and $L_{j,k}$ are multipliers from some spaces $L^p(\mathbb{R}^N)$ to some other spaces $L^q(\mathbb{R}^N)$. For instance, the kernel K_0 satisfies the assumptions of Lizorkin's theorem for $\beta = \frac{2}{2N-1}$. Therefore, the function $\widehat{K_0}$ is a Fourier multiplier from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$. By convolution equations (15) and (16), this enables to improve the L^p-integrability of the functions η and $\nabla(\psi\theta)$, and of their derivatives.

• The second argument follows from Proposition 4. Since the function ∇v belongs to some spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ for $1 < p_0 < N-1 < p_1 < +\infty$, we can use the following proposition to prove the convergence of the function v at infinity.

Proposition 5 ([22]). *Consider a smooth function v on $\mathbb{R}^N$ and assume that $N \geq 3$ and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ where $1 < p_0 < N-1 < p_1 < +\infty$. Then, there is a constant $v_\infty \in \mathbb{C}$ which satisfies*

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

The proof of Theorem 2 is then a consequence of Propositions 4 and 5. That is the reason why we first show Proposition 4.

Proof of Proposition 4. We split the proof in three steps. In the first one, we specify the form of some derivatives of the Fourier transform of the kernel K_0. Our goal is to prove that the kernel K_0 satisfies the assumptions of Lizorkin's theorem in order to show that $\widehat{K_0}$ is a Fourier multiplier from some space $L^p(\mathbb{R}^N)$ to another space $L^q(\mathbb{R}^N)$.

Step 1. *Consider $\alpha \in \{0,1\}^N$. Then, the function $\partial^\alpha \widehat{K_0}$ writes*

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \partial^\alpha \widehat{K_0}(\xi) = \frac{\xi^\alpha P_\alpha(\xi)}{(|\xi|^4 + 2|\xi_\perp|^2)^{1+|\alpha|}}, \tag{25}$$

where P_α is a polynomial function of degree $d_\alpha \leq 2|\alpha| + 2$ which satisfies

(*i*) *For every $j \in \{1, \ldots, N\}$, P_α is even in the variable ξ_j.*

(*ii*) *The term of lowest degree of P_α is equal to $(-1)^{|\alpha|-1}(|\alpha|-1)!4^{|\alpha|}|\xi_\perp|^2$ if $\alpha_1 = 1$, and to $(-1)^{|\alpha|-1}|\alpha|!4^{|\alpha|}\xi_1^2$, if $\alpha_1 = 0$ and $|\alpha| \neq 0$.*

Step 1 follows from an inductive argument on $|\alpha|$. Indeed, if $|\alpha| = 0$ or $|\alpha| = 1$, we compute by formula (17) for every $j \in \{2, \ldots, N\}$,

$$\begin{aligned}
\widehat{K_0}(\xi) &= \frac{|\xi|^2}{|\xi|^4 + 2|\xi_\perp|^2} \\
\partial_1 \widehat{K_0}(\xi) &= \frac{\xi_1(-2|\xi|^4 + 4|\xi_\perp|^2)}{(|\xi|^4 + 2|\xi_\perp|^2)^2} \\
\partial_j \widehat{K_0}(\xi) &= \frac{\xi_j(-2|\xi|^4 + 4\xi_1^2)}{(|\xi|^4 + 2|\xi_\perp|^2)^2}.
\end{aligned}$$

Thus, Step 1 holds in this case.

Now, assume that Step 1 is valid for $|\alpha| = p$ and fix some $\alpha \in \{0,1\}^N$ such that $|\alpha| = p+1$. There are two cases to consider. If $\alpha_1 = 0$, there is some integer $j \in \{2, \ldots, N\}$ such that $\alpha_j = 1$, so we can state

$$\partial^\alpha \widehat{K_0} = \partial_j \partial^\beta \widehat{K_0}$$

with $|\beta| = p$. Applying the inductive assumption, it yields for every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$\partial^\alpha \widehat{K_0}(\xi) = \frac{\xi^\beta}{(|\xi|^4 + 2|\xi_\perp|^2)^{|\alpha|+1}} \Big(\partial_j P_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - (1+|\beta|) P_\beta(\xi)(4\xi_j|\xi|^2 + 4\xi_j) \Big). \tag{26}$$

However, by assumption (i), P_β is even in every variable ξ_k, so there is some polynomial function R_β, even in every variable ξ_k, such that

$$\partial_j P_\beta(\xi) = \xi_j R_\beta(\xi).$$

Moreover, by assumption (ii), R_β is either equal to 0 or the term of lowest degree of R_β is of degree at least equal to one.

Then, let us denote

$$P_\alpha(\xi) = R_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - 4(1+|\beta|)P_\beta(\xi)(|\xi|^2+1). \tag{27}$$

By the inductive assumption, the functions P_β and R_β are even in every variable ξ_k, so by equation (27), P_α is also even in every variable ξ_k. Likewise, the term of lowest degree of P_β is equal to $(-1)^{p-1}p!4^p\xi_1^2$ and, if R_β is not equal to 0, the term of lowest degree of R_β is of degree at least equal to one. Therefore, by equation (27), the term of lowest degree of P_α is $(-1)^p(p+1)!4^{p+1}\xi_1^2$. On the other hand, by the inductive assumption and formula (27), the degree d_α of P_α is less than $2|\alpha|+2$. Finally, equation (25) is a straightforward consequence of equations (26) and (27). Therefore, the proof of the inductive step is valid in case $\alpha_1 = 0$.

In the case $\alpha_1 = 1$, we can always assume that we first derivated $\widehat{K_0}$ by the partial operator ∂_1. Therefore, there is some integer $j \in \{2, \dots, N\}$ such that $\alpha_j = 1$ and we can state

$$\partial^\alpha \widehat{K_0} = \partial_j \partial^\beta \widehat{K_0}$$

with $|\beta| = p$. Applying the inductive assumption, it yields for every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$\partial^\alpha \widehat{K_0}(\xi) = \frac{\xi^\beta}{(|\xi|^4 + 2|\xi_\perp|^2)^{|\alpha|+1}} \Big(\partial_j P_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - 4(1+|\beta|)\xi_j P_\beta(\xi)(1+|\xi|^2) \Big).$$

Likewise, by assumption (i), P_β is even in every variable ξ_k, so there is some polynomial function R_β, even in every variable ξ_k, such that

$$\partial_j P_\beta(\xi) = \xi_j R_\beta(\xi).$$

Moreover, by assumption (ii), the term of lowest degree of R_β is equal to $2(-1)^{p-1}(p-1)!4^p$.

Denoting

$$P_\alpha(\xi) = R_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - 4(1+|\beta|)P_\beta(\xi)(|\xi|^2+1),$$

we can prove equation (25), assumptions (i) and (ii), and compute the suitable bound of the degree of P_α by the same argument as in the case $\alpha_1 = 0$. By induction, this completes the proof of Step 1.

In the second step, we use Step 1 and Lizorkin's theorem to state some properties of the Fourier multipliers $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$.

Step 2. *Let $1 < p < +\infty$. The functions $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ are Fourier multipliers from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$ if $1 < p < N - \frac{1}{2}$, while the functions $\widehat{d^2K_0}$, $\widehat{d^2K_j}$ and $\widehat{d^2L_{j,k}}$ are L^p-multipliers.*

Indeed, consider $\alpha \in \{0,1\}^N$ and set $\beta = \frac{2}{2N-1}$. By equation (25), we compute

$$\prod_{j=1}^N (\xi_j^{\alpha_j+\beta}) \partial^\alpha \widehat{K_0}(\xi) = \prod_{j=1}^N \xi_j^\beta \frac{\xi^{2\alpha} P_\alpha(\xi)}{(|\xi|^4 + 2|\xi_\perp|^2)^{1+|\alpha|}}.$$

Therefore, by Step 1, if $|\xi| \geq 1$,

$$\left|\prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha \widehat{K_0}(\xi)\right| \leq A\frac{|\xi|^{N\beta+4|\alpha|+2}}{|\xi|^{4+4|\alpha|}} \leq A|\xi|^{N\beta-2} \leq A.$$

On the other hand, if $|\xi| \leq 1$, denoting $\xi = \rho\sigma$ where $\rho \geq 0$ and $\sigma \in \mathbb{S}^{N-1}$, we compute by Step 1,

$$\begin{aligned}\left|\prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha \widehat{K_0}(\xi)\right| &\leq A\frac{\rho^{2|\alpha|+N\beta+2}|\sigma_\perp|^{(N-1)\beta+2|\alpha|-2}\max\{\rho^2,|\sigma_\perp|^2\}}{\rho^{2|\alpha|+2}(\rho^2+2|\sigma_\perp|^2)^{1+|\alpha|}}\\ &\leq A\max\{\rho^2,|\sigma_\perp|^2\}^{(2N-1)\beta-2} \leq A.\end{aligned}$$

Thus, it follows that

$$\forall\alpha\in\{0,1\}^N, \xi\mapsto\prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha\widehat{K_0}(\xi)\in L^\infty(\mathbb{R}^N).$$

By Lizorkin's theorem, $\widehat{K_0}$ is a Fourier multiplier from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$ for every $1 < p < N - \frac{1}{2}$.

Moreover, by equations (18) and (19), the Fourier transforms of the functions K_j and $L_{j,k}$ write

$$\begin{aligned}\widehat{K_j}(\xi) &= \frac{\xi_1\xi_j}{|\xi|^2}\widehat{K_0}(\xi)\\ \widehat{L_{j,k}}(\xi) &= \frac{\xi_1^2\xi_j\xi_k}{|\xi|^4}\widehat{K_0}(\xi).\end{aligned}$$

By the standard Riesz operator theory (see for instance the book of E.M. Stein and G. Weiss [49] for more details), the functions $\xi \mapsto \frac{\xi_1\xi_j}{|\xi|^2}$ and $\xi \mapsto \frac{\xi_1^2\xi_j\xi_k}{|\xi|^4}$ are L^p-multipliers for every $p > 1$. Therefore, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ are also Fourier multipliers from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$ for every $1 < p < N - \frac{1}{2}$.

Now, consider the Fourier transform of the kernel ΔK_0. Leibnitz's formula yields for every $\alpha \in \{0,1\}^N$,

$$\partial^\alpha(|\xi|^2\widehat{K_0}(\xi)) = 2\sum_{j=1}^{N}\delta_{\alpha_j,1}\xi_j\partial^{\beta^j}\widehat{K_0}(\xi) + |\xi|^2\partial^\alpha\widehat{K_0}(\xi),$$

where β^j is defined by

$$\forall k\in\{1,\ldots,N\}, \beta_k^j = \begin{cases}\alpha_k, & \text{if } k \neq j,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, we compute

$$\left|\prod_{j=1}^{N}\xi_j^{\alpha_j}\partial^\alpha(|\xi|^2\widehat{K_0}(\xi))\right| \leq A\left(\sum_{j=1}^{N}\left(\frac{|\xi^{2\alpha}||P_{\beta^j}(\xi)|}{(|\xi|^4+2|\xi_\perp|^2)^{|\alpha|}}\right) + \frac{|\xi|^2|\xi^{2\alpha}||P_\alpha(\xi)|}{(|\xi|^4+2|\xi_\perp|^2)^{1+|\alpha|}}\right).$$

By Step 1, if $|\xi| \geq 1$,

$$\left|\prod_{j=1}^{N}\xi_j^{\alpha_j}\partial^\alpha(|\xi|^2\widehat{K_0}(\xi))\right| \leq A\left(\frac{|\xi|^{4|\alpha|}}{|\xi|^{4|\alpha|}} + \frac{|\xi|^{4|\alpha|+4}}{|\xi|^{4|\alpha|+4}}\right) \leq A.$$

Likewise, by Step 1, if $|\xi| < 1$, denoting $\xi = \rho\sigma$ where $\rho \geq 0$ and $\sigma \in \mathbb{S}^{N-1}$,

$$\left|\prod_{j=1}^{N} \xi_j^{\alpha_j} \partial^\alpha(|\xi|^2 \widehat{K_0}(\xi))\right| \leq A \left(\frac{\rho^{2|\alpha|+2}|\sigma_\perp|^{2|\alpha|-2}}{\rho^{2|\alpha|}(\rho^2 + 2|\sigma_\perp|^2)^{|\alpha|}} + \frac{\rho^{4+2|\alpha|}|\sigma_\perp|^{2|\alpha|-2} \max\{|\sigma_\perp|^2, \rho^2\}}{\rho^{2|\alpha|+2}(\rho^2 + 2|\sigma_\perp|^2)^{1+|\alpha|}}\right)$$
$$\leq A,$$

which yields

$$\forall \alpha \in \{0,1\}^N, \xi \mapsto \prod_{j=1}^{N} \xi_j^{\alpha_j} \partial^\alpha \widehat{\Delta K_0}(\xi) \in L^\infty(\mathbb{R}^N).$$

By Lizorkin's theorem, we conclude that $\widehat{\Delta K_0}$ is a L^p-multiplier for every $p > 1$. By the standard Riesz operator theory, it follows that $\widehat{d^2 K_0}$, $\widehat{d^2 K_j}$ and $\widehat{d^2 L_{j,k}}$ are L^p-multipliers for every $p > 1$.

Remark. By standard Riesz operator theory, the functions $\widehat{R_{j,k}}$ are also L^p-multipliers for every $p > 1$.

At this stage, by Proposition 1 and formulae (12) and (13), the functions F and G are in all the spaces $L^p(\mathbb{R}^N)$ for every $p \geq 1$. Therefore, by Proposition 1, Step 2 and equations (15) and (16), the functions η and $\nabla(\psi\theta)$ are in $L^p(\mathbb{R}^N)$ for every $p > \frac{2N-1}{2N-3}$, while their second order derivatives are in $L^p(\mathbb{R}^N)$ for every $p > 1$. Thus, it only remains to prove

Step 3. *The functions $\nabla\eta$ and $d^2(\psi\theta)$ belong to $L^p(\mathbb{R}^N)$ for every $p > \frac{2N-1}{2N-2}$.*

Indeed, consider $p > \frac{2N-1}{2N-2}$. There are some real numbers $q > \frac{2N-1}{2N-3}$ and $r > 1$ such that

$$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{r}\right).$$

In particular, by Gagliardo-Nirenberg inequality, we derive

$$\|\nabla\eta\|_{L^p(\mathbb{R}^N)} \leq A\|\eta\|_{L^q(\mathbb{R}^N)}^{\frac{1}{2}} \|d^2\eta\|_{L^r(\mathbb{R}^N)}^{\frac{1}{2}} < +\infty.$$

Thus, the function $\nabla\eta$ is in $L^p(\mathbb{R}^N)$ for every $p > \frac{2N-1}{2N-2}$. The proof being identical for the function $d^2(\psi\theta)$, we omit it. □

Now, we end the proof of Theorem 2.

Proof of Theorem 2. By Proposition 1, the function ∇v is C^∞ on $\mathbb{R}^N$ and is equal to

$$\nabla v = \left(-\frac{\nabla\eta}{2\sqrt{1-\eta}} + i\sqrt{1-\eta}\nabla(\psi\theta)\right) e^{i(\psi\theta)}$$

on a neighbourhood of infinity. However, by Lemma 1, the function $1 - \eta$ converges to 1 at infinity, so by Proposition 4, there is some real numbers $1 < p_0 < N - 1 < p_1 < +\infty$ such that ∇v belongs to $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$. Therefore, by Proposition 5, there is some constant $\lambda_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} \lambda_\infty.$$

Finally, by Lemma 1, the modulus of λ_∞ is necessarily equal to one. □

4 Conclusion.

To our knowledge, the question of the non-existence of non-constant sonic travelling waves of finite energy remains open in dimension $N \geq 3$. However, we can expect to prove such a conjecture by studying the asymptotic behaviour of the sonic travelling waves. Here, the key idea is to prove integral equation (21) by some integrations by parts. Indeed, let B_R be the ball of centre 0 and of radius $R > 0$ and S_R the related sphere. By multiplying equation (2) by the function v and integrating by parts on B_R, we find

$$\int_{B_R} (|\nabla v|^2 + \eta^2) = \int_{B_R} (\eta + \sqrt{2} i \partial_1 v.v) + \int_{S_R} \partial_\nu v.v. \tag{28}$$

However, the multiplication of (2) by the function iv gives

$$\partial_1 \eta + \sqrt{2} \mathrm{div}(i \nabla v.v) = 0,$$

so, by multiplying by the function x_1 and integrating by parts on B_R,

$$\int_{B_R} (\eta + \sqrt{2} i \partial_1 v.v) = \int_{S_R} x_1 (\nu_1 \eta + \sqrt{2} i \partial_\nu v.v). \tag{29}$$

The sum of equations (28) and (29) is then

$$\int_{B_R} (|\nabla v|^2 + \eta^2) = \int_{S_R} (\partial_\nu v.v + x_1(\nu_1 \eta + \sqrt{2} i \partial_\nu v.v)). \tag{30}$$

The question is now to prove that the integral of the second member of equation (30) tends to 0 when R tends to $+\infty$. One possible argument in this direction is to derive some algebraic decay for the functions η and $\nabla(\psi\theta)$. Actually, it seems rather difficult because Lemma 10 of [24], which gives a crucial decay estimate in the subsonic case, is not yet available for sonic travelling waves.

Appendix. Travelling waves for the Gross-Pitaevskii equation in dimension one.

In this appendix, we classify the travelling waves for the Gross-Pitaevskii equation of finite energy and of speed $c > 0$ in dimension one (see also the article of M. Maris [39] for more details).

Theorem 3. *Assume $N = 1$ and $c > 0$. Let v a solution of finite energy of equation (2). Then,*

- *if $c \geq \sqrt{2}$, v is a constant of modulus one.*
- *if $0 < c < \sqrt{2}$, up to a multiplication by a constant of modulus one and a translation, v is either identically equal to 1, or to the function*

$$v(x) = \sqrt{1 - \frac{2 - c^2}{2\mathrm{ch}^2\Big(\frac{\sqrt{2-c^2}}{2} x\Big)}} \exp\Big(i \arctan\Big(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2 - c^2}}\Big) - i \arctan\Big(\frac{c}{\sqrt{2 - c^2}}\Big)\Big).$$

Proof. Indeed, let us denote $v = v_1 + iv_2$. Equation (2) then writes

$$v_1'' - cv_2' + v_1(1 - v_1^2 - v_2^2) = 0, \tag{31}$$
$$v_2'' + cv_1' + v_2(1 - v_1^2 - v_2^2) = 0. \tag{32}$$

The multiplication of equation (31) by v_2 and of equation (32) by v_1 gives

$$(v_1 v_2' - v_2 v_1')' = \frac{c}{2}\eta'. \tag{33}$$

However, Proposition 1 also holds in the case $N = 1$. In particular, it follows that the functions η and v' uniformly converge to 0 at infinity. Thus, by integrating equation (33), we get

$$v_1 v_2' - v_2 v_1' = \frac{c}{2}\eta. \tag{34}$$

Likewise, we multiply equation (31) by v_1' and equation (32) by v_2' to deduce

$$\left(\frac{|v'|^2}{2}\right)' = \left(\frac{\eta^2}{4}\right)',$$

which yields

$$|v'|^2 = \frac{\eta^2}{2}. \tag{35}$$

Finally, we compute

$$\eta'' = -2|v'|^2 - 2(v_1 v_1'' + v_2 v_2'') = -2|v'|^2 - 2c(v_1 v_2' - v_2 v_1') + 2\eta - 2\eta^2.$$

Therefore, by equations (34) and (35),

$$\eta'' + (c^2 - 2)\eta + 3\eta^2 = 0. \tag{36}$$

Finally, we multiply equation (36) by the function η' and integrate to obtain

$$\eta'^2 + (c^2 - 2)\eta^2 + 2\eta^3 = 0. \tag{37}$$

Now, we consider different cases according to the value of c.

• If $c > \sqrt{2}$, then, by equation (37),

$$(c^2 - 2 + 2\eta)\eta^2 = -\eta'^2 \leq 0.$$

Therefore, for every $x \in \mathbb{R}$, $\eta(x)$ is either equal to 0, or less than $1 - \frac{c^2}{2}$. Since the function η is continuous and in $L^2(\mathbb{R})$, we deduce that η is identically equal to 0. By equation (35), v' also vanishes, which means that v is a constant of modulus one.

• If $c = \sqrt{2}$, then, by equation (37),

$$\eta^3 = -\frac{\eta'^2}{2} \leq 0,$$

so, η is a non positive function on $\mathbb{R}$. Now, assume for the sake of contradiction that there is some real number x_0 such that

$$\eta(x_0) < 0.$$

Since η is smooth on $\mathbb{R}$ by Proposition 1, we deduce that there are some positive real number δ and some integer $\varepsilon \in \{-1, 1\}$ such that

$$\forall x_0 - \delta \leq x \leq x_0 + \delta, \eta'(x) = \varepsilon\sqrt{-2\eta^3(x)}.$$

Denoting $x_1 = x_0 - \varepsilon\sqrt{-\frac{2}{\eta(x_0)}}$, it follows that

$$\forall x_0 - \delta \leq x \leq x_0 + \delta, \eta(x) = -\frac{2}{(x - x_1)^2}.$$

In particular, such a solution cannot be extended to a function in $L^2(\mathbb{R})$, which yields a contradiction and proves that

$$\eta = 0.$$

As in the case $c > \sqrt{2}$, it follows that v is a constant of modulus one.

• Assume finally that $0 < c < \sqrt{2}$ and $\eta \neq 0$ (indeed, if $\eta = 0$, it follows from equation (35) that η is a constant of modulus one). By equation (37),

$$(c^2 - 2 + 2\eta)\eta^2 = -\eta'^2 \leq 0,$$

so,

$$\eta \leq 1 - \frac{c^2}{2}. \tag{38}$$

Now, suppose for the sake of contradiction that there is some real number x_0 such that

$$\eta(x_0) < 0.$$

Since η is smooth on $\mathbb{R}$ by Proposition 1, there are some positive real number δ and some integer $\varepsilon \in \{-1, 1\}$ such that

$$\forall x_0 - \delta \leq x \leq x_0 + \delta, \eta'(x) = \varepsilon\eta(x)\sqrt{2 - c^2 - 2\eta(x)}.$$

Denoting $x_1 = x_0 + \frac{2\varepsilon}{\sqrt{2-c^2}}\coth^{-1}\left(\sqrt{\frac{2-c^2-2\eta(0)}{2-c^2}}\right)$, it yields

$$\forall x_0 - \delta \leq x \leq x_0 + \delta, \eta(x) = -\frac{1 - \frac{c^2}{2}}{\mathrm{sh}^2(\frac{\sqrt{2-c^2}}{2}(x - x_1))}.$$

Since such a solution cannot be extended to a function in $L^2(\mathbb{R})$, it yields a contradiction and proves that

$$\eta \geq 0.$$

Moreover, by equation (38), since the constant function $1 - \frac{c^2}{2}$ is not in $L^2(\mathbb{R})$ and since we made the additional assumption that $\eta \neq 0$, we can assume up to a translation that

$$0 < \eta(0) < 1 - \frac{c^2}{2}.$$

Therefore, there are some positive real number δ and some integer $\varepsilon \in \{-1, 1\}$ such that

$$\forall -\delta \leq x \leq \delta, \eta'(x) = \varepsilon\eta(x)\sqrt{2 - c^2 - 2\eta(x)},$$

which gives

$$\forall -\delta \leq x \leq \delta, \eta(x) = \frac{1 - \frac{c^2}{2}}{\mathrm{ch}^2(\frac{\sqrt{2-c^2}}{2}(x - x_1))}$$

where $x_1 = \frac{2\varepsilon}{\sqrt{2-c^2}}\mathrm{ch}^{-1}\left(\sqrt{\frac{2-c^2}{2\eta(0)}}\right)$. Naturally, this solution can be extended to a smooth function in $L^2(\mathbb{R})$. Therefore, up to another translation, we conclude that

$$\forall x \in \mathbb{R}, \eta(x) = \frac{1 - \frac{c^2}{2}}{\mathrm{ch}^2\left(\frac{\sqrt{2-c^2}}{2}x\right)}. \tag{39}$$

In particular, we find

$$\forall x \in \mathbb{R}, |v(x)| = \sqrt{1 - \eta(x)} \geq \frac{c}{\sqrt{2}} > 0.$$

Therefore, we can construct a smooth lifting θ of v which satisfies

$$\forall x \in \mathbb{R}, v(x) = \rho(x)e^{i\theta(x)}.$$

By equation (34), the function θ verifies the differential equation

$$\theta' = \frac{c\eta}{2 - 2\eta}.$$

Thus, there is some real number θ_0 such that

$$\forall x \in \mathbb{R}, \theta(x) = \theta_0 + \arctan\left(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2-c^2}}\right).$$

By equation (39), up to a multiplication by a constant of modulus one, we finally obtain

$$v(x) = \sqrt{1 - \frac{2-c^2}{2\mathrm{ch}^2\left(\frac{\sqrt{2-c^2}}{2}x\right)}}\exp\left(i\ \arctan\left(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2-c^2}}\right) - i\ \arctan\left(\frac{c}{\sqrt{2-c^2}}\right)\right),$$

which concludes the proof of Theorem 3. □

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Chapitre III

Limit at infinity for subsonic travelling waves in the Gross-Pitaevskii equation.

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Abstract.
We study the decay of the travelling waves of finite energy in the Gross-Pitaevskii equation in dimension larger than three and prove their uniform convergence to a constant of modulus one at infinity.

Résumé.
Nous étudions la limite à l'infini des ondes progressives d'énergie finie dans l'équation de Gross-Pitaevskii en dimension supérieure ou égale à trois et nous montrons leur convergence uniforme vers une constante de module un.

Version française abrégée.

Dans cet article, nous étudions les ondes progressives u de vitesse $c > 0$ pour l'équation de Gross-Pitaevskii

$$i\partial_t u = \Delta u + u(1 - |u|^2)$$

de la forme

$$u(t, x) = v(x_1 - ct, \dots, x_N).$$

L'équation vérifiée par v que nous étudierons désormais est

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{1}$$

L'équation de Gross-Pitaevskii est un modèle physique qui décrit la supraconductivité ou la superfluidité et qui est associé à l'énergie

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2.$$

C.A. Jones et P.H. Roberts [30] se sont intéressés aux ondes progressives d'énergie finie parce qu'elles sont supposées expliquer la dynamique en temps long des solutions générales. Ils ont ainsi conjecturé qu'elles n'existent que lorsque

$$0 < c < \sqrt{2},$$

ce que nous supposerons désormais, et qu'elles ont une limite à l'infini qui est une constante de module un.

F. Béthuel et J.C. Saut [4, 5] les ont étudiées sur le plan mathématique et ont notamment montré leur existence en dimension deux lorsque c est petit, et l'existence de leur limite à l'infini.

Théorème 1 ([4, 5]). *En dimension deux, une onde progressive v pour l'équation de Gross-Pitaevskii de vitesse $0 < c < \sqrt{2}$ et d'énergie finie vérifie à une constante multiplicative de module un près,*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

En dimension supérieure ou égale à trois, F. Béthuel, G. Orlandi et D. Smets [7] ont prouvé leur existence lorsque c est petit. En toute dimension, A. Farina [18] a donné une borne universelle sur leur module. Dans cet article, nous allons compléter leurs travaux en dimension supérieure ou égale à trois par le théorème suivant.

Théorème 2. *En dimension supérieure ou égale à trois, une onde progressive v pour l'équation de Gross-Pitaevskii de vitesse $0 < c < \sqrt{2}$ et d'énergie finie vérifie à une constante multiplicative de module un près,*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

Dans la suite, nous esquisserons la preuve de ce théorème. Nous déterminerons d'abord la régularité des ondes progressives avant d'énoncer un argument général pour l'étude de la limite à l'infini d'une fonction.

Introduction.

In this article, we will focus on the travelling waves of speed $c > 0$ in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1-|u|^2),$$

which are of the form

$$u(t,x) = v(x_1 - ct, \dots, x_N).$$

The simplified equation for v, which we will study now, is

$$ic\partial_1 v + \Delta v + v(1-|v|^2) = 0. \quad (1)$$

The Gross-Pitaevskii equation is a physical model for superconductivity or superfluidity which is associated to the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2.$$

The travelling waves of finite energy are supposed to explain the long time dynamics of general solutions. They were first considered by C.A. Jones and P.H. Roberts [30]: they conjectured that they only exist when

$$0 < c < \sqrt{2},$$

which will be supposed henceforth, and that they have a limit at infinity which is a constant of modulus one.

F. Béthuel and J.C. Saut [4, 5] first studied mathematically those travelling waves: they showed their existence in dimension two when c is small, and also gave a mathematical proof for their decay at infinity.

Theorem 1 ([4, 5]). *In dimension two, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

In dimension larger than three, F. Béthuel, G. Orlandi and D. Smets [7] showed their existence when c is small. In every dimension, A. Farina [18] proved a universal bound for their modulus. In this paper, we will complete these results for the dimensions larger than three by proving the following theorem.

Theorem 2. *In dimension larger than three, a travelling wave v for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

This paper will be organised around the proof of Theorem 2. In the first part, we will study the local and Sobolev regularity of the travelling waves, while in the second part, we will give a general argument to study their decay at infinity.

1 Regularity of travelling waves.

In this part, we will study the regularity of a travelling wave of finite energy and of speed $0 < c < \sqrt{2}$ in dimension $N \geq 2$. We will prove the following proposition thanks to arguments from F. Béthuel and J.C. Saut [4, 5].

Proposition 1. *If v is a solution of equation (1) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy, then, v is smooth, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, +\infty]$.*

Proof. We first establish the following lemma which is valid even if $c \geq \sqrt{2}$.

Lemma 1. *v is smooth, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty]$.*

The proof of Lemma 1 is reminiscent of a bootstrap argument introduced by F. Béthuel and J.C. Saut [4], so, we will only give its sketch, and only in dimension three because the general proof is identical with small changes of Sobolev indices.

We first consider a point z_0 in $\mathbb{R}^3$ and we denote Ω, the unit ball with centre z_0. Then, we consider the solutions v_1 and v_2 of the equations

$$\begin{cases} \Delta v_1 = 0 \text{ on } \Omega, \\ v_1 = v \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v_2 = g(v) := v(1-|v|^2) + ic\partial_1 v \text{ on } \Omega, \\ v_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Since the energy of v is finite, $g(v)$ is uniformly bounded in $L^{\frac{4}{3}}(\Omega)$, which means that $||g(v)||_{L^{\frac{4}{3}}(\Omega)}$ is bounded by a constant which only depends on c and $E(v)$ but not on z_0. By standard elliptic theory and Sobolev embeddings, v_1 and v_2 are then uniformly bounded in $L^4(\Omega)$ and $W^{2,\frac{4}{3}}(\Omega)$ respectively.

If we denote ω, the ball with centre z_0 and with radius $\frac{1}{2}$, by Caccioppoli's inequalities, v_1 is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$ and in $W^{3,\frac{12}{11}}(\omega)$, so, v is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$. Furthermore, we compute

$$\nabla g(v) = \nabla v(1-|v|^2) - 2(v.\nabla v)v + ic\partial_1 \nabla v,$$

so, $\nabla g(v)$ is uniformly bounded in $L^{\frac{12}{11}}(\omega)$. By standard elliptic theory and Sobolev embeddings, we finally get that v is uniformly bounded in $C^{0,\frac{1}{12}}(\omega)$.

Thus, v is continuous and bounded on $\mathbb{R}^3$. However, its gradient $w = \nabla v$ satisfies

$$-\Delta w - ic\partial_1 w + \left(\frac{c^2}{2} + 2\right)w = w(1-|v|^2) - 2(v.w)v + \left(\frac{c^2}{2} + 2\right)w = h(w).$$

On the other hand, by the previous inequalities, $h(w)$ belongs to $L^2(\mathbb{R}^3)$, which gives that w belongs to $H^2(\mathbb{R}^3)$. So, w is continuous and bounded, and by iterating, we can conclude that v is smooth, bounded and that all its derivatives belong to the spaces $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. We then end the proof of Lemma 1 by standard interpolation between L^p-spaces.

We deduce from Lemma 1 the following lemma.

Lemma 2. *The modulus ρ of v satisfies*

$$\rho(x) \underset{|x|\to+\infty}{\to} 1.$$

Indeed, if we denote

$$\eta := 1 - \rho^2,$$

η^2 is uniformly continuous because v is bounded and lipschitzian by Lemma 1. Since $\int_{\mathbb{R}^N} \eta^2$ is finite, η converges uniformly to 0 at infinity, which ends the proof of Lemma 2.

Thus, ρ does not vanish at the neighbourhood of infinity. Therefore, we can write there $v = \rho e^{i\theta}$, and compute the following equations satisfied by ρ and θ,

$$\begin{cases} \operatorname{div}(\rho^2 \nabla\theta) = -\frac{c}{2}\partial_1 \rho^2, \\ -\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2). \end{cases} \tag{2}$$

Thanks to this polar form, we can now conclude the proof of Proposition 1 by the next lemma.

Lemma 3. *The gradient of v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, 2[$.*

This proof is also reminiscent of an article of F. Béthuel and J.C. Saut [5], so, we will only give its sketch. We first notice by Lemma 2 that ρ does not vanish at the neighbourhood of infinity, and, in order to simplify, we will suppose that ρ does not vanish on $\mathbb{R}^N$. The general situation is technically slightly more involved, but follows essentially the same idea (see [24] for more details).

We first denote

$$F = 2\eta^2 - 2c\eta\partial_1\theta + 2|\nabla v|^2,$$

and

$$G = \eta\nabla\theta.$$

Since

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2,$$

by Lemmas 1 and 2, we can establish that F and G are in all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [1, +\infty]$. Moreover, we compute by taking the Fourier transform of equations (2),

$$\forall \xi \in \mathbb{R}^N, \begin{cases} (|\xi|^2 + 2)\widehat{\eta}(\xi) - 2ic\xi_1\widehat{\theta}(\xi) = \widehat{F}(\xi), \\ |\xi|^2\widehat{\theta}(\xi) + \frac{ic}{2}\xi_1\widehat{\eta}(\xi) = -i\sum\limits_{j=1}^{N}\xi_j\widehat{G_j}(\xi). \end{cases}$$

Denoting L_0 and $(L_{j,1})_{1\leq j\leq N}$ the operators associated to the Fourier multipliers

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2},$$

respectively

$$\widehat{R_{j,1}}(\xi) = \frac{\xi_j\xi_1}{|\xi|^2},$$

we compute

$$\eta = L_0\Big(F + 2c\sum_{j=1}^{N} L_{j,1}(G)\Big).$$

Furthermore, by standard Riesz operator theory, the operators $(L_{j,1})_{1\leq j\leq N}$ are multipliers on all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$. On the other hand, $\widehat{K_0}$ is a smooth bounded function on $\mathbb{R}^N \setminus \{0\}$, which satisfies

$$\prod_{j=1}^{N}(\xi_j^{k_j})\partial_1^{k_1}\dots\partial_N^{k_N}\widehat{K_0}(\xi) \in L^\infty(\mathbb{R}^N),$$

as soon as $(k_1, \dots, k_N) \in \{0, 1\}^N$ satisfies

$$0 \leq \sum_{j=1}^{N} k_j \leq N.$$

Therefore, by Lizorkin's theorem [35] (see also [38] for more details), L_0 is a multiplier on all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$ too. By the previous statements on F and G, we conclude that η is in all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$. Therefore, by the equation

$$\forall j \in \{1, \dots, N\}, \partial_j\theta = -\frac{ic}{2}L_{j,1}(\eta) - i\sum_{k=1}^{N} L_{j,k}(G_k)$$

where $(L_{j,k})_{1\leq j,k\leq N}$ is the operator associated to the Fourier multiplier

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2},$$

$\nabla\theta$ is also in all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$.

By iterating this process to all the derivatives of η and $\nabla\theta$ by Lemma 1, we conclude that η and $\nabla\theta$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, +\infty[$. Since $\eta = 1 - \rho^2$ and ρ is in all the spaces $W^{k,\infty}(\mathbb{R}^N)$ for $k \in \mathbb{N}$, and since

$$|\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2,$$

Lemma 3 is proved as well as Proposition 1. $\square$

2 Limit at infinity.

Before concluding the proof of Theorem 2, we establish the following general proposition concerning the limit of a function at infinity.

Proposition 2. *Consider a smooth function v on $\mathbb{R}^N$ and suppose that $N \geq 3$ and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ for $1 < p_0 < N-1 < p_1 < +\infty$. Then, there is a constant $v_\infty \in \mathbb{C}$ which satisfies*

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

Proof. We begin by constructing the limit v_∞. Indeed, we have

$$\int_{\mathbb{S}^{N-1}} \int_1^{+\infty} |\partial_r v(r\xi)| dr d\xi \leq \int_{\mathbb{S}^{N-1}} \left(\int_1^{+\infty} |\nabla v(r\xi)|^{p_0} r^{N-1} dr \right)^{\frac{1}{p_0}} \left(\int_1^{+\infty} r^{-\frac{N-1}{p_0-1}} dr \right)^{\frac{1}{p_0'}} d\xi$$
$$< +\infty,$$

which gives for almost every $\xi \in \mathbb{S}^{N-1}$,

$$\int_1^{+\infty} |\partial_r v(r\xi)| dr < +\infty.$$

Thus, there is a function v_∞ defined on $\mathbb{S}^{N-1}$ such that for almost every $\xi \in \mathbb{S}^{N-1}$,

$$v(r\xi) \underset{r\to+\infty}{\to} v_\infty(\xi).$$

If we denote

$$\forall p \in [p_0, p_1], \forall r \in \mathbb{R}_+^*, \phi_p(r) = r^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p d\xi,$$

the function ϕ_p is smooth on $\mathbb{R}_+^*$, and its derivative satisfies

$$\int_0^{+\infty} |\phi_p'(r)| dr \leq C(||\nabla v||_{L^p(\mathbb{R}^N)}^p + ||\nabla v||_{L^p(\mathbb{R}^N)}^{p-1} ||\nabla v||_{W^{1,p}(\mathbb{R}^N)}) < +\infty.$$

Hence, the function ϕ_p has a limit at infinity, and since

$$\int_0^{+\infty} \phi_p(r) dr = ||\nabla v||_{L^p(\mathbb{R}^N)}^p < +\infty,$$

this limit is zero. Furthermore, if we denote

$$\forall (r,\xi)\in \mathbb{R}_+^*\times \mathbb{S}^{N-1}, v_r(\xi)=v(r\xi),$$

we remark that

$$|\nabla v(r\xi)|^2=|\partial_r v(r\xi)|^2+r^{-2}|\nabla^{\mathbb{S}^{N-1}}v_r(\xi)|^2,$$

which leads finally to

$$r^{N-1-p}\int_{\mathbb{S}^{N-1}}|\nabla^{\mathbb{S}^{N-1}}v_r(\xi)|^p d\xi \underset{r\to+\infty}{\to} 0.$$

Thus, if $N-1<q<\min\{p_1,N\}$, we get for every $r\in\mathbb{R}_+^*$,

$$\begin{aligned}\int_{\mathbb{S}^{N-1}}|v_r-v_\infty|^q &\leq \int_{\mathbb{S}^{N-1}}\left(\int_r^{+\infty}|\partial_r v(s\xi)|ds\right)^q d\xi\\ &\leq C_{N,q}\int_{\mathbb{S}^{N-1}} r^{q-N}\int_r^{+\infty}|\nabla v(s\xi)|^q s^{N-1}dsd\xi\\ &\leq C_{N,q}||\nabla v||^q_{L^q(\mathbb{R}^N)}r^{q-N},\end{aligned}$$

which gives

$$\begin{aligned}||v_r-v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} &= C_N\int_0^{|\mathbb{S}^{N-1}|}t^{-\frac{N-2}{N-1}}|v_r-v_\infty|^*(t)dt\\ &\leq C_{N,q}\left(\int_0^{|\mathbb{S}^{N-1}|}\Big(|v_r-v_\infty|^*(t)\Big)^q dt\right)^{\frac{1}{q}}\\ &\leq C_{N,q}||v_r-v_\infty||_{L^q(\mathbb{S}^{N-1})}\\ &\leq C_{N,q}||\nabla v||^q_{L^q(\mathbb{R}^N)}r^{q-N}.\end{aligned}$$

This proves that $||v_r-v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})}$ tends to 0 when r tends to $+\infty$. Now, we fix $\varepsilon>0$ and we denote for every $r\in\mathbb{R}_+$,

$$\forall\lambda\in\mathbb{R}_+^*, a_r(\lambda)=|\{\xi\in\mathbb{S}^{N-1}, |\nabla^{\mathbb{S}^{N-1}}v_r(\xi)|>\lambda\}|,$$

and

$$\forall t\in\mathbb{R}_+^*, f_r(t)=|\nabla^{\mathbb{S}^{N-1}}v_r|^*(t)=\inf\{\lambda\in\mathbb{R}_+^*, a_r(\lambda)\leq t\}.$$

We showed that there exists $r_\varepsilon\in\mathbb{R}_+^*$ such that

$$\forall r>r_\varepsilon, \forall i\in\{0,1\}, r^{N-1-p_i}\int_{\mathbb{S}^{N-1}}|\nabla^{\mathbb{S}^{N-1}}v_r(\xi)|^{p_i}d\xi\leq\varepsilon^{p_i}.$$

This gives

$$\forall\lambda\in\mathbb{R}_+^*, a_r(\lambda)\leq\min\left\{\frac{\varepsilon^{p_0}}{r^{N-1-p_0}\lambda^{p_0}},\frac{\varepsilon^{p_1}}{r^{N-1-p_1}\lambda^{p_1}}\right\},$$

then,

$$\forall t\in\mathbb{R}_+^*, f_r(t)\leq\min\left\{\frac{\varepsilon}{r^{\frac{N-1}{p_0}-1}t^{\frac{1}{p_0}}},\frac{\varepsilon}{r^{\frac{N-1}{p_1}-1}t^{\frac{1}{p_1}}}\right\}.$$

Thus, we finally get

$$\begin{aligned}||\nabla v_r||_{L^{N-1,1}(\mathbb{S}^{N-1})} &\leq C_N\varepsilon\left(r^{1-\frac{N-1}{p_1}}\int_0^{r^{1-N}}t^{-\frac{N-2}{N-1}-\frac{1}{p_1}}dt+r^{1-\frac{N-1}{p_0}}\int_{r^{1-N}}^{|\mathbb{S}^{N-1}|}t^{-\frac{N-2}{N-1}-\frac{1}{p_0}}dt\right)\\ &\leq C_{N,p_0,p_1}\varepsilon.\end{aligned}$$

This proves that ∇v_r converges to 0 in $L^{N-1,1}(\mathbb{S}^{N-1})$ when r tends to $+\infty$. Since $(v_r)_{r>0}$ converges to v_∞ in $L^{N-1,1}(\mathbb{S}^{N-1})$ and the limit of its gradient is 0 in this same space, we conclude that the gradient of v_∞ is 0. Therefore, v_∞ is constant. Moreover, by a theorem of A. Cianchi and L. Pick [11], there is some constant C which satisfies for every $r > 0$,

$$||v_r - v_\infty||_{L^\infty(\mathbb{S}^{N-1})} \leq C(||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} + ||\nabla^{\mathbb{S}^{N-1}}(v_r - v_\infty)||_{L^{N-1,1}(\mathbb{S}^{N-1})}) \underset{r\to+\infty}{\to} 0.$$

This completes the proof of Proposition 2. □

Now, we conclude the proof of Theorem 2. If v is a travelling wave of finite energy and of speed $c < \sqrt{2}$, it satisfies the hypothesis of Proposition 2 by Proposition 1. Therefore, there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

It remains to show that v_∞ has a modulus equal to one, which follows from Lemma 2.

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Chapitre IV

Decay for subsonic travelling waves in the Gross-Pitaevskii equation.

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Abstract
We study the limit at infinity of the travelling waves of finite energy in the Gross-Pitaevskii equation in dimension larger than two: their uniform convergence to a constant of modulus one and their asymptotic decay.

Résumé
Nous étudions la limite à l'infini des ondes progressives d'énergie finie pour les équations de Gross-Pitaevskii en dimension supérieure ou égale à deux: leur convergence uniforme vers une constante de module un et leur comportement asymptotique.

Introduction.

In this article, we focus on the travelling waves in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2) \tag{1}$$

of the form $u(t, x) = v(x_1 - ct, \dots, x_N)$: the parameter $c \geq 0$ is the speed of the travelling wave. The profile v then satisfies the equation

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{2}$$

The Gross-Pitaevskii equation is a physical model for superconductivity or superfluidity associated to the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2 = \int_{\mathbb{R}^N} e(v). \tag{3}$$

The non-constant travelling waves of finite energy play an important role in the long time dynamics of general solutions and were first considered by C.A. Jones and P.H. Roberts [30]: they conjectured that they only exist when $c < \sqrt{2}$ and that they are axisymmetric around axis x_1. They also proposed an asymptotic development at infinity for the travelling waves up to a multiplicative constant of modulus one. In particular, in dimension two, they conjectured that

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1-\frac{c^2}{2})x_2^2} \tag{4}$$

and in dimension three, that

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1-\frac{c^2}{2})(x_2^2+x_3^2))^{\frac{3}{2}}}, \tag{5}$$

where the real number α is the so-called stretched dipole coefficient.
The non-existence of non-constant travelling waves of finite energy for the case $c > \sqrt{2}$ was recently established in [23]. Therefore, we will suppose throughout that

$$0 \leq c < \sqrt{2}.$$

Concerning existence, F. Béthuel and J.C. Saut [4, 5] first showed the existence of travelling waves in dimension two when c is small, and also gave a mathematical evidence for their limit at infinity.

Theorem ([4, 5]). *In dimension two, a travelling wave for the Gross-Pitaevskii equation of finite energy and of speed $0 \leq c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

In dimension $N \geq 3$, F. Béthuel, G. Orlandi and D. Smets [7] showed their existence when c is small, and in every dimension, A. Farina [18] proved a universal bound for their modulus.
In this paper, we complement the previous analysis by proving the convergence of the travelling waves at infinity in dimension $N \geq 3$ and by giving a first estimate of their asymptotic decay, which is consistent with conjectures (4) and (5) of C.A. Jones and P.H. Roberts [30].
More precisely, we are going to prove the following theorem.

Theorem 1. *In dimension $N \geq 3$, a travelling wave v for the Gross-Pitaevskii equation of finite energy and of speed $0 \leq c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

Moreover, in dimension $N \geq 2$, the function

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

is bounded on $\mathbb{R}^N$.

Remark. In view of conjectures (4) and (5) of C.A. Jones and P.H. Roberts [30], it is likely that Theorem 1 yields the optimal decay rate for $v-1$.
However, we do not know if there is some argument which prevents the solutions to decay faster as it is the case for constant solutions. Actually, it is commonly conjectured that Theorem 1 gives the optimal decay rate of the travelling waves which are non-constant and axisymmetric around axis x_1.

We deduce immediately from Theorem 1 some integrability properties for $v-1$.

Corollary 1. *The function $v-1$ belongs to all the spaces $L^p(\mathbb{R}^N)$ for*

$$\frac{N}{N-1} < p \leq +\infty.$$

Remark. We conjecture that the function $v-1$ does not belong to $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ unless it is constant.

Corollary 1 has interesting consequences in dimension $N \geq 3$ because, in this case, the function $v-1$ belongs to the space $L^2(\mathbb{R}^N)$, and therefore, in view of the energy bound, to the space $H^1(\mathbb{R}^N)$. Thus, the function

$$(x,t) \mapsto v(x_1 - ct, x_2, \ldots, x_N)$$

is solution in $C^0(\mathbb{R}, 1 + H^1(\mathbb{R}^N))$ of the Cauchy problem associated to equation (1) with the initial data

$$u(0,x) = v(x).$$

The next theorem due to F. Béthuel and J.C. Saut [4] asserts that equation (1) is well-posed in this space.

Theorem ([4]). *Let $v_0 \in 1 + H^1(\mathbb{R}^N)$. There is a unique solution $v \in C^0(\mathbb{R}, 1 + H^1(\mathbb{R}^N))$ of equation* (1)*. Moreover, the energy $E(v_0)$ of v_0 is conserved and the solution v depends continuously on the initial data v_0.*

Therefore, we are now able to study the stability of a travelling wave in the space $1 + H^1(\mathbb{R}^N)$, and to understand better the long time dynamics of the time-dependent Gross-Pitaevskii equation.

The proof of Corollary 1 being an immediate consequence of Theorem 1, the paper is organised around the proof of Theorem 1.
In the first part, we study the local smoothness and the Sobolev regularity of a travelling wave v.

Theorem 2. *If v is a solution of finite energy of equation* (2) *in $L^1_{loc}(\mathbb{R}^N)$, then, v is C^∞, bounded, and the functions $\eta := 1 - |v|^2$ and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p \leq +\infty$.*

Remark. We do not know if the functions η and ∇v belong to some spaces $W^{k,1}(\mathbb{R}^N)$: we will only show that all the derivatives of η are in $L^1(\mathbb{R}^N)$. In fact, it is commonly conjectured that η and ∇v do not belong to $L^1(\mathbb{R}^N)$ (except for the constant case), but that all their derivatives are in $L^1(\mathbb{R}^N)$ (see for example the article of C.A. Jones and P.H. Roberts [30] for more details).

By a bootstrap argument adapted from the articles of F. Béthuel and J.C. Saut [4, 5], we first prove that v is C^∞ on $\mathbb{R}^N$ and that η and ∇v belong to all the L^p-spaces for $2 \leq p \leq +\infty$: it follows that the modulus ρ of v converges to 1 at infinity (see Lemma 5 in Section 1.2). In particular, there is some real number R_0 such that

$$\rho \geq \frac{1}{2} \text{ on } {}^cB(0, R_0).$$

We then construct a lifting θ of v on ${}^cB(0, R_0)$, i.e. a function in $C^\infty({}^cB(0, R_0), \mathbb{R})$ such that

$$v = \rho e^{i\theta}.$$

The construction is actually different in dimension $N = 2$, where it involves to determine the topological degree of the function $\frac{v}{\rho}$ at infinity, and in dimension $N \geq 3$ (see Lemma 6 in Section 1.2).
We next compute new equations for the new functions η and $\nabla\theta$: those functions are more suitable to study the asymptotic decay of v. In order to do so, since θ is not defined on $\mathbb{R}^N$, we introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N, [0, 1])$ such that

$$\begin{cases} \psi = 0 \text{ on } \mathrm{B}(0, 2\mathrm{R}_0), \\ \psi = 1 \text{ on } {}^\mathrm{c}\mathrm{B}(0, 3\mathrm{R}_0). \end{cases}$$

All the asymptotic estimates obtained subsequently will be independent of the choice of ψ. The functions η and $\psi\theta$ then satisfy the equations

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}) \tag{6}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{7}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta), \tag{8}$$

and

$$G = i\nabla v.v + \nabla(\psi\theta). \tag{9}$$

An important aspect of equations (6) and (7) is the fact that F and G behave like quadratic functions of η and ∇v at infinity: it allows to apply the bootstrap argument in Lemma 2.

Remark. In this paragraph, we try to motivate the introduction of the lifting θ. Without lifting, equations (6) and (7) may be written as

$$\begin{cases} \Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta\tilde{F} - 2c\partial_1\mathrm{div}(\tilde{\mathrm{G}}) \\ \frac{c}{2}\partial_1\eta + \mathrm{div}(\tilde{\mathrm{G}}) = 0, \end{cases}$$

where

$$\begin{cases} \tilde{F} = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v \\ \tilde{G} = i\nabla v.v. \end{cases}$$

However, $\tilde{F}$ and $\tilde{G}$ do not behave like quadratic functions of η and ∇v at infinity. For instance, the function $\tilde{G}$ is given by

$$\tilde{G} = -\rho^2\nabla\theta$$

at infinity, and behaves like $-\nabla\theta$. It seems rather difficult to determine the asymptotic decay of v with such an equation.

Starting with equations (6) and (7), we can develop an argument due to J.L. Bona and Yi A. Li [8], and A. de Bouard and J.C. Saut [14] (see also the articles of M. Maris [40, 41] for many more details): it relies on the transformation of a partial differential equation in a convolution equation. Actually, equations (6) and (7) can be written as

$$\eta = K_0 * F + 2c \sum_{j=1}^{N} K_j * G_j, \tag{10}$$

where K_0 and K_j are the kernels of Fourier transform,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{11}$$

respectively,

$$\widehat{K_j}(\xi) = \frac{\xi_1 \xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{12}$$

and for every $j \in \{1, \ldots, N\}$,

$$\partial_j(\psi\theta) = \frac{c}{2} K_j * F + c^2 \sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{13}$$

where $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transform,

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2 \xi_j \xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{14}$$

respectively,

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}. \tag{15}$$

Equations (10) and (13) seem more involved than equation (2), but are presumably more adapted to study the Sobolev regularity of the functions η and ∇v, as well as their decay properties. Indeed, concerning regularity, we complete the proof of Theorem 2 by showing that the kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$ are L^p-multipliers for $1 < p < +\infty$. It follows from Lizorkin's theorem [35] and standard arguments on Riesz operators (see for instance the books of J. Duoandikoetxea [16], and E.M. Stein and G. Weiss [49]). We can then deduce from equations (10) and (13) that the functions η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$ (see Proposition 4 in section 1.3).
Finally, we infer from Theorem 2 the convergence of the travelling waves towards a constant of modulus one at infinity.

Corollary 2. *In dimension $N \geq 3$, a travelling wave v for the Gross-Pitaevskii equation of finite energy and of speed $0 \leq c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x| \to +\infty}{\to} 1.$$

As mentioned above, equations (10) and (13) are also presumably more adapted to study the asymptotic decay of the functions η and ∇v. In order to clarify this claim, let us study a simple example. Consider a convolution equation of the form

$$g = K * f,$$

where we suppose that the functions K and f are smooth functions. We want to estimate the algebraic decay of the function g, i.e. to determine all the indices α for which it belongs to the space

$$M_\alpha^\infty(\mathbb{R}^N) = \{u : \mathbb{R}^N \mapsto \mathbb{C} \ / \ \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} = \sup\{|x|^\alpha |u(x)|, x \in \mathbb{R}^N\} < +\infty\},$$

in function of the algebraic decay of K and f. We have the following lemma.

Lemma 1. *Assume K and f are continuous functions on $\mathbb{R}^N$ which are in the space $M_{\alpha_1}^\infty(\mathbb{R}^N)$, respectively $M_{\alpha_2}^\infty(\mathbb{R}^N)$, where $\alpha_1 > N$ and $\alpha_2 > N$. Then, the function g belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$ for*

$$\alpha \leq \min\{\alpha_1, \alpha_2\}.$$

Proof. The proof of Lemma 1 relies on Young's inequalities

$$\begin{aligned} \forall x \in \mathbb{R}^N, |x|^\alpha |g(x)| &\leq |x|^\alpha \int_{\mathbb{R}^N} |K(x-y)||f(y)| dy \\ &\leq A \int_{\mathbb{R}^N} \left(|x-y|^\alpha |K(x-y)||f(y)| + |K(x-y)||y|^\alpha |f(y)|\right) dy \\ &\leq A \left(\|K\|_{M_\alpha^\infty(\mathbb{R}^N)} \|f\|_{L^1(\mathbb{R}^N)} + \|K\|_{L^1(\mathbb{R}^N)} \|f\|_{M_\alpha^\infty(\mathbb{R}^N)} \right). \end{aligned}$$

Since $\alpha_1 > N$ and $\alpha_2 > N$, K and f belong to $L^1(\mathbb{R}^N)$. Thus, if $\alpha \leq \min\{\alpha_1, \alpha_2\}$, the last term is finite and the function g belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$. □

The assumptions $\alpha_1 > N$ and $\alpha_2 > N$ are quite restrictive, but we can generalise this method by using Young's inequalities involving not only the L^1-L^∞ estimate, but the L^p-$L^{p'}$ estimate, and determine the algebraic decay of functions which satisfy such a convolution equation.
Our situation is close to the previous example. Indeed, equations (10) and (13) are of the form

$$(\eta, \nabla(\psi\theta)) = K * F(\eta, \nabla(\psi\theta)),$$

where F behaves like a quadratic function in terms of the variables η and $\nabla(\psi\theta)$.
In order to understand what happens in this case, we consider the non-linear model

$$f = K * f^2,$$

where f and K are both smooth functions. We get

Lemma 2. *Assume K and f are continuous functions on $\mathbb{R}^N$ which are in the space $M_{\alpha_1}^\infty(\mathbb{R}^N)$, respectively $M_{\alpha_2}^\infty(\mathbb{R}^N)$, where $\alpha_1 > N$, $\alpha_2 > \frac{N}{2}$ and $\alpha_1 > \alpha_2$. Then, the function f belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$ for*

$$\alpha \leq \alpha_1.$$

Proof. The proof of Lemma 2 also relies on Young's inequalities

$$\begin{aligned} \forall x \in \mathbb{R}^N, |x|^\alpha |f(x)| &\leq |x|^\alpha \int_{\mathbb{R}^N} |K(x-y)||f(y)|^2 dy \\ &\leq A \int_{\mathbb{R}^N} \left(|x-y|^\alpha |K(x-y)||f(y)|^2 + |K(x-y)||y|^\alpha |f(y)|^2\right) dy \\ &\leq A \left(\|K\|_{M_\alpha^\infty(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)}^2 + \|K\|_{L^1(\mathbb{R}^N)} \|f\|_{M_{\frac{\alpha}{2}}^\infty(\mathbb{R}^N)}^2 \right). \end{aligned}$$

Since $\alpha_1 > N$ and $\alpha_2 > \frac{N}{2}$, K and f belong to $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$. Thus, if $\alpha \leq \min\{\alpha_1, 2\alpha_2\}$, the last term is finite and the function f belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$. By iterating this step, the function f belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$ if $\alpha \leq \min\{\alpha_1, 2^k\alpha_2\}$ for every $k \in \mathbb{N}$, i.e. for $\alpha \leq \alpha_1$. □

Lemma 2 provides a striking optimal decay property for super linear equations. Indeed, assuming f possesses some algebraic decay, then, if f is moreover solution of such a convolution equation, it decays as fast as the kernel. However, some decay of f must be established first, in order to initiate the inductive argument.

Turning back to the functions η and $\nabla(\psi\theta)$ and convolution equations (10) and (13), the situation is a little more involved, since we have a system of equations and since the kernels are singular at the origin. However, the conclusion is similar: the decay of the solution is determined by the decay of the kernel.

Thus, in our case, we will determine the decay at infinity of the kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$, some decay at infinity for the functions η and $\nabla(\psi\theta)$, before getting their optimal decay by the previous inductive argument.

In view of the previous discussion, the second part of the paper will be devoted to the analysis of the kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$. We will estimate their algebraic decay at the origin, where they are singular, and at infinity. It relies on three different arguments.

• We first use an L^1-L^∞ inequality, which generalises the classical one between a function and its Fourier transform. It follows from the next lemma which is presumably well-known to the experts.

Lemma 3. *Let $0 < s < 1$ and $\widehat{f} \in S(\mathbb{R}^N)$. Then, the function $x \mapsto |x|^s f(x)$ is in $C_0^0(\mathbb{R}^N) := \{g \in C^0(\mathbb{R}^N)/g(x) \underset{|x|\to+\infty}{\to} 0\}$, and satisfies for every $x \in \mathbb{R}^N$,*

$$|x|^s f(x) = I_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\hat{f}(y) - \hat{f}(z)}{|y-z|^{N+s}} e^{ix.y} dydz, \tag{16}$$

where we denote

$$I_N = -((2\pi)^{N+1} \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}) u^{-\frac{N}{2}-s} du)^{-1} > 0,$$

and where $J_{\frac{N}{2}-1}$ is the Bessel function defined by

$$\forall u \in \mathbb{R}, J_{\frac{N}{2}-1}(u) = \left(\frac{u}{2}\right)^{\frac{N}{2}-1} \sum_{n=0}^{+\infty} \frac{(-1)^n u^{2n}}{4^n n! \Gamma(n+\frac{N}{2})}.$$

We deduce from Lemma 3 the following theorem.

Theorem 3. *Let $N-2 < \alpha < N$, $n \in \mathbb{N}$ and $(j,k) \in \{1,\ldots,N\}^2$. The functions $d^n K_0$, $d^n K_j$ and $d^n L_{j,k}$ belong to $M_{\alpha+n}^\infty(\mathbb{R}^N)$.*

• We then prove independently that all those functions are bounded even in the critical case, i.e. when $\alpha = N$. This is done by another duality argument in $S'(\mathbb{R}^N)$, and by a standard integration by parts.

Theorem 4. *Let $n \in \mathbb{N}$ and $(j,k) \in \{1,\ldots,N\}^2$. The functions $d^n K_0$, $d^n K_j$ and $d^n L_{j,k}$ belong to $M_{N+n}^\infty(\mathbb{R}^N)$.*

Remark. We conjecture Theorem 4 is optimal, i.e. the functions $|.|^{\alpha+n} d^n K_0$, $|.|^{\alpha+n} d^n K_j$ and $|.|^{\alpha+n} d^n L_{j,k}$ are not bounded on $\mathbb{R}^N$ for $\alpha > N$.

• Finally, we study what we shall call the composed Riesz kernels, i.e. the kernels $R_{j,k}$. We exactly know their form by standard Riesz operator theory (see for example the books of J. Duoandikoetxea [16], and E.M. Stein and G. Weiss [49]). If f is a smooth function and if we denote $g_{j,k} = R_{j,k} * f$ for every $(j,k) \in \{1, \ldots, N\}^2$, we have the formula

$$\begin{aligned} \forall x \in \mathbb{R}^N, g_{j,k}(x) = A_N \int_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y) dy \\ + A_N \int_{|y|\leq 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} (f(x-y) - f(x)) dy. \end{aligned} \tag{17}$$

Therefore, in this section, we do not study the decay of the kernels $R_{j,k}$ at infinity, but directly, the decay of the functions $g_{j,k}$, when the function f belongs to $L^1(\mathbb{R}^N)$ and the functions $|.|^\alpha f$ and $|.|^\alpha \nabla f$ are bounded for some positive number α.

In the third part, we turn to the decay of the functions η and ∇v at infinity. We first give a refined energy estimate due to F. Béthuel, G. Orlandi and D. Smets [7].

Lemma 4. *Let v, a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$. For every $0 \leq c < \sqrt{2}$, there is a strictly positive constant α_c such that the function*

$$R \to R^{\alpha_c} \int_{B(0,R)^c} e(v)$$

is bounded on $\mathbb{R}_+$.

It is the starting point of the whole study of the decay of v at infinity. Indeed, it enables to prove some algebraic decay for the functions η and ∇v, which leads to the following theorem by the inductive method mentioned above.

Theorem 5. *Let $\alpha \in \mathbb{N}^N$. Then, the functions η, $\nabla(\psi\theta)$ and ∇v satisfy*

$$\begin{cases} (\eta, \partial^\alpha \nabla(\psi\theta), \partial^\alpha \nabla v) \in M^\infty_N(\mathbb{R}^N)^3, \\ \partial^\alpha \nabla \eta \in M^\infty_{N+1}(\mathbb{R}^N). \end{cases}$$

Remark. The key result of Theorem 5 is that the algebraic decay of the functions η, $\nabla\eta$ and $\nabla(\psi\theta)$ is imposed by the kernels of the equations they satisfy. We believe that Theorem 5 is optimal for $\alpha = 0$, but not for higher derivatives. The functions $\partial^\alpha \eta$, $\partial^\alpha \nabla(\psi\theta)$ and $\partial^\alpha \nabla v$ are commonly supposed to belong to $M^\infty_{N+|\alpha|}(\mathbb{R}^N)$.

As mentioned above, we can deduce from Theorem 5 some integrability for the derivatives of the function η.

Corollary 3. *Let $\alpha \in \mathbb{N}^N$. Then,*

$$\partial^\alpha \nabla \eta \in L^1(\mathbb{R}^N).$$

The proof of Corollary 3 being an immediate consequence of Theorems 2 and 5, we will omit it, and instead, we will conclude the paper by proving the asymptotic estimate of Theorem 1 for $v - 1$.

1 Regularity and convergence at infinity of travelling waves for the Gross-Pitaevskii equation.

The first part is devoted to the proofs of Theorem 2 and Corollary 2, i.e. to determine the Sobolev regularity and the convergence at infinity of a travelling wave v of finite energy and of speed $0 \leq c < \sqrt{2}$ in dimension $N \geq 2$.
The proofs essentially stem from the articles of F. Béthuel and J.C. Saut [4, 5], and are based on equations (10) and (13). We first determine the Sobolev regularity of η and ∇v for Sobolev exponents $p \in [2, +\infty]$. We then derive properly equations (10) and (13) by introducing some lifting θ of v. This yields the Sobolev regularity of η and ∇v for Sobolev exponents $p \in]1, 2[$ by using some Fourier multiplier theory. At last, Corollary 2 follows from a general argument connecting the existence of a limit at infinity for some function with its Sobolev regularity (see Proposition 5 in section 1.4).

1.1 L^p-integrability for $2 \leq p \leq +\infty$.

We first prove the Sobolev regularity of η and ∇v for Sobolev exponents $p \in [2, +\infty]$. The following proposition holds even if $c \geq \sqrt{2}$.

Proposition 1. *If v is a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$, then, the function v is C^∞, bounded, and the functions η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $2 \leq p \leq +\infty$.*

Proof. We only prove Proposition 1 in dimension three because the general proof is identical with small changes of Sobolev indices. The proof is reminiscent of an article of F. Béthuel and J.C. Saut [4], where it is written in dimension two. It relies on a bootstrap argument.

We first consider a point z_0 in $\mathbb{R}^3$ and we denote Ω, the unit ball with centre z_0. Then, we consider the solutions v_1 and v_2 of the equations

$$\begin{cases} \Delta v_1 = 0 \text{ on } \Omega, \\ v_1 = v \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v_2 = v(1 - |v|^2) + ic\partial_1 v := g(v) \text{ on } \Omega, \\ v_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Since the energy $E(v)$ of v is finite, v is uniformly bounded in $L^4(\Omega)$, which means that the norm of v in $L^4(\Omega)$ is finite and bounded by a constant which only depends on c and $E(v)$, but not on z_0. Thus, $v(1 - |v|^2)$ is uniformly bounded in $L^{\frac{4}{3}}(\Omega)$. Likewise, $\partial_1 v$ is also uniformly bounded in $L^{\frac{4}{3}}(\Omega)$, such as $g(v)$. By standard elliptic theory, v_2 is then uniformly bounded in $W^{2,\frac{4}{3}}(\Omega)$, and by Sobolev embeddings, v_1 is uniformly bounded in $L^4(\Omega)$.

If we denote ω, the ball with centre z_0 and with radius $\frac{1}{2}$, then, by Caccioppoli's inequalities, v_1 is uniformly bounded in $W^{3,\frac{4}{3}}(\omega)$. Thus, v is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$, and, by Sobolev embeddings, in $L^{12}(\omega)$.

Furthermore, we compute

$$\forall j \in \{1, 2, 3\}, \partial_j g(v) = \partial_j v(1 - |v|^2) - 2(v.\partial_j v)v + ic\partial^2_{1,j} v.$$

Therefore, $\partial_j g(v)$ is uniformly bounded in $L^{\frac{4}{3}}(\omega)$, and by standard elliptic theory, v_2 and v are uniformly bounded in $W^{3,\frac{4}{3}}(\omega)$. Finally, by Sobolev embeddings once more, v is uniformly bounded in $C^{0,\frac{3}{4}}(\omega)$, so, v is continuous and bounded on $\mathbb{R}^3$.
However, its gradient $w = \nabla v$ satisfies

$$-\Delta w - ic\partial_1 w + \left(\frac{c^2}{2} + 2\right) w = w(1 - |v|^2) - 2(v.w)v + \left(\frac{c^2}{2} + 2\right) w := h(w).$$

Since $h(w)$ belongs to $L^2(\mathbb{R}^3)$, this proves that w belongs to $H^2(\mathbb{R}^3)$. So, w is continuous and bounded, and by iterating, we conclude that v is C^∞, bounded and that all its derivatives belong to the spaces $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. Proposition 1 then follows from standard interpolation between L^p-spaces. □

Remark. Proposition 1 shows that every weak solution of finite energy of equation (2) is a classical solution.

1.2 Convolution equations.

In this section, we establish the convolution equations, i.e. equations (10) and (13): we will use them to complete the study of the Sobolev regularity of the travelling waves, and to determine their decay at infinity.
We first construct a lifting θ of v. In order to do so, we first prove that v does not vanish at infinity. This follows from Proposition 1.

Lemma 5. *The modulus ρ of v and all its derivatives $\partial^\alpha v$ satisfy*

$$\begin{cases} \rho(x) \underset{|x|\to+\infty}{\to} 1, \\ \partial^\alpha v(x) \underset{|x|\to+\infty}{\to} 0. \end{cases}$$

Remark. Lemma 5 holds even if $c \geq \sqrt{2}$.

Proof. Indeed, on one hand, v is bounded and lipschitzian by Proposition 1, so, η^2 is uniformly continuous on $\mathbb{R}^N$. As $\int_{\mathbb{R}^N} \eta^2$ is finite, we get

$$\eta(x) \underset{|x|\to+\infty}{\to} 0,$$

which gives

$$\rho(x) \underset{|x|\to+\infty}{\to} 1.$$

On the other hand, ∇v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for every $k \in \mathbb{N}$ and $p \in [2, +\infty]$, so, $\partial^\alpha v$ is uniformly continuous and satisfies

$$\int_{\mathbb{R}^N} |\partial^\alpha v|^2 < +\infty,$$

and we get likewise

$$\partial^\alpha v(x) \underset{|x|\to+\infty}{\to} 0.$$

□

Therefore, v does not vanish at the neighbourhood of infinity, and we can construct a smooth lifting of v there.

Lemma 6. *There is some real number* $R_0 \geq 0$ *and a function* $\theta \in C^\infty({}^cB(0,R_0),\mathbb{R})$ *such that*

$$v = \rho e^{i\theta} \text{ on } {}^c\mathrm{B}(0,\mathrm{R}_0).$$

Remark. Lemma 6 holds even if $c \geq \sqrt{2}$.

Proof. By Lemma 5, there is some real number $R_0 \geq 0$ such that ρ satisfies

$$\rho \geq \frac{1}{2} \text{ on } {}^c\mathrm{B}(0,\mathrm{R}_0).$$

Thus, the map $\frac{v}{|v|}$ is a C^∞ function from ${}^cB(0,R_0)$ to the circle $\mathbb{S}^1$.
In dimension $N \geq 3$, the fundamental group $\pi_1(\mathbb{S}^{N-1})$ of the sphere $\mathbb{S}^{N-1}$ is reduced to $\{0\}$, and therefore, there is a function $\theta \in C^\infty({}^cB(0,R_0),\mathbb{R})$ such that

$$v = |v|e^{i\theta} = \rho e^{i\theta}.$$

In dimension $N = 2$, the fundamental group $\pi_1(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$ is $\mathbb{Z}$, so, there is a function $\theta \in C^\infty({}^cB(0,R_0),\mathbb{R})$ such that v is equal to $|v|e^{i\theta}$ on ${}^cB(0,R_0)$, if and only if the topological degree of v on the circle $S(0,R_0)$ is 0.
Let us denote $d \in \mathbb{Z}$, the topological degree of v on this circle. Since v does not vanish on ${}^cB(0,R_0)$, d is the degree of v on each circle $S(0,R)$ for every $R \geq R_0$, and we get

$$2\pi dR = -\int_{S(0,R)} i\partial_\tau\left(\frac{v}{|v|}\right)(\xi).\frac{v(\xi)}{|v(\xi)|}d\xi = -\int_{S(0,R)} \frac{i\partial_\tau v(\xi).v(\xi)}{|v(\xi)|^2}d\xi,$$

whence

$$|d| \leq \frac{1}{2\pi R}\int_{S(0,R)} \frac{|\partial_\tau v(\xi)|}{|v(\xi)|}d\xi \leq \frac{1}{\pi R}\int_{S(0,R)} |\nabla v(\xi)|d\xi \leq \sqrt{\frac{2}{\pi R}}\left(\int_{S(0,R)} |\nabla v(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Since ∇v belongs to $L^2(\mathbb{R}^N)$, there is some real number $R > \max\{1, R_0\}$ such that

$$\int_{S(0,R)} |\nabla v(\xi)|^2 d\xi \leq 1,$$

which gives

$$|d| \leq \sqrt{\frac{2}{\pi}} < 1.$$

As $d \in \mathbb{Z}$, it yields

$$d = 0,$$

and there is a function $\theta \in C^\infty({}^cB(0,R_0),\mathbb{R})$ such that

$$v = \rho e^{i\theta}.$$

□

Now, we can compute equations (6) and (7) on $\mathbb{R}^N$. Indeed, we introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N,[0,1])$ such that

$$\begin{cases} \psi = 0 \text{ on } \mathrm{B}(0,2\mathrm{R}_0), \\ \psi = 1 \text{ on } {}^c\mathrm{B}(0,3\mathrm{R}_0), \end{cases}$$

and we then prove

Proposition 2. *If $v := v_1 + iv_2$ is a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$, the functions η and $\psi\theta$ satisfy the equations*

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}), \tag{6}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{7}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta), \tag{8}$$

and

$$G = -v_1\nabla v_2 + v_2\nabla v_1 + \nabla(\psi\theta). \tag{9}$$

Remark. Proposition 2 holds even if $c \geq \sqrt{2}$.

Proof. Denoting $v = v_1 + iv_2$, we have by equation (2),

$$\Delta v_1 - c\partial_1 v_2 + v_1(1 - |v|^2) = 0, \tag{18}$$
$$\Delta v_2 + c\partial_1 v_1 + v_2(1 - |v|^2) = 0. \tag{19}$$

We then compute

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -2\Delta|\nabla v|^2 - 2\Delta(v.\Delta v) - 2\Delta\eta + c^2\partial^2_{1,1}\eta.$$

By equations (18) and (19), we have on one hand

$$v.\Delta v = v_1\Delta v_1 + v_2\Delta v_2 = c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - |v|^2\eta,$$

and on the other hand,

$$c\partial_1\eta = -2c(v_1\partial_1 v_1 + v_2\partial_1 v_2) = 2(\Delta v_2 v_1 - \Delta v_1 v_2) = 2\mathrm{div}(\nabla v_2 v_1 - \nabla v_1 v_2). \tag{20}$$

Therefore, we get

$$\begin{aligned}\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = & -2\Delta|\nabla v|^2 - 2\Delta\eta^2 - 2c\Delta(v_1\partial_1 v_2 - v_2\partial_1 v_1) \\ & + 2c\partial_1\mathrm{div}(v_1\nabla v_2 - v_2\nabla v_1) \\ = & -\Delta(2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta)) \\ & + 2c\partial_1\mathrm{div}(v_1\nabla v_2 - v_2\nabla v_1 - \nabla(\psi\theta)) \\ = & -\Delta F - 2c\partial_1\mathrm{div}(G),\end{aligned}$$

which gives equation (6).
For equation (7), we introduce the function $\psi\theta$ in equation (20) and we get

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\nabla v_1 v_2 - \nabla v_2 v_1 + \nabla(\psi\theta)) = \frac{c}{2}\partial_1\eta + \mathrm{div}(G).$$

□

Finally, so as to study equations (6) and (7), we transform them in convolution equations.

Proposition 3. *The functions η and $\nabla(\psi\theta)$ satisfy the equations*

$$\eta = K_0 * F + 2c\sum_{j=1}^{N} K_j * G_j, \tag{10}$$

$$\partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{13}$$

where K_0, K_j, $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transform,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{11}$$

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{12}$$

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{14}$$

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}. \tag{15}$$

Though equations (10) and (13) look more involved than equation (2), they simplify a lot the study of the regularity and of the decay of v in the next sections.

1.3 L^p-integrability for $1 < p < 2$.

In this section, we complete the proof of Theorem 2 by proving the following proposition in the case $c < \sqrt{2}$.

Proposition 4. *If v is a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$, then the functions η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

Proof. The proof is reminiscent of an article of F. Béthuel and J.C. Saut [5]. It relies on equations (10) and (13). We first study the Sobolev regularity of the functions F and G for Sobolev exponents $p \in [1, +\infty]$.

Step 1. *F and G belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$.*

By formulae (8) and (9), F and G are equal to

$$F = 2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta)$$

and

$$G = -v_1\nabla v_2 + v_2\nabla v_1 + \nabla(\psi\theta).$$

Therefore, by Proposition 1, they are C^∞ on $\mathbb{R}^N$, and it is sufficient to prove that they belong to all the spaces $W^{k,p}({}^cB(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$.
On the set ${}^cB(0, 3R_0)$, F is equal to

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1\theta.$$

On one hand, by Proposition 1, η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $2 \leq p \leq +\infty$.
On the other hand, ρ is higher than $\frac{1}{2}$ on the set ${}^cB(0, 3R_0)$ by definition of R_0 (see the

proof of Lemma 6), and v belongs to all the spaces $W^{k,\infty}(\mathbb{R}^N)$ for $k \in \mathbb{N}$. Therefore, the map $\nabla(\psi\theta)$, given by

$$\nabla(\psi\theta) = \frac{iv.\nabla v}{|v|^2}$$

at infinity, also belongs to all the spaces $W^{k,p}({}^c B(0, 3R_0))$ for $k \in \mathbb{N}$ and $2 \leq p \leq +\infty$.
As F is a quadratic function of η, $\nabla(\psi\theta)$ and ∇v, it is in all the spaces $W^{k,p}({}^c B(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$.
Likewise, the function G is given by

$$G = \eta\nabla(\psi\theta)$$

on the set ${}^c B(0, 3R_0)$, and it is also a quadratic function of η and $\nabla(\psi\theta)$. Thus, G belongs to all the spaces $W^{k,p}({}^c B(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$.

We then establish a first property of the Gross-Pitaevskii kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$.

Step 2. *The functions $\widehat{K_0}$, $\widehat{K_j}$, $\widehat{L_{j,k}}$ and $\widehat{R_{j,k}}$ are L^p-multipliers for $1 < p < +\infty$.*

Step 2 follows from Lizorkin's theorem [35].

Theorem ([35]). *Let $\widehat{K}$ a bounded function in $C^N(\mathbb{R}^N \setminus \{0\})$ and assume*

$$\prod_{j=1}^{N} (\xi_j^{k_j}) \partial_1^{k_1} \dots \partial_N^{k_N} \widehat{K}(\xi) \in L^\infty(\mathbb{R}^N),$$

as soon as $(k_1, \dots, k_N) \in \{0, 1\}^N$ satisfies

$$0 \leq \sum_{j=1}^{N} k_j \leq N.$$

Then, $\widehat{K}$ is a L^p-multiplier for $1 < p < +\infty$.

By a straightforward computation, $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ satisfy all the hypothesis of Lizorkin's theorem, so, they are L^p-multipliers for $1 < p < +\infty$.
By standard Riesz operator theory, the functions $\widehat{R_{j,k}}$ are L^p-multipliers too (see for example the books of J. Duoandikoetxea [16] and E.M. Stein and G. Weiss [49]).

Step 3. *η and $\nabla(\psi\theta)$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

By Steps 1 and 2, and equations (10) and (13), η and $\nabla(\psi\theta)$ belong to $L^p(\mathbb{R}^N)$ for $1 < p < 2$. We then iterate the proof for all the derivatives of η and $\nabla(\psi\theta)$ using the equations

$$\partial^\alpha \eta = K_0 * \partial^\alpha F + 2c \sum_{j=1}^{N} K_j * \partial^\alpha G_j, \tag{21}$$

$$\partial^\alpha \partial_j(\psi\theta) = \frac{c}{2} K_j * \partial^\alpha F + c^2 \sum_{k=1}^{N} L_{j,k} * \partial^\alpha G_k + \sum_{k=1}^{N} R_{j,k} * \partial^\alpha G_k, \tag{22}$$

for every $\alpha \in \mathbb{N}^N$. By Step 1, $\partial^\alpha F$ and $\partial^\alpha G$ belong to all the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p \leq +\infty$: Step 3 then follows from Step 2 and equations (21) and (22).

Step 4. *∇v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

The function v being C^∞ on $\mathbb{R}^N$ by Proposition 1, it is sufficient to prove that ∇v belongs to all the spaces $W^{k,p}({}^cB(0,3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$.
In order to do so, we first claim that $\nabla\rho$ belongs to the spaces $W^{k,p}({}^cB(0,3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$. Indeed, ρ is given by

$$\rho = \sqrt{1-\eta}.$$

By the proof of Lemma 6, η is higher than $\frac{3}{4}$ on the set ${}^cB(0,3R_0)$, so, by Step 3 and by the L^p-chain rule theorem, $\nabla\rho$ belongs to all the spaces $W^{k,p}({}^cB(0,3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$.
Thus, ρ and $\nabla(\psi\theta)$ belong to all the spaces $W^{k,\infty}({}^cB(0,3R_0))$ for $k \in \mathbb{N}$, and $\nabla\rho$ and $\nabla(\psi\theta)$ belong to all the spaces $W^{k,p}({}^cB(0,3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$. Since ∇v is given by

$$\nabla v = \nabla\rho e^{i\psi\theta} + i\rho\nabla(\psi\theta)e^{i\psi\theta}$$

at infinity, by Leibnitz' formula and the L^p-chain rule theorem, ∇v belongs to all the spaces $W^{k,p}({}^cB(0,3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$. □

1.4 Convergence at infinity in dimension $N \geq 3$.

We now deduce Corollary 2 from Theorem 2. Indeed, by the following proposition, the convergence at infinity of a travelling wave v follows from its regularity.

Proposition 5. *Let $v \in C^2(\mathbb{R}^N)$, and suppose that $N \geq 3$ and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$, where*

$$1 < p_0 < N-1 < p_1 < +\infty.$$

Then, there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

Proof. Proposition 5 relies on a radial construction of the limit v_∞: we focus on the functions $(v_r)_{r>0}$ defined by

$$\forall \xi \in \mathbb{S}^{N-1}, v_r(\xi) = v(r\xi).$$

We first prove their convergence almost everywhere towards a measurable function v_∞ on $\mathbb{S}^{N-1}$ when r tends to $+\infty$. We then show the uniformity of this convergence by a standard embedding theorem involving Lorentz' spaces, and we conclude by showing that v_∞ is a constant function.
At first, we construct the limit v_∞. We compute

$$\begin{aligned}\int_{\mathbb{S}^{N-1}}\int_1^{+\infty}|\partial_r v(r\xi)|drd\xi &\leq \int_{\mathbb{S}^{N-1}}\left(\int_1^{+\infty}|\nabla v(r\xi)|^{p_0}r^{N-1}dr\right)^{\frac{1}{p_0}}\left(\int_1^{+\infty}r^{-\frac{N-1}{p_0-1}}dr\right)^{\frac{1}{p_0'}}d\xi\\ &\leq A_{N,p_0}\left(\int_{{}^cB(0,1)}|\nabla v(x)|^{p_0}dx\right)^{\frac{1}{p_0}} < +\infty,\end{aligned}$$

so, for almost every $\xi \in \mathbb{S}^{N-1}$,

$$\int_1^{+\infty}|\partial_r v(r\xi)|dr < +\infty.$$

Hence, there is a measurable function v_∞ on $\mathbb{S}^{N-1}$ such that for almost every $\xi \in \mathbb{S}^{N-1}$,

$$v_r(\xi) \underset{r\to+\infty}{\to} v_\infty(\xi).$$

We now claim

Lemma 7. *v_∞ is the limit in $L^\infty(\mathbb{S}^{N-1})$ of the functions $(v_r)_{r>0}$ when r tends to $+\infty$, i.e.*

$$||v_r - v_\infty||_{L^\infty(\mathbb{S}^{N-1})} \underset{r\to+\infty}{\to} 0.$$

Indeed, denote

$$\forall p \in [p_0, p_1], \forall r > 0, I_p(r) = r^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p d\xi.$$

The function I_p is C^1 on $\mathbb{R}_+^*$ and its derivative satisfies

$$\forall r > 0, |I_p'(r)| \leq (N-1)r^{N-2} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p d\xi + pr^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^{p-1} |\partial_r \nabla v(r\xi)| d\xi,$$

so,

$$\int_0^{+\infty} |I_p'(r)| dr \leq A(||\nabla v||^p_{L^p(\mathbb{R}^N)} + ||\nabla v||^{p-1}_{L^p(\mathbb{R}^N)} ||\nabla v||_{W^{1,p}(\mathbb{R}^N)}) < +\infty.$$

Hence, I_p has a limit at $+\infty$, and since

$$\int_0^{+\infty} I_p(r) dr = ||\nabla v||^p_{L^p(\mathbb{R}^N)} < +\infty,$$

this limit is zero.
Furthermore, we notice that

$$|\nabla v(r\xi)|^2 = |\partial_r v(r\xi)|^2 + r^{-2} |\nabla^{\mathbb{S}^{N-1}} v_r(\xi)|^2,$$

where $\nabla^{\mathbb{S}^{N-1}} v_r$ denotes the gradient of the function v_r on the sphere $\mathbb{S}^{N-1}$. It yields

$$r^{N-1-p} \int_{\mathbb{S}^{N-1}} |\nabla^{\mathbb{S}^{N-1}} v_r(\xi)|^p d\xi \underset{r\to+\infty}{\to} 0. \tag{23}$$

So, we know at least partly the L^p-convergence of the gradients of the functions v_r. We now estimate the L^q-convergence of the functions v_r to prove their uniform convergence by using embedding theorems.
Thus, if $p_0 \leq q < \min\{p_1, N\}$, we get for every $r > 0$,

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} |v_r(\xi) - v_\infty(\xi)|^q d\xi &\leq \int_{\mathbb{S}^{N-1}} \left(\int_r^{+\infty} |\partial_r v(s\xi)| ds \right)^q d\xi \\ &\leq \left(\frac{q-1}{N-q} \right)^{q-1} \int_{\mathbb{S}^{N-1}} r^{q-N} \int_r^{+\infty} |\nabla v(s\xi)|^q s^{N-1} ds d\xi \\ &\leq A_{N,q} ||\nabla v||^q_{L^q(\mathbb{R}^N)} r^{q-N}. \end{aligned} \tag{24}$$

By assertions (23) and (24), the functions v_r converge to v_∞ in $L^q(\mathbb{S}^{N-1})$ for every $q \in [p_0, \min\{p_1, N\}[$, and their gradient converge to 0 in $L^q(\mathbb{S}^{N-1})$ for every $q \in [p_0, N-1]$. Hence, the functions v_r converge to v_∞ in $W^{1,q}(\mathbb{S}^{N-1})$ for every $q \in [p_0, N-1]$, and since their gradient converge to 0, the gradient of v_∞ in $\mathcal{D}'(\mathbb{S}^{N-1})$ is 0, i.e. the function v_∞ is constant.

Actually, by standard Sobolev embedding theorem, the spaces $W^{1,q}(\mathbb{S}^{N-1})$ do not embed in $L^{\infty}(\mathbb{S}^{N-1})$ for any $q \in [p_0, N-1]$: that is the reason why we introduce the Lorentz space $L^{N-1,1}(\mathbb{S}^{N-1})$.
At first, let us recall briefly the definition of this space. Consider a measurable function f on $\mathbb{S}^{N-1}$ and define its distribution function λ_f by

$$\forall t > 0, \lambda_f(t) := \mu(\{\xi \in \mathbb{S}^{N-1}, |f(\xi)| > t\}),$$

where μ is the standard measure of $\mathbb{S}^{N-1}$, and its decreasing rearrangement f^* by

$$\forall t > 0, f^*(t) := \inf\{s > 0, \lambda_f(s) \leq t\}.$$

The Lorentz space $L^{N-1,1}(\mathbb{S}^{N-1})$ is then the set of all measurable functions f such that

$$\|f\|_{L^{N-1,1}(\mathbb{S}^{N-1})} := \int_0^{+\infty} t^{\frac{1}{N-1}-1} f^*(t)dt < +\infty.$$

The interest of this space relies on the theorem of A. Cianchi and L. Pick [11].

Theorem ([11]). *Denote*

$$W(L^{N-1,1}(\mathbb{S}^{N-1})) := \{u \in L^{N-1,1}(\mathbb{S}^{N-1}), \nabla^{\mathbb{S}^{N-1}} u \in L^{N-1,1}(\mathbb{S}^{N-1})\}.$$

Then,

$$W(L^{N-1,1}(\mathbb{S}^{N-1})) \hookrightarrow L^{\infty}(\mathbb{S}^{N-1}),$$

i.e. there is some constant $C > 0$ such that for every function $f \in W(L^{N-1,1}(\mathbb{S}^{N-1}))$,

$$\|f\|_{L^{\infty}(\mathbb{S}^{N-1})} \leq C(\|f\|_{L^{N-1,1}(\mathbb{S}^{N-1})} + \|\nabla^{\mathbb{S}^{N-1}} f\|_{L^{N-1,1}(\mathbb{S}^{N-1})}).$$

Remark. In fact, A. Cianchi and L. Pick [11] proved a stronger result (Theorem 3.5 and Remark 3.7 there), which is not useful here, but which explains why we introduce the Lorentz space $L^{N-1,1}(\mathbb{S}^{N-1})$.
Let X, a rearrangement-invariant Banach function space on the sphere $\mathbb{S}^{N-1}$, and denote

$$W(X) = \{u \in X, \nabla^{\mathbb{S}^{N-1}} u \in X\}.$$

Then, $W(X)$ embeds in $L^{\infty}(\mathbb{S}^{N-1})$ if and only if

$$X \hookrightarrow L^{N-1,1}(\mathbb{S}^{N-1}).$$

Thus, in some sense, $W(L^{N-1,1}(\mathbb{S}^{N-1}))$ is the largest space (among the admissible $W(X)$) which embeds in $L^{\infty}(\mathbb{S}^{N-1})$: that is the reason why the space $L^{N-1,1}(\mathbb{S}^{N-1})$ appears naturally in our proof.

By Cianchi and Pick's theorem, it only remains to prove that the functions v_r and their gradients converge to v_∞, respectively ∇v_∞ in $L^{N-1,1}(\mathbb{S}^{N-1})$. Indeed, by assertion (24), we have for every $N-1 < q < \min\{p_1, N\}$,

$$\begin{aligned}
||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} &= \int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}} |v_r - v_\infty|^*(t)dt \\
&\leq \left(\int_0^{|\mathbb{S}^{N-1}|} |v_r - v_\infty|^{*q}(t)dt\right)^{\frac{1}{q}} \left(\int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{q'(N-2)}{N-1}} dt\right)^{\frac{1}{q'}} \\
&\leq A_{N,q} ||v_r - v_\infty||_{L^q(\mathbb{S}^{N-1})} \\
&\leq A_{N,q} ||\nabla v||^q_{L^q(\mathbb{R}^N)} r^{q-N} \\
&\underset{r \to +\infty}{\to} 0.
\end{aligned}$$

Now, fix $\varepsilon > 0$. By assertion (23), there is some real number $r_\varepsilon > 0$ such that

$$\forall r > r_\varepsilon, \forall q \in \{p_0, p_1\}, r^{N-1-q} \int_{\mathbb{S}^{N-1}} |\nabla^{\mathbb{S}^{N-1}} v_r(\xi)|^q d\xi \leq \varepsilon^q.$$

Thus, denoting $\lambda_r = \lambda_{\nabla^{\mathbb{S}^{N-1}} v_r}$ and $f_r = |\nabla^{\mathbb{S}^{N-1}} v_r|^*$, we obtain

$$\forall t > 0, \lambda_r(t) \leq \min\left\{ \frac{\varepsilon^{p_0}}{r^{N-1-p_0} t^{p_0}}, \frac{\varepsilon^{p_1}}{r^{N-1-p_1} t^{p_1}} \right\},$$

and

$$\forall t > 0, f_r(t) \leq \min\left\{ \frac{\varepsilon}{r^{\frac{N-1}{p_0}-1} t^{\frac{1}{p_0}}}, \frac{\varepsilon}{r^{\frac{N-1}{p_1}-1} t^{\frac{1}{p_1}}} \right\}.$$

Finally, we compute

$$\begin{aligned} ||\nabla^{\mathbb{S}^{N-1}} v_r||_{L^{N-1,1}(\mathbb{S}^{N-1})} &= \int_0^{|\mathbb{S}^{N-1}|} f_r(t) t^{-\frac{N-2}{N-1}} dt \\ &\leq \varepsilon r^{1-\frac{N-1}{p_1}} \int_0^{r^{1-N}} t^{-\frac{N-2}{N-1}-\frac{1}{p_1}} dt \\ &+ \varepsilon r^{1-\frac{N-1}{p_0}} \int_{r^{1-N}}^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}-\frac{1}{p_0}} dt \\ &\leq A_{N,p_0,p_1} \varepsilon. \end{aligned}$$

It yields that $\nabla^{\mathbb{S}^{N-1}} v_r$ converges to 0 in $L^{N-1,1}(\mathbb{S}^{N-1})$ when r tends to $+\infty$. By Cianchi and Pick's theorem, we then get

$$||v_r - v_\infty||_{L^\infty(\mathbb{S}^{N-1})} \underset{r \to +\infty}{\to} 0,$$

which ends the proof of Lemma 7.

The proof of Proposition 5 is then complete because the functions v_r converge uniformly to v_∞ by Lemma 7, and because the proof of Lemma 7 yields that v_∞ is a constant function. □

Corollary 2 then follows from Theorem 2 and Proposition 5.

Proof of Corollary 2. If v is a travelling wave of finite energy and of speed $0 \leq c < \sqrt{2}$, it satisfies the assumptions of Proposition 5 by Theorem 2. Therefore, there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x| \to +\infty}{\to} v_\infty.$$

By Lemma 5, the modulus of v_∞ is one. □

Remark. To simplify the notations, and since the solutions are defined up to a rotation, we will assume from now on that

$$v_\infty = 1.$$

2 Linear estimates for the Gross-Pitaevskii kernels.

In the second part, we estimate the algebraic decay of the kernels associated to the Gross-Pitaevskii equation K_0, K_j, $L_{j,k}$ and $R_{j,k}$, i.e. the exponents α for which the functions $|.|^\alpha K_0$, $|.|^\alpha K_j$, $|.|^\alpha L_{j,k}$ and $|.|^\alpha R_{j,k}$ are bounded on $\mathbb{R}^N$. We then deduce some L^p-regularity for those kernels.

2.1 Inequalities L^1-L^∞.

In this section, for sake of completeness, we first prove Lemma 3, which is presumably well-known to the experts. We then deduce three generalisations of it for functions which are not necessarily in $S(\mathbb{R}^N)$. The first one concerns the functions in the fractional Sobolev space $W^{s,1}(\mathbb{R}^N)$ defined by

$$W^{s,1}(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N), \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}} dydz < +\infty\} \tag{25}$$

for $0 < s < 1$, the second one, the functions in the fractional Deny-Lions space $D^{s,1}(\mathbb{R}^N)$ defined by

$$D^{s,1}(\mathbb{R}^N) := \{u \in L^{p_s}(\mathbb{R}^N), \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}} dydz < +\infty\} \tag{26}$$

for $0 < s < 1$: they are both useful to study the algebraic decay of the kernels K_0, K_j and $L_{j,k}$. The last one concerns the functions in the homogeneous fractional Sobolev space $\dot{W}^{s,1}(\mathbb{R}^N)$, whose definition is more involved: it is likely to be the largest space in which the L^1-L^∞ estimate of Lemma 3 holds.

Proof of Lemma 3. Let $\widehat{f}$, a function in $S(\mathbb{R}^N)$. At first, f is also in $S(\mathbb{R}^N)$, so, the function $x \mapsto |x|^s f(x)$ is in $C^0_0(\mathbb{R}^N)$. Now , fix $x \in \mathbb{R}^N$. We get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z)-\hat{f}(y)}{|z-y|^{N+s}} e^{ix.y} dydz &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(y+t)-\hat{f}(y)}{|t|^{N+s}} e^{ix.y} dydt \\ &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(\sigma) e^{i(x.y-\sigma.y)} \frac{e^{-it.\sigma}-1}{|t|^{N+s}} d\sigma\right) dy\right) dt.\end{aligned}$$

We then compute

$$\int_{\mathbb{R}^N} \frac{e^{-it.\sigma}-1}{|t|^{N+s}} dt$$

by a general formula for the Fourier transform of radial functions (see for example the book of L. Schwartz [48]):

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{e^{-it.\sigma}-1}{|t|^{N+s}} dt &= 2\pi \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi r|\sigma|) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})}(r|\sigma|)^{\frac{N}{2}-1}) r^{-s-\frac{N}{2}} |\sigma|^{1-\frac{N}{2}} dr \\ &= 2\pi|\sigma|^s \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}) u^{-\frac{N}{2}-s} du.\end{aligned}$$

So, if we denote

$$A_N = 2\pi \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}) u^{-\frac{N}{2}-s} du < 0,$$

we get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z)-\hat{f}(y)}{|z-y|^{N+s}} e^{ix.y} dydz &= A_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\sigma)|\sigma|^s e^{i(x.y-\sigma.y)} d\sigma dy \\ &= A_N \int_{\mathbb{R}^N} \widehat{|.|^s f}(y) e^{iy.x} dy \\ &= (2\pi)^N A_N f(x)|x|^s.\end{aligned}$$

Thus, formula (16) holds for every function $\widehat{f} \in S(\mathbb{R}^N)$. □

We assumed in Lemma 3 that $\widehat{f}$ is a smooth function in $S(\mathbb{R}^N)$. However, we can extend Lemma 3 at least in three ways by an argument of density.

• Consider first the fractional Sobolev space $W^{s,1}(\mathbb{R}^N)$ defined by (25) for every $0 < s < 1$. $W^{s,1}(\mathbb{R}^N)$ is a Banach space for the norm

$$\|u\|_{W^{s,1}(\mathbb{R}^N)} := \|u\|_{L^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}} dydz,$$

in which the space $S(\mathbb{R}^N)$ is dense (see the books of J. Peetre [44] and H. Triebel [51] for many more details: $W^{s,1}(\mathbb{R}^N)$ is equal to the Besov space $B^s_{1,1}(\mathbb{R}^N)$).
We deduce from the property of density of $S(\mathbb{R}^N)$ and from Lemma 3 the next corollary.

Corollary 4. *Let $0 < s < 1$ and $\widehat{f} \in W^{s,1}(\mathbb{R}^N)$. Then, the function $x \mapsto |x|^s f(x)$ is in $C^0_0(\mathbb{R}^N)$ and satisfies*

$$\| \, |.|^s f\|_{L^\infty(\mathbb{R}^N)} \leq I_N \|\widehat{f}\|_{W^{s,1}(\mathbb{R}^N)}, \tag{27}$$

where I_N is the constant given by Lemma 3.

Proof. Let $\widehat{f} \in W^{s,1}(\mathbb{R}^N)$. Since $S(\mathbb{R}^N)$ is dense in $W^{s,1}(\mathbb{R}^N)$, there is a sequence $(\widehat{f_n})_{n\in\mathbb{N}}$ of functions of $S(\mathbb{R}^N)$ such that

$$\|\widehat{f} - \widehat{f_n}\|_{W^{s,1}(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0.$$

Thus, by Lemma 3, the sequence of functions

$$g_n : x \mapsto g_n(x) = |x|^s f_n(x)$$

is a Cauchy sequence in the space $C^0_0(\mathbb{R}^N)$. Therefore, there is a function $g \in C^0_0(\mathbb{R}^N)$ such that

$$\|g_n - g\|_{L^\infty(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0.$$

By assumption, the functions $\widehat{f_n}$ converge to $\widehat{f}$ in $L^1(\mathbb{R}^N)$, so, the functions f_n converge to f in $L^\infty(\mathbb{R}^N)$, and up to an extraction, almost everywhere. It follows that

$$g = |.|^s f.$$

By Lemma 3, we have for every $n \in \mathbb{N}$,

$$\| \, |.|^s f_n\|_{L^\infty(\mathbb{R}^N)} \leq I_N \|\widehat{f_n}\|_{W^{s,1}(\mathbb{R}^N)},$$

which yields inequality (27) by taking the limit $n \to +\infty$. □

• Actually, we are going to work on functions which do not belong to the space $W^{s,1}(\mathbb{R}^N)$. That is the reason why we introduce a second space in which Lemma 3 holds. By standard Sobolev embeddings, we know that

$$W^{s,1}(\mathbb{R}^N) \hookrightarrow L^{p_s}(\mathbb{R}^N)$$

for every $0 < s < 1$ and $p_s = \frac{N}{N-s}$. Thus, we are led to consider the fractional Deny-Lions space $D^{s,1}(\mathbb{R}^N)$ defined by (26) for every $0 < s < 1$. $D^{s,1}(\mathbb{R}^N)$ is also a Banach space for the norm

$$\|u\|_{W^{s,1}(\mathbb{R}^N)} := \|u\|_{L^{p_s}(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}} dydz,$$

and the space $S(\mathbb{R}^N)$ is also dense in $D^{s,1}(\mathbb{R}^N)$ (see the books of J. Peetre [44] and H. Triebel [51] for many more details).
We deduce from the property of density of $S(\mathbb{R}^N)$ and from Lemma 3 the next corollary.

Corollary 5. *Let* $0 < s < 1$ *and* $\widehat{f} \in D^{s,1}(\mathbb{R}^N)$. *Then, the function* $x \mapsto |x|^s f(x)$ *is in* $C_0^0(\mathbb{R}^N)$ *and satisfies*

$$\| \, |.|^s f\|_{L^\infty(\mathbb{R}^N)} \le I_N \|\widehat{f}\|_{D^{s,1}(\mathbb{R}^N)},$$

where I_N *is the constant given by Lemma 3.*

Proof. The proof being nearly identical to the proof of Corollary 4, we omit it. The main difference is that the functions $\widehat{f_n}$ do not converge to $\widehat{f}$ in $L^1(\mathbb{R}^N)$ anymore. However, they converge to $\widehat{f}$ in $L^{p_s}(\mathbb{R}^N)$: since $p_s \le 2$, the functions f_n converge to f in $L^{p'_s}(\mathbb{R}^N)$ where $p'_s = \frac{p_s}{p_s - 1}$, so, up to an extraction, they also converge to f almost everywhere. Corollary 5 then follows from the same arguments as in the proof of Corollary 4. □

• Finally, we introduce a last space to which the conclusion of Lemma 3 can be extended: the homogeneous fractional Sobolev space $\dot{W}^{s,1}(\mathbb{R}^N)$. Its definition is rather involved. We first consider the space

$$Z(\mathbb{R}^N) = \{u \in S(\mathbb{R}^N) / \forall \alpha \in \mathbb{N}^N, \partial^\alpha \widehat{u}(0) = 0\},$$

and its topological dual space $Z'(\mathbb{R}^N)$. We are going to identify $Z'(\mathbb{R}^N)$ with the factor space $S'(\mathbb{R}^N)/P(\mathbb{R}^N)$, where $P(\mathbb{R}^N)$ denotes the set of all polynomial functions on $\mathbb{R}^N$. In this case, an element of $Z'(\mathbb{R}^N)$ is a class of tempered distributions defined modulo a polynomial function: we will denote $\dot{u}$, a representative of the class u in $S'(\mathbb{R}^N)$. The space $\dot{W}^{s,1}(\mathbb{R}^N)$ is then given by

$$\dot{W}^{s,1}(\mathbb{R}^N) = \{u \in Z'(\mathbb{R}^N) / \inf_{P \in P(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\dot{u}(z) + P(z) - \dot{u}(y) - P(y)|}{|z - y|^{N+s}} dydz \right) < +\infty\}$$

for every $0 < s < 1$. $\dot{W}^{s,1}(\mathbb{R}^N)$ is a Banach space for the norm

$$\|u\|_{\dot{W}^{s,1}(\mathbb{R}^N)} := \inf_{P \in P(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\dot{u}(z) + P(z) - \dot{u}(y) - P(y)|}{|z - y|^{N+s}} dydz \right).$$

The space $Z(\mathbb{R}^N)$ is dense in $\dot{W}^{s,1}(\mathbb{R}^N)$ and $\dot{W}^{s,1}(\mathbb{R}^N)$ is continuously embedded in $Z'(\mathbb{R}^N)$ (see the book of J. Peetre [44] and H. Triebel [51] for many more details: $\dot{W}^{s,1}(\mathbb{R}^N)$ is equal to the homogeneous Besov space $\dot{B}^s_{1,1}(\mathbb{R}^N)$).
We deduce from the property of density of $Z(\mathbb{R}^N)$ and from Lemma 3 the following corollary.

Corollary 6. *Let* $0 < s < 1$ *and* $\widehat{f} \in \dot{W}^{s,1}(\mathbb{R}^N)$. *There exists a distribution* $\tilde{f}$ *in the class* f *such that the function* $x \mapsto |x|^s \tilde{f}(x)$ *is in* $C_0^0(\mathbb{R}^N)$ *and satisfies*

$$\| \, |.|^s \tilde{f}\|_{L^\infty(\mathbb{R}^N)} \le I_N \|\widehat{f}\|_{\dot{W}^{s,1}(\mathbb{R}^N)}, \tag{28}$$

where I_N *is the constant given by Lemma 3.*

Remark. We must clarify some points. $\widehat{f}$ is a class of distributions modulo a polynomial function. Thus, f is also a class of tempered distributions, but modulo a finite linear combination of the Dirac mass δ_0 in 0 and of some of its derivatives: we will denote $\tilde{f}$, a representative of the class f in $S'(\mathbb{R}^N)$.

Proof. Let $\widehat{f} \in \dot{W}^{s,1}(\mathbb{R}^N)$. $Z(\mathbb{R}^N)$ is dense in $\dot{W}^{s,1}(\mathbb{R}^N)$, so, there is a sequence $(\widehat{f_n})_{n \in \mathbb{N}}$ of functions of $Z(\mathbb{R}^N)$ such that

$$\|\widehat{f} - \widehat{f_n}\|_{\dot{W}^{s,1}(\mathbb{R}^N)} \underset{n \to +\infty}{\to} 0. \tag{29}$$

Thus, $(\widehat{f_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\dot{W}^{s,1}(\mathbb{R}^N)$, so, by Lemma 3, the sequence of functions

$$g_n : x \mapsto g_n(x) = |x|^s f_n(x)$$

is a Cauchy sequence in the space $C^0_0(\mathbb{R}^N)$. Therefore, there is a function $g \in C^0_0(\mathbb{R}^N)$ such that

$$\|g_n - g\|_{L^\infty(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0.$$

On the other hand, since $\dot{W}^{s,1}(\mathbb{R}^N)$ is continuously embedded in $Z'(\mathbb{R}^N)$, assertion (29) yields that

$$\widehat{f_n} \underset{n\to+\infty}{\to} \widehat{f} \text{ in } Z'(\mathbb{R}^N).$$

Thus, if we consider a function $\phi \in S(\mathbb{R}^N)$ such that

$$\widehat{|.|^s\phi} \in Z(\mathbb{R}^N),$$

i.e. a function $\phi \in S(\mathbb{R}^N)$ such that $|.|^s\phi$ is in $C^\infty(\mathbb{R}^N)$ and

$$\forall \alpha \in \mathbb{N}^N, \partial^\alpha(|.|^s\phi)(0) = 0,$$

we get

$$\begin{aligned}
< g, \phi > &= \lim_{n\to+\infty} < |.|^s f_n, \phi > \\
&= (2\pi)^{-N} \lim_{n\to+\infty} < \widehat{f_n}, \widehat{|.|^s\phi} > \\
&= (2\pi)^{-N} < \widehat{f}, \widehat{|.|^s\phi} > \\
&= < |.|^s f, \phi > .
\end{aligned}$$

We deduce that there is some representative $\tilde{f}$ in the class of f which is in $C^0_0(\mathbb{R}^N \setminus \{0\})$ and which satisfies

$$|.|^s \tilde{f} = g \text{ on } \mathbb{R}^N \setminus \{0\}.$$

Since g is in $C^0_0(\mathbb{R}^N)$, $\frac{g}{|.|^s}$ is in $L^1_{loc}(\mathbb{R}^N)$, so, it is a tempered distribution. Consequently, $\tilde{f} - \frac{g}{|.|^s}$ is also a tempered distribution whose support is included in the set $\{0\}$.
By Schwartz lemma, it is a finite linear combination of δ_0 and of some of its derivatives, i.e. the classes of $\tilde{f}$ and $\frac{g}{|.|^s}$ modulo a finite linear combination of δ_0 and of some of its derivatives are the same. Up to the choice of a new representative $\tilde{f}$ in the class f, we will assume that we have exactly

$$\tilde{f} = \frac{g}{|.|^s} \text{ in } S'(\mathbb{R}^N).$$

Then, $\tilde{f}$ is in $L^1_{loc}(\mathbb{R}^N)$, and $|.|^s\tilde{f}$ is a tempered distribution in $L^1_{loc}(\mathbb{R}^N)$ which satisfies

$$g = |.|^s \tilde{f} \text{ on } \mathbb{R}^N.$$

Finally, $|.|^s\tilde{f}$ is in $C^0_0(\mathbb{R}^N)$, and since for every $n \in \mathbb{N}$,

$$\|\, |.|^s f_n\|_{L^\infty(\mathbb{R}^N)} \le I_N \|\widehat{f_n}\|_{\dot{W}^{s,1}(\mathbb{R}^N)},$$

estimate (28) holds by taking the limit $n \to +\infty$. □

2.2 First estimates for the Gross-Pitaevskii kernels.

In this section, we deduce from Lemma 3 and Corollaries 4, 5 and 6 some L^∞-estimates for the Gross-Pitaevskii kernels, i.e. Theorem 3.

Proof of Theorem 3. We first report some properties of the functions K_0, K_j, $L_{j,k}$ and of their derivatives.

Step 1. *Let $(n,p) \in \mathbb{N}^2$ and f, either the function $d^p\widehat{d^nK_0}$, $d^p\widehat{d^nK_j}$ or $d^p\widehat{d^nL_{j,k}}$. f is a rational fraction on $\mathbb{R}^N$, whose denominator only vanishes at 0 and such that*

$$|.|^{p-n} f \in L^\infty(B(0,1)),$$

and

$$|.|^{p-n+2} f \in L^\infty(B(0,1)^c).$$

Step 1 follows from a straightforward inductive argument based on formulae (11), (12) and (14): we only give its sketch. For instance, for $n = 0$, by formula (11), the function $\widehat{K_0}$ is a rational fraction equal to

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2},$$

so, it satisfies the estimates of Step 1. Moreover, its derivative $\partial_j \widehat{K_0}$ is

$$\partial_j \widehat{K_0}(\xi) = \frac{2\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2} - \frac{4\xi_j|\xi|^4 + 4\xi_j|\xi|^2 - 2c^2\delta_{j,1}\xi_1|\xi|^2}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2}.$$

It is also a rational fraction which satisfies the conclusion of Step 1. The proof then follows from a straightforward induction on p.

Remark. We infer from Step 1 that the behaviour of all those kernels is identical, and in order to simplify the proof, we focus on the function d^nK_0.

We notice that $d^{N-1+n}\widehat{d^nK_0}$ belongs to $L^1(\mathbb{R}^N)$, so, by the standard L^1-L^∞ inequality, $|.|^{N-1+n} d^nK_0$ is bounded on $\mathbb{R}^N$.
To prove the other estimates, we then derive

Step 2. *Let $s \in]0,1[$ and $n \in \mathbb{N}$. The functions*

$$|.|^{N-2+s+n} d^nK_0$$

are bounded on $\mathbb{R}^N$.

Indeed, we apply Corollary 5 to the function

$$\widehat{f} = d^{N-2+n}\widehat{d^nK_0}.$$

We first notice by Step 1 that $\widehat{f}$ is in $L^p(\mathbb{R}^N)$ for $1 < p < \frac{N}{N-2}$. Since $1 < p_s < \frac{N}{N-2}$ for every $0 < s < 1$, $\widehat{f}$ is in $L^{p_s}(\mathbb{R}^N)$ for every $0 < s < 1$ and it only remains to compute

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt \\
&= \int_{\mathbb{R}^N}\left(\int_{|t|\le 1} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dt\right)dy \\
&+ \int_{|t|>1}\left(\int_{|y|>2|t|} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt \\
&+ \int_{|t|>1}\left(\int_{|y|\le 2|t|} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt.
\end{aligned}$$

For the first integral, we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\left(\int_{|t|\le 1} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dt\right)dy &\le \int_0^1\left(\int_{\mathbb{R}^N}\left(\int_{|t|\le 1} \frac{|\nabla\hat{f}(y+\sigma t)|}{|t|^{N+s-1}}dt\right)dy\right)d\sigma \\
&\le \left(\int_{\mathbb{R}^N}|\nabla\widehat{f}(z)|dz\right)\left(\int_{|t|\le 1}\frac{dt}{|t|^{N+s-1}}\right) \\
&\le A\int_{\mathbb{R}^N}|d^{N-1+n}\widehat{d^n K_0}(\xi)|d\xi < +\infty,
\end{aligned}$$

for the second one,

$$\begin{aligned}
&\int_{|t|>1}\left(\int_{|y|>2|t|} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt \\
&\le \int_0^1\left(\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{|\nabla\hat{f}(y+\sigma t)|}{|t|^{N+s-1}}dy\right)dt\right)d\sigma \\
&\le A\int_0^1\left(\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{dy}{|y+\sigma t|^{N+1}}\right)\frac{dt}{|t|^{N+s-1}}\right)d\sigma \\
&\le A\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{dy}{(|y|-|t|)^{N+1}}\right)\frac{dt}{|t|^{N+s-1}} \\
&\le A\left(\int_{|t|>1}\frac{dt}{|t|^{N+s}}\right)\left(\int_{|u|>2}\frac{du}{(|u|-1)^{N+1}}\right) < +\infty,
\end{aligned}$$

and for the last one,

$$\begin{aligned}
&\int_{|t|>1}\left(\int_{|y|\le 2|t|} \frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt \\
&\le 2\int_{|t|>1}\left(\int_{|y|\le 3|t|}|\widehat{f}(y)|dy\right)\frac{dt}{|t|^{N+s}} \\
&\le A\int_{|t|>1}\left(\int_{|y|\le 1}\frac{dy}{|y|^{N-2}} + \int_{1<|y|\le 3|t|}\frac{dy}{|y|^N}\right)\frac{dt}{|t|^{N+s}} \\
&\le A\left(\int_{|t|>1}\frac{dt}{|t|^{N+s}}\right)\left(\int_{|y|\le 1}\frac{dy}{|y|^{N-2}}\right) + A\int_{|t|>1}\frac{\ln(3|t|)}{|t|^{N+s}}dt < +\infty.
\end{aligned}$$

Thus, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz<+\infty,$$

so, $\widehat{f}$ is in $D^{s,1}(\mathbb{R}^N)$. By Corollary 5, $|.|^{N-2+s+n}d^nK_0$ is then bounded on $\mathbb{R}^N$ for every $0<s<1$.

We complete the proof by the next similar step.

Step 3. *Let $s\in]0,1[$ and $n\in\mathbb{N}$. The functions*

$$|.|^{N-1+s+n}d^nK_0$$

are bounded on $\mathbb{R}^N$.

The proof relies on Corollary 4 for the function

$$\hat{f}=d^{N-1+n}\widehat{d^nK_0}.$$

By Step 1, $\widehat{f}$ is in $L^1(\mathbb{R}^N)$ and we compute likewise

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt\\ &= \int_{\mathbb{R}^N}\left(\int_{|t|\geq 1}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dt\right)dy\\ &+\int_{|t|<1}\left(\int_{|y|\leq 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt\\ &+\int_{|t|<1}\left(\int_{|y|>2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt.\end{aligned}$$

For the first integral, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\int_{|t|\geq 1}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dt\right)dy &\leq 2\left(\int_{\mathbb{R}^N}|\hat{f}(z)|dz\right)\left(\int_{|t|\geq 1}\frac{dt}{|t|^{N+s}}\right)\\ &\leq 2\left(\int_{\mathbb{R}^N}|d^{N-1+n}\widehat{d^nK_0}(z)|dz\right)\left(\int_{|t|\geq 1}\frac{dt}{|t|^{N+s}}\right)\\ &<+\infty,\end{aligned}$$

for the second one,

$$\begin{aligned}\int_{|t|<1}\left(\int_{|y|\leq 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt &\leq 2\int_{|t|<1}\left(\int_{|y|\leq 3|t|}|\hat{f}(y)|dy\right)\frac{dt}{|t|^{N+s}}\\ &\leq A\int_{|t|<1}\left(\int_{|y|\leq 3|t|}\frac{dy}{|y|^{N-1}}\right)\frac{dt}{|t|^{N+s}}\\ &\leq A\left(\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\right)\left(\int_{|u|\leq 3}\frac{du}{|u|^{N-1}}\right)\\ &<+\infty,\end{aligned}$$

and for the last one,

$$\begin{aligned}
&\int_{|t|<1}\left(\int_{|y|>2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dy\right)dt\\
&\leq \int_0^1\left(\int_{|t|<1}\left(\int_{|y|>2|t|}|\nabla\widehat{f}(y+\sigma t)|dy\right)\frac{dt}{|t|^{N+s-1}}\right)d\sigma\\
&\leq A\int_{|t|<1}\left(\int_{2>|y|>2|t|}\frac{dy}{(|y|-|t|)^N}+\int_{|y|>2}\frac{dy}{(|y|-|t|)^{N+2}}\right)\frac{dt}{|t|^{N+s-1}}\\
&\leq A\int_{|t|<1}\left(\int_2^{\frac{2}{|t|}}\frac{u^{N-1}}{(u-1)^N}du\right)\frac{dt}{|t|^{N+s-1}}+A\left(\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\right)\left(\int_{|y|>2}\frac{dy}{(|y|-1)^{N+2}}\right)\\
&\leq A\int_{|t|<1}\frac{|\ln(t)|}{|t|^{N+s-1}}dt+A\left(\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\right)\left(\int_{|y|>2}\frac{dy}{(|y|-1)^{N+2}}\right)<+\infty.
\end{aligned}$$

Thus, we also get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz<+\infty,$$

so, $\hat{f}$ is in $W^{s,1}(\mathbb{R}^N)$. By Corollary 4, $|.|^{N+s-1+n}d^nK_0$ is bounded on $\mathbb{R}^N$ for every $0<s<1$, which completes the proofs of Step 3 and Theorem 3. □

Remark. Here, the key ingredient is the form of the Fourier transform $\widehat{K}$ of the kernels.

- $\widehat{K}$ is a rational fraction;
- $\widehat{K}$ is only singular at the origin, where the singularity is of the form $\underset{\xi\to 0}{O}(\frac{1}{|\xi|^\alpha})$;
- at infinity, $\widehat{K}$ is of the form $\underset{|\xi|\to+\infty}{O}(\frac{1}{|\xi|^\beta})$, where $\beta>\alpha$.

We can obtain the algebraic decay of all the kernels whose Fourier transform satisfies similar assumptions by the same argument.

Before improving those estimates, we deduce some L^p-integrability for the Gross-Pitaevskii kernels.

Corollary 7. *Let $(j,k)\in\{1,\dots,N\}^2$. The functions K_0, K_j and $L_{j,k}$ belong to all the spaces $L^p(\mathbb{R}^N)$ for*

$$1<p<\frac{N}{N-2},$$

and their gradients, for

$$1\leq p<\frac{N}{N-1}.$$

Proof. It follows from the estimates of Theorem 3. □

Remark. We conjecture Corollary 7 is optimal, i.e.

- the functions K_0, K_j and $L_{j,k}$ do not belong either to $L^1(\mathbb{R}^N)$, nor to $L^{\frac{N}{N-2}}(\mathbb{R}^N)$;
- their gradients do not belong to $L^{\frac{N}{N-1}}(\mathbb{R}^N)$.

2.3 Critical estimates for the Gross-Pitaevskii kernels.

In this section, we improve the linear estimates of Theorem 3 by proving Theorem 4. It seems very similar to Theorem 3, but its proof is quite different. Indeed, we conjecture that the functions $|.|^{N+n}d^nK_0$, $|.|^{N+n}d^nK_j$ and $|.|^{N+n}d^nL_{j,k}$ do not tend to 0 at infinity. Thus, we cannot prove Theorem 4 from a general inequality deduced from the density of $S(\mathbb{R}^N)$: it would mean that $|.|^{N+n}d^nK_0$, $|.|^{N+n}d^nK_j$ and $|.|^{N+n}d^nL_{j,k}$ tend to 0 at infinity. Actually, the proof relies on the following lemma.

Lemma 8. *Let $1 \leq j \leq N$. The function*

$$x \mapsto x_j f(x)$$

is bounded on $B(0,1)^c$ for every $f \in S'(\mathbb{R}^N)$ such that

(i) $\widehat{f}$ is of class C^2 on $\mathbb{R}^N \setminus \{0\}$,

(ii) $(|.|^{N+1} + |.|^{N-1})\widehat{f}$ is bounded on $\mathbb{R}^N$,

(iii) $(|.|^{N+2} + |.|^{N})\partial_j\widehat{f}$ is bounded on $\mathbb{R}^N$,

(iv) $(|.|^{N+3} + |.|^{N+1})\partial_j\partial_k\widehat{f}$ are bounded on $\mathbb{R}^N$ for $1 \leq k \leq N$.

Proof. Indeed, we establish the formula

Step 1. *Let $\lambda > 0$. The following equality holds almost everywhere*

$$\begin{aligned} x_j f(x) =& \frac{i}{(2\pi)^N}\Bigg(\int_{B(0,\lambda)^c} \partial_j\widehat{f}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j\widehat{f}(\xi)(e^{ix.\xi}-1)d\xi \\ &+\frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j\widehat{f}(\xi)e^{ix.\xi}d\xi\Bigg). \end{aligned} \tag{30}$$

Let $g \in S(\mathbb{R}^N)$. We have

$$< x_j f, \widehat{g} >=< f, x_j\widehat{g} >= -i < f, \widehat{\partial_j g} >= -i < \widehat{f}, \partial_j g > .$$

By assumption (ii), $\widehat{f}$ is in $L^1(\mathbb{R}^N)$, so, we can write

$$< x_j f, \widehat{g} >= -i\int_{\mathbb{R}^N} \widehat{f}(\xi)\partial_j g(\xi)d\xi,$$

and by integrating by parts, we deduce

$$\begin{aligned} < x_j f, \widehat{g} >= -i < \widehat{f}, \partial_j g > = i\int_{B(0,\lambda)^c} \partial_j\widehat{f}(\xi)g(\xi)d\xi + i\int_{B(0,\lambda)} \partial_j\widehat{f}(\xi)(g(\xi)-g(0))d\xi \\ + \frac{ig(0)}{\lambda}\int_{S(0,\lambda)} \xi_j\widehat{f}(\xi)d\xi. \end{aligned}$$

Since g is in $S(\mathbb{R}^N)$, it satisfies

$$g(\xi) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{g}(x)e^{ix.\xi}dx,$$

which yields

$$< x_j f, \widehat{g} > = \frac{i}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{g}(x) \left(\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi \right.$$
$$\left. + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi \right) dx.$$

As the function

$$x \mapsto \int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix\cdot\xi} d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi$$

belongs to $L^1_{loc}(\mathbb{R}^N)$, by standard duality, formula (30) is valid almost everywhere. To proceed further, we estimate each term of formula (30).

Step 2. *The following inequalities hold for every $x \in \mathbb{R}^N$ and $\lambda > 0$,*

$$\begin{cases} |\int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi| \leq A\lambda |x|, \\ |\int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) e^{ix.\xi} d\xi| \leq A\lambda, \end{cases}$$

where A is a real number independent of x and λ.

Indeed, on one hand, we know

$$\forall u \in \mathbb{R}, |e^{iu} - 1| \leq A|u|,$$

and therefore,

$$\left| \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi \right| \leq A|x| \int_{B(0,\lambda)} |\partial_j \widehat{f}(\xi)||\xi| d\xi.$$

By assumption (iii), we get

$$\left| \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi \right| \leq A|x| \int_{B(0,\lambda)} \frac{d\xi}{|\xi|^{N-1}} \leq A\lambda |x|.$$

On the other hand, we deduce likewise from assumption (ii),

$$\left| \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) e^{ix.\xi} d\xi \right| \leq A \int_{S(0,\lambda)} \frac{d\xi}{|\xi|^{N-2}} \leq A\lambda.$$

Therefore, it only remains a single integral to estimate.

Step 3. *The following inequality holds for every $x \in B(0,1)^c$ and $0 < \lambda < 1$,*

$$\left| \int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi \right| \leq A \left(1 + \frac{1}{\lambda |x|} \right),$$

where A is a real number independent of x and λ.

Indeed, we have

$$\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi = \int_{B(0,1)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi.$$

For the first integral, we deduce from assumption (iii),

$$\left| \int_{B(0,1)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi \right| \leq \int_{B(0,1)^c} |\partial_j \widehat{f}(\xi)| d\xi \leq A.$$

For the second one, by assumption,

$$|x| > 1,$$

so, there is some integer $1 \leq k \leq N$ such that

$$|x_k| \geq \frac{|x|}{N}.$$

By integrating by parts, we then get

$$\begin{aligned}
\int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi &= \frac{1}{ix_k} \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) \partial_k (e^{ix.\xi}) d\xi \\
&= \frac{1}{ix_k} \left(- \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \partial_k \widehat{f}(\xi) e^{ix.\xi} d\xi \right. \\
&\quad + \int_{S(0,1)} \partial_j \widehat{f}(\xi) e^{ix.\xi} \xi_k d\xi \\
&\quad \left. - \frac{1}{\lambda} \int_{S(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} \xi_k d\xi \right),
\end{aligned}$$

and by assumptions (iii) and (iv),

$$\begin{aligned}
\left| \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi \right| &\leq \frac{N}{|x|} \left(A \int_{B(0,1)\setminus B(0,\lambda)} \frac{d\xi}{|\xi|^{N+1}} + A + \frac{A}{\lambda} \int_{S(0,\lambda)} \frac{d\xi}{|\xi|^{N-1}} \right) \\
&\leq \frac{A}{\lambda |x|} + A.
\end{aligned}$$

Thus, we get

$$\left| \int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi \right| \leq \frac{A}{\lambda |x|} + A.$$

Finally, by Steps 1, 2 and 3, we get for every $x \in B(0,1)^c$ and $0 < \lambda < 1$,

$$|x_j f(x)| \leq A\lambda |x| + \frac{A}{\lambda |x|} + A.$$

By choosing

$$\lambda = \frac{1}{|x|},$$

we obtain the result of Lemma 8. □

Now, we can deduce the proof of Theorem 4.

Proof of Theorem 4. By Step 1 of the proof of Theorem 3, the functions $d^{N-1+n}\widehat{d^n K_0}$, $d^{N-1+n}\widehat{d^n K_j}$ and $d^{N-1+n}\widehat{d^n L_{j,k}}$ satisfy the four assumptions of Lemma 8, which implies Theorem 4. □

2.4 Estimates for the composed Riesz kernels.

We focus next on the kernels $R_{j,k}$, for which we have the explicit expression (17). Indeed, if f is a smooth function, and if $g_{j,k}$ is the function defined by

$$\forall \xi \in \mathbb{R}^N, \widehat{g_{j,k}}(\xi) = \widehat{R_{j,k}}(\xi)\hat{f}(\xi),$$

we have

$$\forall x \in \mathbb{R}^N, g_{j,k}(x) = A_N \int_{|y|>1} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} f(x-y)dy \\ + A_N \int_{|y|\leq 1} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} (f(x-y) - f(x))dy.$$

Therefore, we do not need to study the decay of the kernels $R_{j,k}$ directly, and instead, we may restrict ourselves to the decay of the functions $g_{j,k}$ with suitable assumptions on f. In that context, we recall some useful facts, which are presumably well-known to the experts. For sake of completeness, we also mention the proofs.

Proposition 6. *Let f a function C^1 on $\mathbb{R}^N$ which belongs to $L^p(\mathbb{R}^N)$ for $1 < p \leq +\infty$, and suppose there is*

$$\delta \in]0, N]$$

such that for every $\beta \in [0, \delta[$,

$$\begin{cases} |.|^\beta f \in L^\infty(\mathbb{R}^N), \\ |.|^\beta \nabla f \in L^\infty(\mathbb{R}^N). \end{cases}$$

Then, the functions

$$|.|^\beta g_{j,k} \in L^\infty(\mathbb{R}^N)$$

for every $(j,k) \in \{1, \ldots, N\}^2$ and for every $\beta \in [0, \delta[$.

Proof. Recalling formula (17), we first denote

$$g_{j,k}(x) = A_N \int_{|y|>1} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} f(x-y)dy \\ + A_N \int_{|y|\leq 1} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} (f(x-y) - f(x))dy \\ = I_1(x) + I_2(x).$$

Then, if we fix $\beta \in [0, \delta[$, we get

$$|x|^\beta |I_1(x)| \leq A \int_{|y|>1} \frac{|x-y|^\beta |f(x-y)|}{|y|^N} dy + A \int_{|y|>1} \frac{|f(x-y)|}{|y|^{N-\beta}} dy.$$

Hence, if $p > \frac{N}{N-\beta}$, we have

$$\int_{|y|>1} \frac{|f(x-y)|}{|y|^{N-\beta}} dy \leq ||f||_{L^{p'}(\mathbb{R}^N)} \left(\int_{|y|>1} \frac{dy}{|y|^{p(N-\beta)}} \right)^{\frac{1}{p}} < +\infty,$$

and if $\beta < \delta - \varepsilon$ and $|x| > 4$, then,

$$\begin{aligned}\int_{|y|>1} \frac{|x-y|^\beta |f(x-y)|}{|y|^N} dy &\leq A \int_{|y|>1} \frac{dy}{|y|^N |x-y|^\varepsilon} \\ &\leq \frac{A}{|x|^\varepsilon} \int_{|t|>\frac{1}{|x|}} \frac{dt}{|t|^N |\frac{x}{|x|} - t|^\varepsilon} \\ &\leq \frac{A}{|x|^\varepsilon} \int_{\frac{1}{|x|}<|t|<\frac{1}{2}} \frac{dt}{|t|^N} + \frac{A}{|x|^\varepsilon} \int_{\frac{1}{2}<|t|<\frac{3}{2}} \frac{dt}{|\frac{x}{|x|} - t|^\varepsilon} \\ &+ \frac{A}{|x|^\varepsilon} \int_{|t|>\frac{3}{2}} \frac{dt}{|t|^N (|t|-1)^\varepsilon} \\ &\leq \frac{A \ln |x|}{|x|^\varepsilon} + A + \frac{A}{|x|^\varepsilon} \int_{|t-\frac{x}{|x|}|<\frac{1}{2}} \frac{dt}{|\frac{x}{|x|} - t|^\varepsilon} \\ &\leq \frac{A \ln |x|}{|x|^\varepsilon} + A < +\infty,\end{aligned}$$

whereas, if $|x| \leq 4$, we get

$$\int_{|y|>1} \frac{|x-y|^\beta |f(x-y)|}{|y|^N} dy \leq A \int_{1<|y|<5} \frac{dy}{|y|^N} + A \int_{|y|>5} \frac{dy}{|y|^N (|y|-4)^\varepsilon} < +\infty.$$

Thus, $|.|^\beta I_1$ is bounded on $\mathbb{R}^N$. Likewise, we have for I_2,

$$|x|^\beta I_2(x) \leq A \int_{|y|\leq 1} \frac{|x-y|^\beta |f(x-y) - f(x)|}{|y|^N} dy + A \int_{|y|\leq 1} \frac{|f(x-y) - f(x)|}{|y|^{N-\beta}} dy.$$

On one hand, if $\beta < \delta - \varepsilon$, we compute

$$\begin{aligned}\int_{|y|\leq 1} \frac{|x-y|^\beta |f(x-y) - f(x)|}{|y|^N} dy &\leq ||\nabla f||_{L^\infty(B(x,1))} (|x|+1)^\beta \int_{|y|\leq 1} \frac{dy}{|y|^{N-1}} \\ &\leq \frac{A}{(1+|x|)^\varepsilon} < +\infty.\end{aligned}$$

On the other hand, we get if $\beta = 0$,

$$\int_{|y|\leq 1} \frac{|f(x-y) - f(x)|}{|y|^N} dy \leq A \int_{|y|\leq 1} \frac{dy}{|y|^{N-1}} < +\infty,$$

whereas if $\beta > 0$,

$$\int_{|y|\leq 1} \frac{|f(x-y) - f(x)|}{|y|^{N-\beta}} dy \leq A \int_{|y|\leq 1} \frac{dy}{|y|^{N-\beta}}.$$

Therefore, $|.|^\beta I_2$ is also bounded on $\mathbb{R}^N$, such as $|.|^\beta g_{j,k}$. □

Remark. In fact, a similar proposition holds for the Riesz kernels.

Actually, we will make use of the next more precise proposition in the critical case. It is also presumably well-known to the experts, but for sake of completeness, we also mention the proof.

Proposition 7. *Let f a function C^1 on $\mathbb{R}^N$ which belongs to $L^1(\mathbb{R}^N)$, and suppose that*

$$\begin{cases} (1+|.|^N) f \in L^\infty(\mathbb{R}^N), \\ (1+|.|^{N+1}) \nabla f \in L^\infty(\mathbb{R}^N). \end{cases}$$

Then, the functions

$$|.|^N g_{j,k} \in L^\infty(\mathbb{R}^N)$$

for every $(j,k) \in \{1,\dots,N\}^2$.

Proof. Recalling formula (17) once more, we notice

$$\begin{aligned} g_{j,k}(x) = A_N \int_{|y|>\frac{|x|}{4}, |x-y|>\frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y) dy \\ + A_N \int_{|x-y|\leq \frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y) dy \\ + A_N \int_{|y|\leq \frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} (f(x-y) - f(x)) dy \\ = I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

For the first integral, we compute

$$\begin{aligned} |I_1(x)| &\leq A_N \int_{|y|>\frac{|x|}{4}, |x-y|>\frac{|x|}{4}} \frac{dy}{|y|^N |x-y|^N} \\ &\leq \frac{A_N}{|x|^N} \int_{|z|>\frac{1}{4}, |\frac{x}{|x|}-z|>\frac{1}{4}} \frac{dz}{|z|^N |\frac{x}{|x|} - z|^N} \\ &\leq \frac{A_N}{|x|^N} \int_{|z|>\frac{1}{4}, |e_1-z|>\frac{1}{4}} \frac{dz}{|z|^N |e_1 - z|^N} \\ &\leq \frac{A_N}{|x|^N}, \end{aligned}$$

for the second one,

$$\begin{aligned} |I_2(x)| &\leq \frac{A_N}{|x|^N} \int_{|x-y|\leq \frac{|x|}{4}} |f(x-y)| dy \\ &\leq \frac{A_N}{|x|^N} \int_{|t|<\frac{|x|}{4}} |f(t)| dt \\ &\leq \frac{A_N}{|x|^N}, \end{aligned}$$

and for the last one,

$$\begin{aligned} |I_3(x)| &\leq A_N \int_{|y|\leq \frac{|x|}{4}} |y|^{1-N} |x|^{-N-1} dy \\ &\leq \frac{A_N}{|x|^N}. \end{aligned}$$

Thus, $|.|^N g_{j,k}$ is bounded on $\mathbb{R}^N$. □

3 Decay at infinity.

In this last part, we study the algebraic decay of the functions η, $\nabla(\psi\theta)$, ∇v and of their derivatives by the inductive argument yet explained in the introduction (see Lemmas 1 and 2), which was introduced by J.L. Bona and Yi A. Li [8], and A. de Bouard and J.C.

Saut [14] (see also the articles of M. Maris [40, 41] for many more details).
We first prove a refined energy estimate based on Lemma 4, which provides some algebraic decay for the functions η, $\nabla(\psi\theta)$ and ∇v. Then, by convolution equations (10) and (13), we deduce inductively Theorem 5 which gives some decay rates for all those functions.

3.1 A refined energy estimate.

We first give an energy estimate for v thanks to arguments from F. Béthuel, G. Orlandi and D. Smets [7]. It will yield in the next section some algebraic decay for the functions η, $\nabla(\psi\theta)$ and ∇v.

Proposition 8. *If v is a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$, there exists some real number $\alpha > 0$ such that the integral*

$$\int_{\mathbb{R}^N} |x|^\beta e(v)(x)dx$$

is finite for every $0 \leq \beta < \alpha$.

The proof relies on Lemma 4 proved by F. Béthuel, G. Orlandi and D. Smets [7] for small c. For sake of completeness, we mention the proof of Lemma 4 for every $0 \leq c < \sqrt{2}$.

Proof of Lemma 4. We first invoke Lemma 6 to choose some real number R so large that

$$v = \rho e^{i\theta} \text{ on } B(0,R)^c.$$

By equation (2), we then compute

$$-\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2), \tag{31}$$

$$\mathrm{div}(\rho^2\nabla\theta) = -\frac{c}{2}\partial_1\rho^2, \tag{32}$$

on the set $B(0,R)^c$.
Then, fix $\lambda > R$ and denote $\Omega = B(0,\lambda) \setminus B(0,R)$, and $\theta_R = \frac{1}{|\mathbb{S}_R|}\int_{\mathbb{S}_R}\theta$. We first multiply equation (31) by $\rho^2 - 1$, which gives by integrating by parts,

$$\begin{aligned} 2\int_\Omega \rho|\nabla\rho|^2 - \int_{\mathbb{S}_\lambda} \partial_\nu\rho(\rho^2-1) + \int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) \\ + \int_\Omega \rho(\rho^2-1)|\nabla\theta|^2 + c\int_\Omega \rho(\rho^2-1)\partial_1\theta = -\int_\Omega \rho(\rho^2-1)^2. \end{aligned} \tag{33}$$

We already know that $\partial_\nu\rho(\rho^2-1)$ belongs to $L^1(B(0,R)^c)$, so, we can construct an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$, and such that

$$\int_{\mathbb{S}_{\lambda_n}} \partial_\nu\rho(\rho^2-1) \underset{n\to+\infty}{\to} 0.$$

By taking the limit at infinity in equation (33), we get

$$\begin{aligned} 2\int_{B(0,R)^c} \rho|\nabla\rho|^2 + \int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) + \int_{B(0,R)^c} \rho(\rho^2-1)|\nabla\theta|^2 \\ + c\int_{B(0,R)^c} \rho(\rho^2-1)\partial_1\theta = -\int_{B(0,R)^c} \rho(\rho^2-1)^2. \end{aligned} \tag{34}$$

We also get such a result by multiplying equation (32) by $\theta - \theta_R$ and by integrating by parts,

$$\int_\Omega \rho^2|\nabla\theta|^2 - \int_{\mathbb{S}_\lambda} \rho^2\partial_\nu\theta(\theta-\theta_R) + \int_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R)$$
$$= -\frac{c}{2}\int_\Omega (\rho^2-1)\partial_1\theta + \frac{c}{2}\int_{\mathbb{S}_\lambda} (\rho^2-1)\nu_1(\theta-\theta_R) - \frac{c}{2}\int_{\mathbb{S}_R} (\rho^2-1)\nu_1(\theta-\theta_R).$$

By Theorem 2, $\nabla\theta$ and $1-\rho^2$ belong to $L^{\frac{N}{N-1}}(B(0,R)^c)$, so, we can construct another increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$, and such that

$$\lambda_n \int_{\mathbb{S}_{\lambda_n}} (|\nabla\theta|^{\frac{N}{N-1}} + |1-\rho^2|^{\frac{N}{N-1}}) \underset{n\to\infty}{\to} 0.$$

Since

$$\begin{cases} |\int_{\mathbb{S}_\lambda} \rho^2\partial_\nu\theta(\theta-\theta_R)| \leq A\int_{\mathbb{S}_\lambda} |\partial_\nu\theta| \leq A(\lambda\int_{\mathbb{S}_\lambda}|\partial_\nu\theta|^{\frac{N}{N-1}})^{\frac{N-1}{N}} \\ |\int_{\mathbb{S}_\lambda} (\rho^2-1)\nu_1(\theta-\theta_R)| \leq A\int_{\mathbb{S}_\lambda} |1-\rho^2| \leq A(\lambda\int_{\mathbb{S}_\lambda}|1-\rho^2|^{\frac{N}{N-1}})^{\frac{N-1}{N}}, \end{cases}$$

we get

$$\begin{aligned} &\int_{B(0,R)^c} \rho^2|\nabla\theta|^2 + \int_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R) \\ &= -\frac{c}{2}\left(\int_{B(0,R)^c} (\rho^2-1)\partial_1\theta + \int_{\mathbb{S}_R} (\rho^2-1)\nu_1(\theta-\theta_R)\right). \end{aligned} \tag{35}$$

By adding equations (34) and (35), we infer

$$\begin{aligned} \int_{B(0,R)^c} e(v) = &-\frac{c}{2}\int_{B(0,R)^c} \rho(\rho^2-1)\partial_1\theta - \frac{1}{2}\int_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R) \\ &- \frac{c}{4}\int_{\mathbb{S}_R} (\theta-\theta_R)(\rho^2-1)\nu_1 + \int_{B(0,R)^c} (1-\rho)\left(\frac{|\nabla\rho|^2}{2} + \frac{(1-\rho^2)^2}{4}\right) \\ &- \frac{c}{4}\int_{B(0,R)^c} (1-\rho)(\rho^2-1)\partial_1\theta - \frac{1}{4}\int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) \\ &+ \frac{1}{4}\int_{B(0,R)^c} \rho(1-\rho^2)|\nabla\theta|^2. \end{aligned} \tag{36}$$

It remains to evaluate each term in the right member of equation (36). For the first one, we can write

$$\left|\frac{c}{2}\int_{B(0,R)^c} \rho(\rho^2-1)\partial_1\theta\right| \leq \frac{c}{\sqrt{2}}\int_{B(0,R)^c} \left(\frac{\rho^2\partial_1\theta^2}{2} + \frac{(1-\rho^2)^2}{4}\right) \leq \frac{c}{\sqrt{2}}\int_{B(0,R)^c} e(v).$$

For the next one, we get by Sobolev-Poincaré inequality,

$$\begin{aligned} \left|\frac{1}{2}\int_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R)\right| &\leq A\left(\int_{\mathbb{S}_R} \rho^2\partial\nu\theta^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}_R} (\theta-\theta_R)^2\right)^{\frac{1}{2}} \\ &\leq AR\left(\int_{\mathbb{S}_R} \rho^2\partial\nu\theta^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}_R} \partial_\nu\theta^2\right)^{\frac{1}{2}} \\ &\leq AR\int_{\mathbb{S}_R} e(v), \end{aligned}$$

and likewise,

$$\begin{cases} |\frac{c}{4}\int_{\mathbb{S}_R}(\theta-\theta_R)(\rho^2-1)| \le AR\int_{\mathbb{S}_R} e(v) \\ |\int_{\mathbb{S}_R}\partial_\nu\rho(\rho^2-1)| \le A\int_{\mathbb{S}_R} e(v). \end{cases}$$

In order to estimate the other terms, we fix $\varepsilon > 0$, and by Lemma 5, we choose R sufficiently large such as $|\rho - 1|$ and $|\nabla\theta|$ are less than ε on the domain $B(0,R)^c$. For such an R, we have

$$\begin{cases} |\int_{B(0,R)^c}(1-\rho)\left(\frac{|\nabla\rho|^2}{2}+\frac{(1-\rho^2)^2}{4}\right)| \le \varepsilon\int_{B(0,R)^c} e(v) \\ |\frac{c}{4}\int_{B(0,R)^c}(1-\rho)(\rho^2-1)\partial_1\theta| \le A\varepsilon\int_{B(0,R)^c} e(v) \\ |\frac{1}{4}\int_{B(0,R)^c}\rho(1-\rho^2)|\nabla\theta|^2| \le A\varepsilon\int_{B(0,R)^c} e(v) \end{cases}$$

which finally gives,

$$\int_{B(0,R)^c} e(v) \le \left(\frac{c}{\sqrt{2}} + A\varepsilon\right)\int_{B(0,R)^c} e(v) + AR\int_{\mathbb{S}_R} e(v).$$

If ε is sufficiently small such as

$$\frac{c}{\sqrt{2}} + A\varepsilon < 1,$$

it yields

$$\int_{B(0,R)^c} e(v) \le A_c R\int_{\mathbb{S}_R} e(v).$$

Denoting $J(R) = \int_{B(0,R)^c} e(v)$, we get for R sufficiently large

$$J(R) \le -A_c R J'(R)$$

which gives

$$J(R) \le \frac{C}{R^{\frac{1}{A_c}}}.$$

Lemma 4 then holds for $\alpha_c = \frac{1}{A_c}$. □

Finally, we deduce Proposition 8 from Lemma 4.

Proof of Proposition 8. The case $\beta = 0$ being immediate, we choose $\beta \in]0, \alpha_c[$ and compute

$$\begin{aligned} \int_{\mathbb{R}^N}|x|^\beta e(v)(x)dx &= \int_0^{+\infty} r^\beta\int_{\mathbb{S}_r} e(v)dr \\ &= -\left[r^\beta\int_r^{+\infty}\int_{\mathbb{S}_\rho} e(v)d\rho\right]_0^{+\infty} + \beta\int_0^{+\infty} r^{\beta-1}\left(\int_r^{+\infty}\int_{\mathbb{S}_\rho} e(v)d\rho\right)dr \\ &= \beta\int_0^{+\infty} r^{\beta-1}\left(\int_{B(0,r)^c} e(v)\right)dr < +\infty. \end{aligned}$$

□

Remark. Proposition 8 is crucial to initialise the proof of the next section.

3.2 Decay of the functions η and ∇v.

In this section, we prove Theorem 5, which gives some algebraic decay for the functions η, $\nabla(\psi\theta)$, ∇v and their derivatives.
The proof of Theorem 5 essentially follows inductively from the arguments developed in the introduction in Lemmas 1 and 2. However, as mentioned above, it is more involved, since we have to consider a system of convolution equations and to handle the singularities of the convolution kernels near the origin. Thus, we will split the argument in four subsections.
In subsection 3.2.1, we show that the functions η and ∇v belong to some spaces $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small. It provides the initialisation needed by Lemma 2.
In subsection 3.2.2, we apply the inductive argument of Lemma 2 to equations (10) and (13) to improve the algebraic decay of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v.
In subsection 3.2.3, we deduce inductively some algebraic decay for the derivatives of the functions η, $\nabla(\psi\theta)$ and ∇v by the same argument.
Finally, in subsection 3.2.4, we improve once more the decay rate of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v by using the critical estimates of Theorem 4 instead of Theorem 3, and Proposition 7 instead of Proposition 6.

3.2.1 Initialisation of the proof of Theorem 5.

In this first subsection, we deduce some algebraic decay for the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v from Proposition 8.

Proposition 9. *There exists some real number $\alpha > 0$ such that*

$$(\eta, \nabla\eta, \nabla(\psi\theta), \nabla v) \in M_\beta^\infty(\mathbb{R}^N)^4$$

for every $0 \leq \beta < \alpha$.

Proof. The proof relies on equations (10),

$$\eta = K_0 * F + 2c \sum_{j=1}^{N} K_j * G_j,$$

and (13),

$$\partial_j(\psi\theta) = \frac{c}{2} K_j * F + c^2 \sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k.$$

We estimate each term of those equations beginning by equation (10).

Step 1.1. *Let $j \in \{1, \ldots, N\}$. Then,*

- $K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$,

for β sufficiently small.

Indeed, we have for $0 \leq \beta < N$ and for every $x \in \mathbb{R}^N$,

$$|x|^\beta |K_0 * F(x)| \leq A \left(\int_{\mathbb{R}^N} |x-y|^\beta |K_0(x-y)||F(y)|dy + \int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta |F(y)|dy \right).$$

On one hand, by Theorem 3,

$$|.|^\beta K_0 \in L^p(\mathbb{R}^N),$$

for

$$\frac{N}{N-\beta} < p < \frac{N}{N-\beta-2}$$

if $0 \leq \beta < N-2$, and for

$$p > \frac{N}{N-\beta}$$

if $N-2 \leq \beta < N$. For such a p, by Theorem 2, F is in $L^{p'}(\mathbb{R}^N)$, so, we get by Young's inequality,

$$\|(|.|^\beta K_0) * F\|_{L^\infty(\mathbb{R}^N)} \leq \| \, |.|^\beta K_0\|_{L^p(\mathbb{R}^N)} \|F\|_{L^{p'}(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 7,

$$K_0 \in L^q(\mathbb{R}^N)$$

for every $1 < q < \frac{N}{N-2}$, and by Proposition 8, there is some real number $\alpha > 0$ such that

$$\forall \beta \in [0, \alpha[, \int_{\mathbb{R}^N} |.|^\beta (|F| + |G|) < +\infty.$$

Then, consider $\beta \in [0, \frac{2\alpha}{N}[$. There is $1 < q < \frac{N}{N-2}$ such that

$$\beta q' < \alpha.$$

As F tends to 0 at infinity by Lemma 5, we deduce

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |F|^{q'} \leq A \int_{\mathbb{R}^N} |.|^{\beta q'} |F| < +\infty.$$

Thus, for every $\beta \in [0, \frac{2\alpha}{N}[$, we get

$$\|K_0 * (|.|^\beta F)\|_{L^\infty(\mathbb{R}^N)} \leq \|K_0\|_{L^q(\mathbb{R}^N)} \| \, |.|^\beta F\|_{L^{q'}(\mathbb{R}^N)}.$$

Therefore, the function $K_0 * (|.|^\beta F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^\beta K_0 * F$: the proof being identical for the functions $|.|^\beta K_j * G_j$ by replacing F by G_j, we omit it.

By equation (10) and Step 1.1, η belongs to $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small.

To prove the remaining results, we turn to the function $\nabla \eta$ which satisfies the equation

$$\nabla \eta = \nabla K_0 * F + 2c \sum_{j=1}^{N} \nabla K_j * G_j \tag{37}$$

and we establish similarly

Step 1.2. *Let $j \in \{1, \ldots, N\}$. Then,*

- $\nabla K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $\nabla K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$,

for β sufficiently small.

Indeed, we have for $0 \leq \beta < N+1$ and for every $x \in \mathbb{R}^N$,

$$|x|^\beta |\nabla K_0 * F(x)| \leq A \int_{\mathbb{R}^N} \Big(|x-y|^\beta |\nabla K_0(x-y)||F(y)| + |\nabla K_0(x-y)||y|^\beta |F(y)| \Big) \, dy.$$

On one hand, by Theorem 3,

$$|.|^\beta \nabla K_0 \in L^p(\mathbb{R}^N),$$

for

$$\frac{N}{N+1-\beta} < p < \frac{N}{N-1-\beta}$$

if $0 \leq \beta < N-1$, and for

$$p > \frac{N}{N+1-\beta}$$

if $N-1 \leq \beta < N+1$. For such a p, by Theorem 2, F is in $L^{p'}(\mathbb{R}^N)$, so, we get by Young's inequality,

$$\|(|.|^\beta \nabla K_0) * F\|_{L^\infty(\mathbb{R}^N)} \leq \| \, |.|^\beta \nabla K_0\|_{L^p(\mathbb{R}^N)} \|F\|_{L^{p'}(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 7,

$$\nabla K_0 \in L^q(\mathbb{R}^N)$$

for $1 \leq q < \frac{N}{N-1}$, and by Proposition 8, there is some real number $\alpha > 0$ such that

$$\forall \beta \in [0, \alpha[, \int_{\mathbb{R}^N} |.|^\beta (|F| + |G|) < +\infty.$$

Then, consider $\beta \in [0, \frac{\alpha}{N}[$. There is $1 \leq q < \frac{N}{N-1}$ such that

$$\beta q' < \alpha.$$

As F tends to 0 at infinity by Lemma 5, we deduce

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |F|^{q'} \leq A \int_{\mathbb{R}^N} |.|^{\beta q'} |F| < +\infty.$$

Thus, for every $\beta \in [0, \frac{\alpha}{N}[$, we get

$$\|\nabla K_0 * (|.|^\beta F)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \| \, |.|^\beta F\|_{L^{q'}(\mathbb{R}^N)}.$$

Hence, $\nabla K_0 * (|.|^\beta F)$ is bounded on $\mathbb{R}^N$, such as $|.|^\beta \nabla K_0 * F$: the proof being identical for $|.|^\beta \nabla K_j * G_j$ by replacing F by G_j, we omit it.

By equation (37) and Step 1.2, $\nabla \eta$ belongs to $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small.

We then turn to the function $\nabla(\psi\theta)$ and study equation (13). The study of the terms involving the kernels K_j and $L_{j,k}$ is strictly identical to Step 1.1, and gives

Step 1.3. *Let $(j,k) \in \{1, \ldots, N\}^2$. Then,*

- $K_j * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $L_{j,k} * G_k \in M_\beta^\infty(\mathbb{R}^N)$,

for β sufficiently small.

It only remains to evaluate the functions $R_{j,k} * G_k$.

Step 1.4. *Let* $(j,k) \in \{1,\dots,N\}^2$. *Then,*

$$R_{j,k} * G_k \in M_\beta^\infty(\mathbb{R}^N),$$

for β *sufficiently small.*

Indeed, on one hand, by Steps 1.1 and 1.2, the functions $|.|^\beta \eta$ and $|.|^\beta \nabla\eta$ are bounded on $\mathbb{R}^N$ for β sufficiently small.
On the other hand, $\nabla(\psi\theta)$ is C^∞ on $\mathbb{R}^N$ and is given by

$$\nabla(\psi\theta) = \frac{iv.\nabla v}{|v|^2}$$

at infinity. However, by Theorem 2, ∇v and $d^2 v$ are bounded on $\mathbb{R}^N$, and by Lemma 5,

$$|v(x)| = \rho(x) \underset{|x|\to+\infty}{\to} 1,$$

so, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$ are bounded on $\mathbb{R}^N$.
At last, G is C^∞ on $\mathbb{R}^N$ and is equal to

$$G = \eta\nabla(\psi\theta)$$

at infinity, so, $|.|^\beta G$ and $|.|^\beta \nabla G$ are bounded on $\mathbb{R}^N$ for β sufficiently small. As G and ∇G belong to all the spaces $L^p(\mathbb{R}^N)$ by Step 1 of the proof of Proposition 4, it follows from Proposition 6 that $|.|^\beta R_{j,k} * G_k$ is bounded for β sufficiently small.
By equation (13) and Steps 1.3 and 1.4, $\nabla(\psi\theta)$ then belongs to $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small.
We complete the proof of Proposition 9 by deducing that

$$\nabla v \in M_\beta^\infty(\mathbb{R}^N)$$

for β sufficiently small. Indeed, by Theorem 2, ∇v is C^∞ on $\mathbb{R}^N$ and satisfies at infinity

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2 = \frac{|\nabla\eta|^2}{4\rho^2} + \rho^2|\nabla(\psi\theta)|^2.$$

Since

$$\rho(x) \underset{|x|\to+\infty}{\to} 1$$

by Lemma 5, we infer from the study of $\nabla\eta$ and $\nabla(\psi\theta)$ that $|.|^\beta\nabla v$ is bounded on $\mathbb{R}^N$ for β sufficiently small. □

3.2.2 Inductive argument for the decay of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v.

We then improve by the inductive argument of Lemma 2 the decay rate of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v.

Proposition 10. *Assume there is some real number* $\alpha > 0$ *such that*

$$(\eta, \nabla\eta, \nabla(\psi\theta), \nabla v) \in M_\beta^\infty(\mathbb{R}^N)^4,$$

for

$$\beta \in [0, \alpha[.$$

Then,

$$(\eta, \nabla(\psi\theta), \nabla v) \in M_\beta^\infty(\mathbb{R}^N)^3,$$

for

$$\beta \in [0, \min\{N, 2\alpha\}[,$$

and

$$\nabla\eta \in M_\beta^\infty(\mathbb{R}^N),$$

for

$$\beta \in [0, \min\{N+1, 2\alpha\}[.$$

Proof. The proof is quite similar to the previous one. We first use the quadratic form of F and G.

Step 2.1. *The function*

$$|.|^\beta(|F| + |G|)$$

is bounded for every

$$\beta \in [0, 2\alpha[.$$

By formulae (8) and (9), F and G are C^∞ on $\mathbb{R}^N$ and are given by

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta),$$

and

$$G = \eta\nabla(\psi\theta)$$

at infinity. Step 2.1 then follows directly from the assumptions of Proposition 10.

Now, we study the function η by equation (10).

Step 2.2. *Let $j \in \{1, \dots, N\}$ and $\beta \in [0, \min\{N, 2\alpha\}[$. Then,*

- $K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$.

Indeed, we have likewise for $0 \leq \beta < N$ and for every $x \in \mathbb{R}^N$,

$$|x|^\beta|K_0 * F(x)| \leq A\left(\int_{\mathbb{R}^N} |x-y|^\beta|K_0(x-y)||F(y)|dy + \int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta|F(y)|dy\right).$$

On one hand, we have already proved in the proof of Step 1.1 that for every $\beta \in [0, N[$,

$$\|(|.|^\beta K_0) * F\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 7,

$$K_0 \in L^q(\mathbb{R}^N)$$

for $1 < q < \frac{N}{N-2}$, so, we get for every $\beta \in [0, 2\alpha[$,

$$\|K_0 * (|.|^\beta F)\|_{L^\infty(\mathbb{R}^N)} \leq \|K_0\|_{L^q(\mathbb{R}^N)}\| \ |.|^\beta F\|_{L^{q'}(\mathbb{R}^N)}.$$

By Step 2.1, there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |F|^{q'} < +\infty.$$

Thus, the function $K_0 * (|.|^\beta F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^\beta K_0 * F$: the proof being identical for the functions $|.|^\beta K_j * G_j$ by replacing F by G_j, we omit it.

By Step 2.2 and equation (10), Proposition 10 holds for the function η.
We then estimate the function $\nabla \eta$ by equation (37).

Step 2.3. *Let $j \in \{1, \dots, N\}$ and $\beta \in [0, \min\{2\alpha, N+1\}[$. Then,*

- $\nabla K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $\nabla K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$.

In Step 1.2, we showed that

$$(|.|^\beta \nabla K_0) * F \in L^\infty(\mathbb{R}^N)$$

for $\beta \in [0, N+1[$. We also deduce from Corollary 7 that for $q \in [1, \frac{N}{N-1}[$ sufficiently small and for every $\beta \in [0, 2\alpha[$,

$$\|\nabla K_0 * (|.|^\beta F)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \| \, |.|^\beta F\|_{L^{q'}(\mathbb{R}^N)} < +\infty.$$

Similarly, the functions $\nabla K_j * (|.|^\beta G_j)$ and $(|.|^\beta \nabla K_j) * G_j$ are bounded for $\beta \in [0, \min\{N+1, 2\alpha\}[$, which completes the proof of Step 2.3.

The result of Proposition 10 for the function $\nabla \eta$ follows from Step 2.3 and equation (37), and we can turn to the function $\nabla(\psi\theta)$, which satisfies equation (13). The study of the terms involving the kernels K_j and $L_{j,k}$ is strictly identical to Steps 2.2 and 2.3.

Step 2.4. *Let $(j,k) \in \{1, \dots, N\}^2$. Then,*

- $K_j * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $L_{j,k} * G_k \in M_\beta^\infty(\mathbb{R}^N)$,

for every $\beta \in [0, \min\{N, 2\alpha\}[$.

Thus, it only remains to evaluate the functions $R_{j,k} * G_k$.

Step 2.5. *Let $(j,k) \in \{1, \dots, N\}^2$ and $\beta \in [0, \min\{N, 2\alpha\}[$. Then,*

$$R_{j,k} * G_k \in M_\beta^\infty(\mathbb{R}^N).$$

Indeed, by Steps 2.2 and 2.3, the functions $|.|^\beta \eta$ and $|.|^\beta \nabla \eta$ are bounded on $\mathbb{R}^N$ for $\beta \in [0, \min\{N, 2\alpha\}[$, so, the functions $|.|^\beta G$ and $|.|^\beta \nabla G$ are also bounded on $\mathbb{R}^N$ for β in this range. Since G and ∇G belong to the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p \leq +\infty$ by Step 1 of the proof of Proposition 4, by Proposition 6, the functions $|.|^\beta R_{j,k} * G_k$ are bounded for β in this range.

Subsequently, by Steps 2.4 and 2.5, and equation (13), $\nabla(\psi\theta)$ is in $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0, \min\{N, 2\alpha\}[$.

We conclude the proof of Proposition 10 by showing that

$$\nabla v \in M^\infty_\beta(\mathbb{R}^N)$$

for $\beta \in [0, \min\{N, 2\alpha\}[$. Indeed, by Theorem 2, ∇v is C^∞ on $\mathbb{R}^N$ and satisfies at infinity

$$|\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2 = \frac{|\nabla \eta|^2}{4\rho^2} + \rho^2 |\nabla(\psi\theta)|^2.$$

Since

$$\rho(x) \underset{|x|\to+\infty}{\to} 1$$

by Lemma 5, it follows from the study of $\nabla\eta$ and $\nabla(\psi\theta)$ that $|.|^\beta \nabla v$ is bounded on $\mathbb{R}^N$ for $0 \leq \beta < \min\{N, 2\alpha\}$. □

3.2.3 Inductive argument for the decay of the derivatives of the functions η, $\nabla(\psi\theta)$ and ∇v.

We deduce from Propositions 9 and 10 that

$$(\eta, \nabla(\psi\theta), \nabla v) \in M^\infty_\beta(\mathbb{R}^N)^3,$$

for every $\beta \in [0, N[$ and

$$\nabla\eta \in M^\infty_\beta(\mathbb{R}^N),$$

for every $\beta \in [0, N+1[$. We now estimate the decay of the derivatives of η, $\nabla(\psi\theta)$ and ∇v.

Proposition 11. *Let $\alpha \in \mathbb{N}^N$. Then,*

$$(\eta, \partial^\alpha \nabla(\psi\theta), \partial^\alpha \nabla v) \in M^\infty_\beta(\mathbb{R}^N)^3,$$

for every $\beta \in [0, N[$ and

$$\partial^\alpha \nabla\eta \in M^\infty_\beta(\mathbb{R}^N),$$

for every $\beta \in [0, N+1[$.

Proof. The proof is by induction on $|\alpha| \in \mathbb{N}$: the case $\alpha = 0$ follows from Propositions 9 and 10.
Now, assume that Proposition 11 holds for every $|\alpha| \leq p$ and fix $\alpha \in \mathbb{N}^N$ such that $|\alpha| = p+1$. As in the proof of Proposition 10, we first estimate F and G.

Step 3.1. *The function*

$$|.|^\beta (|\partial^\gamma F| + |\partial^\gamma G|)$$

is bounded for every $\beta \in [0, N[$ and for every $\gamma \in \mathbb{N}^N$ such that $|\gamma| = p+1$.

Step 3.1 relies on Leibnitz' formula and on the quadratic form of F and G.
F is a C^∞ function on $\mathbb{R}^N$ given by

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta)$$

at infinity. By Leibnitz' formula, we compute

$$\partial^\gamma F = 2 \sum_{\delta \leq \gamma} c_{\delta,\gamma} [\partial^{\gamma-\delta} \nabla v . \partial^\delta \nabla v + \partial^{\gamma-\delta}\eta . \partial^\delta \eta - c\partial^{\gamma-\delta}\eta . \partial^\delta \partial_1(\psi\theta)],$$

where the coefficients $c_{\delta,\gamma}$ are positive integers.
On one hand, by the assumption of induction,

$$|.|^\beta(|\partial^\delta \nabla v| + |\partial^\delta \eta| + |\partial^\delta \partial_1(\psi\theta)|) \in L^\infty(\mathbb{R}^N)$$

for $\delta \leq \gamma$ and $\delta \neq \gamma$, and for $\beta \in [0, N[$.
On the other hand, by Theorem 2, $\partial^\gamma \nabla v$, $\partial^\gamma \eta$ and $\partial^\gamma \partial_1(\psi\theta)$ are bounded on $\mathbb{R}^N$, so,

$$|.|^\beta |\partial^\gamma F| \in L^\infty(\mathbb{R}^N)$$

for every $\beta \in [0, N[$.
Likewise, G is a C^∞ function on $\mathbb{R}^N$ given by

$$G = \eta \nabla(\psi\theta)$$

at infinity, so, by the same argument, $|.|^\beta \partial^\gamma G$ is bounded on $\mathbb{R}^N$ for $\beta \in [0, N[$.

We then study the function $\partial^\alpha \nabla \eta$, which satisfies

$$\partial^\alpha \nabla \eta = \nabla K_0 * \partial^\alpha F + 2c \sum_{j=1}^N \nabla K_j * \partial^\alpha G_j. \tag{38}$$

Step 3.2. *Let $j \in \{1, \dots, N\}$ and $\beta \in [0, N[$. Then,*

- $\nabla K_0 * \partial^\alpha F \in M^\infty_\beta(\mathbb{R}^N)$,
- $\nabla K_j * \partial^\alpha G_j \in M^\infty_\beta(\mathbb{R}^N)$.

By Step 3.1, the proof is similar to the proof of Step 2.3. By Step 1 of the proof of Proposition 4, $\partial^\alpha F$ and $\partial^\alpha G_j$ are in all the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p \leq +\infty$ as well as F and G_j. Therefore, we omit it.

Thus, $\partial^\alpha \nabla \eta$ belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta \in [0, N[$.

Now, we turn to the function $\partial^\alpha \partial_j(\psi\theta)$, which satisfies

$$\partial^\alpha \partial_j(\psi\theta) = \frac{c}{2} K_j * \partial^\alpha F + c^2 \sum_{k=1}^N L_{j,k} * \partial^\alpha G_k + \sum_{k=1}^N R_{j,k} * \partial^\alpha G_k. \tag{39}$$

By Step 3.1, the study of the terms involving the kernels K_j and $L_{j,k}$ is strictly identical to Steps 2.2, 2.3, 2.4 or 3.2.

Step 3.3. *Let $(j, k) \in \{1, \dots, N\}^2$. Then,*

- $K_j * \partial^\alpha F \in M^\infty_\beta(\mathbb{R}^N)$,
- $L_{j,k} * \partial^\alpha G_k \in M^\infty_\beta(\mathbb{R}^N)$,

for every $\beta \in [0, N[$.

It only remains to evaluate the functions $R_{j,k} * \partial^\alpha G_k$.

Step 3.4. *Let $(j, k) \in \{1, \dots, N\}^2$ and $\beta \in [0, N[$. Then,*

$$R_{j,k} * \partial^\alpha G_k \in M^\infty_\beta(\mathbb{R}^N).$$

Indeed, let $H_k = \partial^\alpha G_k$. H_k belongs to all the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p \leq +\infty$ by Step 1 of the proof of Proposition 4, and $|.|^\beta H_k$ is bounded on $\mathbb{R}^N$ for every $\beta \in [0, N[$ by Step 3.1.
In order to apply Proposition 6, we claim that $|.|^\beta \nabla H_k$ is bounded on $\mathbb{R}^N$ for every $\beta \in [0, N[$. It follows from Leibnitz' formula as well as in the proof of Step 3.1. Indeed, by formula (9), we have at infinity

$$\nabla G_k = \nabla\eta.\partial_k(\psi\theta) + \eta.\nabla\partial_k(\psi\theta).$$

By Leibnitz' formula, we get

$$\nabla H_k = \sum_{\delta \leq \alpha} c_{\delta,\alpha}(\partial^\delta \nabla\eta.\partial^{\alpha-\delta}\partial_k(\psi\theta) + \partial^\delta\eta.\partial^{\alpha-\delta}\nabla\partial_k(\psi\theta)).$$

The terms involving the highest derivatives are $\partial^\alpha\nabla\eta.\partial_k(\psi\theta)$, $\nabla\eta.\partial^\alpha\partial_k(\psi\theta)$, $\partial^\alpha\eta.\nabla\partial_k(\psi\theta)$, $\eta.\nabla\partial_k(\psi\theta)$. All of them belong to $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0, N[$ because of the assumption of induction and of Step 1 of the proof of Proposition 4. The other terms are also in $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0, N[$ by the same argument. Therefore, $|.|^\beta \nabla H_k$ is bounded on $\mathbb{R}^N$ for every $\beta \in [0, N[$ and we can apply Proposition 6 to end the proof of Step 3.4.

Subsequently, by Steps 3.3 and 3.4, and equation (39), $\partial^\alpha\nabla(\psi\theta)$ is in $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0, N[$.

Then, by Steps 3.2, 3.3 and 3.4, we claim that

$$\partial^\alpha \nabla v \in M_\beta^\infty(\mathbb{R}^N)$$

for $\beta \in [0, N[$. Indeed, ∇v is C^∞ on $\mathbb{R}^N$ and is given by

$$\nabla v = \frac{\nabla\eta}{2\rho} e^{i\psi\theta} + i\rho\nabla(\psi\theta)e^{i\psi\theta}$$

at infinity. Our claim then follows from Theorem 2, Lemma 5, Steps 3.2, 3.3 and 3.4, the chain rule theorem and Leibnitz' formula once more.

At last, we improve Step 3.1 so as to improve the estimate for the function $\partial^\alpha\nabla\eta$.

Step 3.5. *The function*

$$|.|^\beta(|\partial^\gamma F| + |\partial^\gamma G|)$$

is bounded for every $\beta \in [0, 2N[$ and for every $\gamma \in \mathbb{N}^N$ such that $|\gamma| = p + 1$.

The proof is similar to the proof of Step 3.1.
For instance, F is a C^∞ function on $\mathbb{R}^N$ given by

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta)$$

at infinity. By Leibnitz' formula, we compute again

$$\partial^\gamma F = 2\sum_{\delta \leq \gamma} c_{\delta,\gamma}[\partial^{\gamma-\delta}\nabla v.\partial^\delta\nabla v + \partial^{\gamma-\delta}\eta.\partial^\delta\eta - c\partial^{\gamma-\delta}\eta.\partial^\delta\partial_1(\psi\theta)].$$

On one hand, by the assumption of induction, we know

$$|.|^\beta(|\partial^\delta\nabla v| + |\partial^\delta\eta| + |\partial^\delta\partial_1(\psi\theta)|) \in L^\infty(\mathbb{R}^N)$$

for $\delta \leq \gamma$ and $\delta \neq \gamma$, and for $\beta \in [0, N[$.
On the other hand, by Steps 3.2, 3.3 and 3.4,

$$|.|^\beta (|\partial^\gamma \nabla v| + |\partial^\gamma \eta| + |\partial^\gamma \partial_1 (\psi\theta)|) \in L^\infty(\mathbb{R}^N)$$

for every $\beta \in [0, N[$, so,

$$|.|^\beta |\partial^\gamma F| \in L^\infty(\mathbb{R}^N)$$

for every $\beta \in [0, 2N[$.
The proof is identical for $\partial^\gamma G$.

We then deduce from equation (38)

Step 3.6. *Let $j \in \{1, \dots, N\}$ and $\beta \in [0, N+1[$. Then,*

- $\nabla K_0 * \partial^\alpha F \in M^\infty_\beta(\mathbb{R}^N)$,
- $\nabla K_j * \partial^\alpha G_j \in M^\infty_\beta(\mathbb{R}^N)$.

The proof is identical to the proof of Step 3.2 by replacing Step 3.1 by Step 3.5, so, we omit it.

By equation (38), $\partial^\alpha \nabla \eta$ belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta \in [0, N+1[$, which completes the inductive argument of the proof of Proposition 11. □

3.2.4 Critical decay of the functions η, $\nabla(\psi\theta)$ and ∇v.

At last, we study the critical case, i.e. the case $\beta = N$ or $\beta = N + 1$.

Proposition 12. *Let $\alpha \in \mathbb{N}^N$. Then,*

$$(\eta, \partial^\alpha \nabla(\psi\theta), \partial^\alpha \nabla v) \in M^\infty_N(\mathbb{R}^N)^3,$$

and

$$\partial^\alpha \nabla \eta \in M^\infty_{N+1}(\mathbb{R}^N).$$

Proof. The proof is similar to the proofs of Propositions 9, 10 and 11. We first recall some estimates for the functions F and G.

Step 4.1. *The function*

$$|.|^\beta (|\partial^\alpha F| + |\partial^\alpha G|)$$

is bounded on $\mathbb{R}^N$ for every $\alpha \in \mathbb{N}^N$ and $\beta \in [0, 2N[$.

The proof of Step 4.1 is the same as the proof of Step 3.5, so, we omit it.

We then turn to the function η, i.e. to equation (10).

Step 4.2. *Let $j \in \{1, \dots, N\}$. Then,*

- $K_0 * F \in M^\infty_N(\mathbb{R}^N)$,
- $K_j * G_j \in M^\infty_N(\mathbb{R}^N)$.

Indeed, we have for every $x \in \mathbb{R}^N$,

$$|x|^N |K_0 * F(x)| \leq A \left(\int_{\mathbb{R}^N} |x-y|^N |K_0(x-y)||F(y)|dy + \int_{\mathbb{R}^N} |K_0(x-y)||y|^N |F(y)|dy \right).$$

On one hand, by Theorem 4 and Step 1 of the proof of Proposition 4,

$$\|(|.|^N K_0) * F\|_{L^\infty(\mathbb{R}^N)} \leq \| \, |.|^N K_0\|_{L^\infty(\mathbb{R}^N)} \|F\|_{L^1(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 7,

$$K_0 \in L^q(\mathbb{R}^N)$$

for $1 < q < \frac{N}{N-2}$, so,

$$\|K_0 * (|.|^N F)\|_{L^\infty(\mathbb{R}^N)} \leq \|K_0\|_{L^q(\mathbb{R}^N)} \| \, |.|^N F\|_{L^{q'}(\mathbb{R}^N)}.$$

By Step 4.1, there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\| \, |.|^N F\|_{L^{q'}(\mathbb{R}^N)} < +\infty,$$

so, the function $K_0 * (|.|^N F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^N K_0 * F$: the proof being identical for the functions $|.|^N K_j * G_j$ by replacing F by G_j, we omit it.

By Step 4.2 and equation (10), Proposition 12 holds for the function η.
For the functions $\partial^\alpha \nabla \eta$, we study equation (38).

Step 4.3. *Let $j \in \{1, \ldots, N\}$. Then,*

- $\nabla K_0 * \partial^\alpha F \in M^\infty_{N+1}(\mathbb{R}^N)$,
- $\nabla K_j * \partial^\alpha G_j \in M^\infty_{N+1}(\mathbb{R}^N)$.

Indeed, we have for every $x \in \mathbb{R}^N$,

$$|x|^{N+1} |\nabla K_0 * \partial^\alpha F(x)|$$
$$\leq A \int_{\mathbb{R}^N} (|x-y|^{N+1} |\nabla K_0(x-y)||\partial^\alpha F(y)| + |\nabla K_0(x-y)||y|^{N+1} |\partial^\alpha F(y)|)dy.$$

On one hand, by Theorem 4 and Step 1 of the proof of Proposition 4,

$$\|(|.|^{N+1} \nabla K_0) * \partial^\alpha F\|_{L^\infty(\mathbb{R}^N)} \leq \| \, |.|^{N+1} \nabla K_0\|_{L^\infty(\mathbb{R}^N)} \|\partial^\alpha F\|_{L^1(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 7,

$$\nabla K_0 \in L^q(\mathbb{R}^N)$$

for $1 \leq q < \frac{N}{N-1}$, so,

$$\|\nabla K_0 * (|.|^{N+1} \partial^\alpha F)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \| \, |.|^{N+1} \partial^\alpha F\|_{L^{q'}(\mathbb{R}^N)}.$$

By Step 4.1, there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\| \, |.|^{N+1} \partial^\alpha F\|_{L^{q'}(\mathbb{R}^N)} < +\infty,$$

so, the function $\nabla K_0 * (|.|^{N+1} \partial^\alpha F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^{N+1} \nabla K_0 * \partial^\alpha F$: the proof being identical for the functions $|.|^{N+1} \nabla K_j * \partial^\alpha G_j$ by replacing $\partial^\alpha F$ by $\partial^\alpha G_j$, we omit it.

By Step 4.3 and equation (38), Proposition 12 also holds for the function $\partial^\alpha \nabla \eta$.
We then deduce a similar estimate for $\partial^\alpha \partial_j(\psi\theta)$ by equation (39). We first study the terms involving the kernels K_j and $L_{j,k}$.

Step 4.4. *Let* $(j,k) \in \{1,\dots,N\}^2$. *Then,*

- $K_j * \partial^\alpha F \in M_N^\infty(\mathbb{R}^N)$,
- $L_{j,k} * \partial^\alpha G_k \in M_N^\infty(\mathbb{R}^N)$.

The proof is identical to the proof of Steps 4.2 and 4.3, so, we omit it.

Finally, it only remains to evaluate the functions $R_{j,k} * G_k$.

Step 4.5. *Let* $(j,k) \in \{1,\dots,N\}^2$. *Then,*

$$R_{j,k} * \partial^\alpha G_k \in M_N^\infty(\mathbb{R}^N).$$

Indeed, by Step 4.1 and Step 1 of the proof of Proposition 4, $\partial^\alpha G$ and $\partial^\alpha \nabla G$ belong to $L^1(\mathbb{R}^N)$, and $|.|^N \partial^\alpha G$ and $|.|^{N+1}\partial^\alpha \nabla G$ are bounded on $\mathbb{R}^N$: Step 4.5 then follows from Proposition 7.

Steps 4.4 and 4.5 yield the critical decay of $\partial^\alpha \nabla(\psi\theta)$, and we can end the proofs of Proposition 12 and of Theorem 5 by computing the critical decay of the functions $\partial^\alpha \nabla v$. Indeed, ∇v is C^∞ on $\mathbb{R}^N$ and is given by

$$\nabla v = \frac{\nabla \eta}{2\rho} e^{i\psi\theta} + i\rho\nabla(\psi\theta)e^{i\psi\theta}$$

at infinity. The critical decay of $\partial^\alpha \nabla v$ then follows from Theorem 2, Lemma 5, Steps 4.3, 4.4 and 4.5, the chain rule theorem and Leibnitz' formula. □

3.3 Asymptotic decay for the function v.

In this last section, we complete the proof of Theorem 1. We already showed the convergence at infinity of v towards a complex number of modulus one in Corollary 2. We are now in position to prove the second part of Theorem 1.

Proposition 13. *The function* $|.|^{N-1}(v-1)$ *is bounded on* $\mathbb{R}^N$.

Proof. Indeed, by Theorem 5, the function $|.|^N \nabla v$ is bounded on $\mathbb{R}^N$. Since

$$\forall x \in \mathbb{R}^N \setminus \{0\}, v(x) - 1 = -\int_{|x|}^{+\infty} \partial_r v\left(\frac{sx}{|x|}\right) ds,$$

we get

$$\forall x \in \mathbb{R}^N \setminus \{0\}, |v(x)-1| \leq A \int_{|x|}^{+\infty} \frac{ds}{s^N} \leq \frac{A}{|x|^{N-1}},$$

which ends the proofs of Proposition 13 and of Theorem 1. □

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Chapitre V

Asymptotics for subsonic travelling waves in the Gross-Pitaevskii equation.

Abstract.
We investigate the asymptotic behaviour of the subsonic travelling waves of finite energy in the Gross-Pitaevskii equation in dimension larger than two. In particular, we give their first order development at infinity in the case they are axisymmetric, and link it to their energy and momentum.

Introduction.

1 Motivations.

In this article, we focus on the travelling waves in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2) \tag{1}$$

of the form $u(t, x) = v(x_1 - ct, \ldots, x_N)$. The parameter $c \geq 0$ represents the speed of the travelling wave, which moves in direction x_1. The equation for v, which we will consider now, writes

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{2}$$

The Gross-Pitaevskii equation is a physical model for the Bose-Einstein condensation, which is associated at least formally to the so-called Ginzburg-Landau energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |v|^2)^2, \tag{3}$$

and to the momentum

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla v.v. \tag{4}$$

Equation (1) presents an hydrodynamic form

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathrm{v}) = 0, \\ \rho(\partial_t \mathrm{v} + \mathrm{v}.\nabla \mathrm{v}) + \nabla \rho^2 = \rho \nabla \left(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}\right), \end{cases} \tag{5}$$

obtained by using the Madelung transform [37]

$$u = \sqrt{\rho} e^{i\theta},$$

and denoting

$$\mathrm{v} = -2\nabla\theta.$$

Equations (5) are similar to Euler equations for an irrotational ideal fluid with pressure $p(\rho) = \rho^2$. In particular, the speed of the sound waves near the constant solution $u = 1$ is

$$c_s = \sqrt{2}.$$

The travelling waves of finite energy play an important role in the long time dynamics of general solutions. They were thoroughly studied by C.A. Jones, S.J. Putterman and P.H. Roberts [29, 30]. They conjectured that there exist non-constant travelling waves of finite energy only in the subsonic case

$$0 < c < \sqrt{2}.$$

F. Béthuel and J.C. Saut [4, 5] first investigated mathematically this conjecture. In dimension two, they showed the existence of non-constant travelling waves of finite energy for small values of c, and for a sequence of values of $c < \sqrt{2}$ tending to $\sqrt{2}$. They also proved their non-existence for $c = 0$ in every dimension. Their work was complemented in dimension larger than three by F. Béthuel, G. Orlandi and D. Smets [7] and D. Chiron [9], who also showed their existence when c is small. On the other hand, we proved their non-existence for every $c > \sqrt{2}$ [23]. Thus, the problem of their non-existence only remains open in the sonic case $c = \sqrt{2}$ (see [25] however for more details). We will deliberately omit this case and only consider from now on the subsonic travelling waves, i.e. we will assume

$$0 < c < \sqrt{2}.$$

Under this assumption and the additional hypothesis the travelling waves are axisymmetric around axis x_1, C.A. Jones, S.J. Putterman and P.H. Roberts [29, 30] characterised their behaviour at infinity by giving their first order development up to a multiplicative constant of modulus one. In dimension two, they derived a formal asymptotic expansion

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2}) x_2^2} \tag{6}$$

and in dimension three,

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}}. \tag{7}$$

Here, the constant α is the stretched dipole coefficient linked to the energy $E(v)$ and to the scalar momentum in direction x_1, $p(v) = P_1(v)$, by the formulae

$$2\pi\alpha\sqrt{1 - \frac{c^2}{2}} = cE(v) + 2\left(1 - \frac{c^2}{4}\right) p(v) \tag{8}$$

in dimension two, and

$$4\pi\alpha = \frac{c}{2} E(v) + 2p(v) \tag{9}$$

in dimension three.

The goal of this paper is to provide a rigorous derivation of the asymptotic behaviour described in (6), (7), (8) and (9), and a generalisation to every dimension $N \geq 2$.

2 Main results.

Our main results are summed up in the next three theorems. The first one is the most general. We consider any subsonic travelling waves of finite energy in any dimension $N \geq 2$, and prove the existence of their first order development at infinity (which is consistent with conjectures (6) and (7) in dimensions two and three). Moreover, we compute a linear partial differential equation satisfied by the first order term of their asymptotic expansion.

Theorem 1. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$. There exist a complex number λ_∞ of modulus one and a smooth function v_∞ defined from the sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}$ such that*

$$|x|^{N-1}(v(x) - \lambda_\infty) - i\lambda_\infty v_\infty \left(\frac{x}{|x|}\right) \underset{|x| \to +\infty}{\to} 0 \text{ uniformly.}$$

Moreover, the function v_∞ satisfies the following linear partial differential equation on $\mathbb{S}^{N-1}$,

$$\Delta_{\mathbb{S}^{N-1}} v_\infty - \frac{c^2}{2} \partial_1^{\mathbb{S}^{N-1}}(\partial_1^{\mathbb{S}^{N-1}} v_\infty) + c^2(N-1)\sigma_1 \partial_1^{\mathbb{S}^{N-1}} v_\infty + (N-1)(1 + \frac{c^2}{2}(1-(N+1)\sigma_1^2))v_\infty = 0. \tag{10}$$

Remarks. 1. Subsequently, we will always assume that

$$\lambda_\infty = 1.$$

Indeed, if this is not the case, we can study the function $\lambda_\infty^{-1} v$ instead of v: it is also a travelling wave of finite energy and of speed c which satisfies equation (2).

2. Equation (10) is defined on the sphere $\mathbb{S}^{N-1}$ immersed in the space $\mathbb{R}^N$. In order to clarify its sense, we need to explicit some notations for derivations on $\mathbb{S}^{N-1}$. Thus, consider some function $f \in C^\infty(\mathbb{S}^{N-1}, \mathbb{C})$. The notation $\partial_i^{\mathbb{S}^{N-1}}$ is then defined by

$$\forall i \in \{1, \ldots, N\}, \forall x \in \mathbb{S}^{N-1}, \partial_i^{\mathbb{S}^{N-1}} f(x) = \lim_{t \to 0} \frac{f(\frac{x+te_i}{|x+te_i|}) - f(x)}{t},$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^N$. The operator $\Delta_{\mathbb{S}^{N-1}}$ is the usual Laplace-Beltrami operator on the sphere $\mathbb{S}^{N-1}$, given by

$$\forall x \in \mathbb{S}^{N-1}, \Delta_{\mathbb{S}^{N-1}} f(x) = \sum_{i=1}^{N} \partial_i^{\mathbb{S}^{N-1}}(\partial_i^{\mathbb{S}^{N-1}} f)(x).$$

Our next theorems specify the form of the limit function v_∞ in two cases: for the axisymmetric travelling waves, which only depend on the variables x_1 and

$$x_\perp = \sqrt{\sum_{i=2}^{N} x_i^2},$$

in every dimension $N \geq 2$, and for every travelling wave in dimension $N = 2$. In both cases, equation (10) reduces to an ordinary differential equation of second order, which is entirely integrable. In particular, it yields a proof of conjectures (6), (7), (8) and (9) in the axisymmetric case.

Theorem 2. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$, axisymmetric around axis x_1. Then, there exists some constant α such that the function v_∞ is given by*

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}}}. \tag{11}$$

Moreover, the constant α is equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \Big(1 - \frac{c^2}{2}\Big)^{\frac{N-3}{2}} \Big(\frac{4-N}{2} cE(v) + \Big(2 + \frac{N-3}{2} c^2\Big) p(v)\Big). \tag{12}$$

Likewise, in dimension two, we can describe explicitly the asymptotic behaviour of every travelling wave.

Theorem 3. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension two of finite energy and speed $0 < c < \sqrt{2}$. Then, there exist some constants α and β such that the function v_∞ is given by*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2}}. \tag{13}$$

Moreover, the constants α and β are equal to

$$\begin{aligned} \alpha &= \frac{1}{2\pi\sqrt{1 - \frac{c^2}{2}}} \Big(cE(v) + \Big(2 - \frac{c^2}{2}\Big) p(v)\Big), \\ \beta &= \frac{\sqrt{1 - \frac{c^2}{2}}}{\pi} P_2(v). \end{aligned} \tag{14}$$

Remarks. 1. There is a difficulty in the definition of $\vec{P}(v)$. Indeed, the integral which appears in definition (4) is not always convergent for a travelling wave of finite energy. In order to state formulae (12) and (14) rigorously, we define the momentum $\vec{P}(v)$ as

$$\vec{P}(v) = \frac{1}{2} \int_{\mathbb{R}^N} i\nabla v.(v - 1), \tag{15}$$

and the scalar momentum in direction x_1 by

$$p(v) = \frac{1}{2} \int_{\mathbb{R}^N} i\partial_1 v.(v - 1). \tag{16}$$

By [24], those integrals are well-defined in the case of travelling waves of finite energy. However, we will give another equivalent definition of the momentum which is more suitable in our context (see subsection 3.1 of the introduction).

2. Theorem 3 is consistent with the axisymmetric case: assuming $\beta = 0$, we recover the axisymmetric solution of Theorem 2 with the same value of the stretched dipole coefficient α.

The integration of equation (10) seems rather involved in dimension $N \geq 3$: we are not able to compute an explicit formula for the function v_∞ from equation (10). However, we conjecture its expression as follows.

Conjecture 1. *Let v be a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$. Then, there exist some constants α, β_2, ..., β_N such that the function v_∞ is equal to*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + \sum_{j=2}^{N} \beta_j \frac{\sigma_j}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}.$$

Moreover, the constants α and β_j are equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \left(1 - \frac{c^2}{2}\right)^{\frac{N-3}{2}} \left(\frac{4-N}{2} cE(v) + \left(2 + \frac{N-3}{2} c^2\right) p(v)\right),$$

$$\beta_j = \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}} \left(1 - \frac{c^2}{2}\right)^{\frac{N-1}{2}} P_j(v).$$

Remark. In the second part, we will motivate this conjecture. Notice that, in case Conjecture 1 holds, it yields every possible asymptotic behaviour of a travelling wave v of finite energy in the Gross-Pitaevskii equation. In particular, the first order term v_∞ of the asymptotic expansion of v is completely determined by some integral quantities α, β_2, ..., β_N, related to the energy $E(v)$ and the momentum $\vec{P}(v)$.
This raises an interesting question. Consider N real numbers a_1, ..., a_N: is it possible to construct a travelling wave v such that the values of the integral quantities α, β_2, ..., β_N are exactly equal to a_1, ..., a_N ? In other words, is it possible to construct travelling waves v whose asymptotic behaviour correspond to any possible one given by Conjecture 1, or are there other restrictions for the possible asymptotic behaviours ?
To our knowledge, those questions remain open problems. Indeed, the existence results of F. Béthuel and J.C. Saut [4, 5] in dimension two, and F. Béthuel, G. Orlandi and D. Smets [7] and D. Chiron [9] in dimension $N \geq 3$ prove the existence of travelling waves which are assumed to be axisymmetric. However, in this case, we can show that the constants β_2, ..., β_N are all equal to 0 (which is consistent with Theorem 2). Therefore, we do not know any travelling wave for which the values of β_2, ..., β_N are not 0. Thus, a first step to answer to our questions could be to prove the existence of travelling waves which are not axisymmetric.

One of the main interests of Theorems 1, 2 and 3 is their **sharpness**. In order to clarify this claim, we must recall some recent mathematical results. F. Béthuel and J.C. Saut [4, 5] first investigated the asymptotic behaviour of subsonic travelling waves in dimension two. They gave a mathematical evidence for their convergence towards a constant of modulus one at infinity. We complemented their work in [22] by proving the same convergence in every dimension $N \geq 3$. Finally, in [24], we gave a first estimate of their decay at infinity (which is moreover an important starting point of the analysis in this paper).

Theorem 4 ([24]). *In dimension $N \geq 2$, for every travelling wave v for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$, the function*

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

is bounded on $\mathbb{R}^N$.

Theorems 1, 2, 3 and 4 are then sharp because the decay rate at infinity they give is optimal. There exist some travelling waves v such that the function

$$x \mapsto |x|^{\beta}(v(x) - 1)$$

is not bounded on $\mathbb{R}^N$ for any $\beta > N-1$: the decay exponent $N-1$ is the best possible in general (although some travelling waves, the constant ones for instance, can decay faster at infinity). The proof of the existence of such travelling waves v follows from two arguments. The first one is the proof of the existence of non-constant axisymmetric travelling waves by F. Béthuel and J.C. Saut [4, 5] in dimension two, and F. Béthuel, G. Orlandi and D. Smets [7] and D. Chiron [9] in dimension $N \geq 3$. The second one relies on the next corollary of Theorem 2.

Corollary 1. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$, axisymmetric around axis x_1. Then, the constant α is equal to 0 if and only if v is a constant travelling wave.*

Therefore, if we now consider a non-constant axisymmetric travelling wave v, by Theorem 2 and Corollary 1, the function v_∞ is not identically equal to 0 on $\mathbb{S}^{N-1}$. In particular, by Theorem 1, it means that the function

$$x \mapsto |x|^\beta (v(x) - 1)$$

is not bounded on $\mathbb{R}^N$ for any $\beta > N-1$, which shows the sharpness of Theorems 1, 2, 3 and 4.

Now, in the hope of clarifying the proof of Theorem 1 and in order to specify general arguments which could prove fruitful for other equations, we are going to explain how to infer such a theorem.

3 Sketch of the proof of Theorem 1.

Theorem 1 deals with the asymptotic expansion of a travelling wave. We construct the limit at infinity of some function, in our case the function

$$x \mapsto |x|^{N-1}(v(x) - 1),$$

prove that the convergence is uniform and then compute a partial differential equation satisfied by the limit function.

3.1 A new formulation of equation (2).

In [24], we already investigated the asymptotic behaviour of the travelling waves v in the Gross-Pitaevskii equation. In particular, we derived Theorem 4 just mentioned above. The proof of this theorem relies on a new formulation of equation (2), also relevant here, which we are going to recall concisely. The first argument is to state the local smoothness and the Sobolev regularity of a subsonic travelling wave v (see also the articles of F. Béthuel and J.C. Saut in dimension two [4, 5], and of A. Farina [18]).

Proposition 1 ([24]). *If v is a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$, then, v is C^∞, bounded, and the functions $\eta := 1 - |v|^2$ and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p \leq +\infty$.*

It follows that the modulus ρ of v converges to 1 at infinity. In particular, there is some real number R_0 such that

$$\rho \geq \frac{1}{2} \text{ on } B(0, R_0)^c.$$

Since the energy $E(v)$ is finite, it follows (up to a standard degree argument in dimension two) that we may construct a lifting θ of v on $B(0,R_0)^c$, that is a function in $C^\infty(B(0,R_0)^c,\mathbb{R})$ such that

$$v=\rho e^{i\theta}.$$

We next compute new equations for the new functions η and $\nabla\theta$: since θ is not well-defined on $\mathbb{R}^N$, we introduce a cut-off function $\psi\in C^\infty(\mathbb{R}^N,[0,1])$ such that

$$\begin{cases}\psi=0 \text{ on } \mathrm{B}(0,2\mathrm{R}_0),\\ \psi=1 \text{ on } \mathrm{B}(0,3\mathrm{R}_0)^{\mathrm{c}}.\end{cases}$$

All the asymptotic estimates obtained in [22, 24] are independent of the choice of R_0 and ψ, and it will also be the case here. We finally deduce

$$\Delta^2\eta-2\Delta\eta+c^2\partial^2_{1,1}\eta=-\Delta F-2c\partial_1\mathrm{div}(\mathrm{G}),\tag{17}$$

and

$$\Delta(\psi\theta)=\frac{c}{2}\partial_1\eta+\mathrm{div}(\mathrm{G}),\tag{18}$$

where

$$F=2|\nabla v|^2+2\eta^2-2ci\partial_1 v.v-2c\partial_1(\psi\theta),\tag{19}$$

and

$$G=i\nabla v.v+\nabla(\psi\theta).\tag{20}$$

Remark. At this stage, we can state another definition of the momentum

$$\vec{P}(v)=\frac{1}{2}\int_{\mathbb{R}^N}(i\nabla v.v+\nabla(\psi\theta)),$$

and of the scalar momentum in direction x_1

$$p(v)=\frac{1}{2}\int_{\mathbb{R}^N}(i\partial_1 v.v+\partial_1(\psi\theta)).$$

A straightforward computation shows that those new definitions are equivalent to the previous ones given by formulae (15) and (16). In the following, we will always use them in preference to formulae (15) and (16) since they seem more suitable in our context.

It follows from those new definitions and from formulae (19) and (20) that the functions F and G are almost quadratic functions of η and ∇v, related to the density of energy and of momentum. This is an important aspect of equations (17) and (18): they link our new functions η and θ to some superlinear quantities F and G, which have a relevant interpretation in terms of quantities conserved by the Gross-Pitaevskii equations. In particular, the superlinear nature of the nonlinearities is a key ingredient to establish the asymptotic properties of the travelling waves. It motivates the introduction of the new variables η and θ.

3.2 Convolution equations.

It is well-known that the asymptotic properties of solutions to linear partial differential equations are related to the behaviour at infinity of their kernels, and this, for a large deal, also remains valid for many nonlinear problems. Our approach is reminiscent of the article of J.L. Bona and Yi A. Li [8], and also appeared in the articles of A. de Bouard and

J.C. Saut [14], and M. Maris [40, 41]. It consists in transforming the partial differential equations satisfied by the travelling wave (equation (2) in our context) in some convolution equations. In the case of the travelling waves for the Gross-Pitaevskii equation, we already computed such convolution equations in [22, 24]. They follow from equations (17) and (18) and write

$$\eta = K_0 * F + 2c \sum_{j=1}^{N} K_j * G_j, \tag{21}$$

where K_0 and K_j are the kernels of Fourier transform,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{22}$$

respectively

$$\widehat{K_j}(\xi) = \frac{\xi_1 \xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{23}$$

and for every $j \in \{1, \ldots, N\}$,

$$\partial_j(\psi\theta) = \frac{c}{2} K_j * F + c^2 \sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{24}$$

where $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transform,

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2 \xi_j \xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{25}$$

respectively

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}. \tag{26}$$

Equations (21) and (24) are convolution equations with terms of the form $K * f$. The functions K are kernels with explicit Fourier transforms which are rational fractions. The functions f are nonlinear functions of η, ∇v and $\nabla(\psi\theta)$.

Our purpose is now to compute the limit at infinity of various weighted functions, for instance

$$x \mapsto |x|^N \eta(x).$$

By the previous convolution equations, it reduces to get the limit at infinity of functions of the type

$$x \mapsto |x|^p K * f(x) = \int_{\mathbb{R}^N} |x|^p K(x-y) f(y) dy, \tag{27}$$

where p is equal to N, K refers to one of the kernels K_0, K_j, $L_{j,k}$ or $R_{j,k}$ and f to the functions F or G. We will handle this problem, which is of independent interest[1], by invoking the dominated convergence theorem. Here, a main part of the analysis is devoted to study the properties of the kernel K, leaving the nonlinear nature of the function f aside for the moment.[2]

[1] A similar analysis will be carried out on the solitary waves for the Kadomtsev-Petviashvili equation (see [27]).

[2] If the function f had compact support, then the limit at infinity of $K * f$ would be directly deduced from the limit of K. In our subsequent analysis, we also have to take into account the decay of f using nonlinear arguments.

3.3 Main properties of the kernels and pointwise convergence at infinity.

In this section, we derive a number of results for our model function (27), which enter directly in the proof of Theorem 1 and which rely on the dominated convergence theorem as mentioned above. More precisely, we wish to establish limits of functions of the form (27), as $|x| \to +\infty$, depending on the value of p and the form of K and f.

Step 1. *Pointwise convergence of the kernels.*

The first step is to prove the pointwise convergence when $|x|$ tends to $+\infty$ of the integrand, i.e.

$$y \mapsto |x|^p K(x-y),$$

where the function K is a kernel whose Fourier transform is known explicitly, actually in our case a rational fraction (the second step being the domination of the integrand).

Remark. It can depend on the direction of the convergence $\sigma = \frac{x}{|x|}$: denoting $x = R\sigma$ where $R > 0$ and $\sigma \in \mathbb{S}^{N-1}$, we are reduced to study the pointwise convergence of the functions

$$y \mapsto R^p K(R\sigma - y)$$

when R tends to $+\infty$ for every $\sigma \in \mathbb{S}^{N-1}$.

Our argument relies on the properties of the Fourier transform of the kernel K. Indeed, we introduce the space of functions

$$\widehat{\mathcal{K}}(\mathbb{R}^N) = \{u \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}), \forall i \in \mathbb{N}, d^i u \in M_i^\infty(\mathbb{R}^N) \cap M_{i+2}^\infty(\mathbb{R}^N)\},$$

where $M_\alpha^\infty(\mathbb{R}^N)$ is defined by

$$M_\alpha^\infty(\mathbb{R}^N) = \{u : \mathbb{R}^N \mapsto \mathbb{C} \ / \ \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} = \sup\{|x|^\alpha |u(x)|, x \in \mathbb{R}^N\} < +\infty\},$$

for every $\alpha > 0$.

Remark. The choice of the spaces is suggested by the form of the Fourier transforms of the kernels K_0, K_j and $L_{j,k}$. They belong to $\widehat{\mathcal{K}}(\mathbb{R}^N)$ by formulae (22), (23) and (25). However, we can introduce some variants for other equations.

Now, we can specify the pointwise convergence of some functions whose Fourier transforms are in $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Indeed, we claim

Theorem 5. *Let $\alpha \in \mathbb{N}^N$ and $K \in S'(\mathbb{R}^N, \mathbb{C})$. Assume its Fourier transform $\widehat{K}$ is a rational fraction*

$$\widehat{K} = \frac{P}{Q},$$

which belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$ and such that

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, Q(\xi) \neq 0.$$

Then, there exists a measurable function $K_\infty^\alpha \in L^\infty(\mathbb{S}^{N-1}, \mathbb{C})$ such that

$$\forall (\sigma, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N, R^{N+|\alpha|} \partial^\alpha K(R\sigma - y) \underset{R \to +\infty}{\to} K_\infty^\alpha(\sigma). \tag{28}$$

Remark. In particular, we prove the pointwise convergence of all the derivatives of the kernels K which satisfy the assumptions of Theorem 5: it will be very useful in the following.

As mentioned above, Theorem 5 relies on the Fourier transform of the kernels K through the next lemma which already appeared in [24].

Lemma 1. *Let $(\sigma, y, R) \in \mathbb{S}^{N-1} \times \mathbb{R}^N \times \mathbb{R}_+^*$ and assume $|y| < R$ and $\sigma_j \neq 0$ for some integer $1 \leq j \leq N$. Consider a tempered distribution $K \in S'(\mathbb{R}^N, \mathbb{C})$ such that its Fourier transform is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Then, we have*

$$R^N K(R\sigma - y) = \frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N} \Bigg(\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma - y)} d\xi + \int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi) (e^{i\xi.(R\sigma-y)} - 1) d\xi + R \int_{S(0,\frac{1}{R})} \xi_j \partial_j^{N-1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \Bigg). \tag{29}$$

The proof of Theorem 5 then follows from applying the dominated convergence theorem to formula (29).

There are many other ways to study the convergences as in (28), but the use of the Fourier transforms of the kernels seems well-adapted to the context of partial differential equations, where we know them explicitly. However, in some cases, we know the explicit expression of the kernel K. It allows to bypass Theorem 5 for the computation of the limit of (27) by direct computations. This is the case for the so-called composed Riesz kernels $R_{j,k}$. Indeed, if f is a smooth function and if we denote $g_{j,k} = R_{j,k} * f$, we compute

$$\forall x \in \mathbb{R}^N, g_{j,k}(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \int_{|x-y|>1} \frac{\delta_{j,k}|x-y|^2 - N(x-y)_j(x-y)_k}{|x-y|^{N+2}} f(y) dy + \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \int_{|x-y|\leq 1} \frac{\delta_{j,k}|x-y|^2 - N(x-y)_j(x-y)_k}{|x-y|^{N+2}} (f(y) - f(x)) dy. \tag{30}$$

Here, the difficulty to apply the dominated convergence theorem does not come from the limit at infinity of the kernels, but instead, from the domination of this convergence.

Step 2. *Domination of the convergence.*

The second step is to dominate the integrand, given by

$$y \mapsto |x|^p K(x-y) f(y),$$

independently of $x \in \mathbb{R}^N$. In order to do so, we assume for instance that f is a smooth function on $\mathbb{R}^N$ with some algebraic decay, i.e. f and some of its derivatives belong to some space $C^\infty(\mathbb{R}^N) \cap M_\alpha^\infty(\mathbb{R}^N)$ for some real number $\alpha > 0$.

Remark. The choice of such assumptions is suggested by the algebraic decay of the functions F and G. Indeed, in [24], we computed the algebraic decay of the functions η, $\nabla(\psi\theta)$ and ∇v by an argument due to J.L. Bona and Yi A. Li [8], and A. de Bouard and J.C Saut [14] (see also the articles of M. Maris [40, 41] for many more details).

Proposition 2 ([24]). *Let $\alpha \in \mathbb{N}^N$. Then, the functions η, $\nabla(\psi\theta)$ and ∇v satisfy*

- $(\eta, \partial^\alpha \nabla(\psi\theta), \partial^\alpha \nabla v) \in M_N^\infty(\mathbb{R}^N)^3$,

- $\partial^\alpha \nabla \eta \in M^\infty_{N+1}(\mathbb{R}^N)$.

By Propositions 1 and 2, and formulae (19) and (20), the functions F and G are smooth on $\mathbb{R}^N$ and belong to $M^\infty_{2N}(\mathbb{R}^N)$, which explains the choice of the assumptions on f. However, it is possible to introduce some variants for other equations.

Under such assumptions for the function f, it remains to dominate the kernel K. It may be straightforward when we know its exact expression (for instance, in the case of the composed Riesz kernels by formula (30)). However, a suitable approach seems once more to estimate the algebraic decay of K. In many cases, we know the Fourier transform of K. Therefore, we can invoke some formula like (29) to obtain their algebraic decay. In [24], we handled this difficulty for the so-called Gross-Pitaevskii kernels K_0, K_j and $L_{j,k}$, and for their derivatives.

Proposition 3 ([24]). *Let $N-2 < \alpha \leq N$, $n \in \mathbb{N}$ and $(j,k) \in \{1, \ldots, N\}^2$. The functions $d^n K_0$, $d^n K_j$ and $d^n L_{j,k}$ belong to $M^\infty_{\alpha+n}(\mathbb{R}^N)$.*

Proving such a proposition for the kernel K (with possible different rates of decay) and using the assumptions on the function f with a suitable value of α enables to dominate the function

$$y \mapsto |x|^p K(x-y) f(y)$$

on $\mathbb{R}^N$. We can then apply the dominated convergence theorem to get the pointwise convergence at infinity of (27), that is the existence of the limit of the function

$$R \mapsto R^p K * f(R\sigma)$$

when R tends to $+\infty$ for every $\sigma \in \mathbb{S}^{N-1}$.

We can illustrate this argument for the travelling waves for Gross-Pitaevskii equation, where it can be applied to equations (21) and (24). In this case, the kernels K_0, K_j and $L_{j,k}$ satisfy the assumptions of Theorem 5 by formulae (22), (23) and (25). Therefore, we can compute their limit at infinity by Theorem 5. Moreover, they belong to the space of functions

$$\mathcal{K}(\mathbb{R}^N) = \{u \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}), \forall n \in \mathbb{N}, \forall \alpha \in]N-2, N], d^n u \in M^\infty_{\alpha+n}(\mathbb{R}^N)\}$$

by Proposition 3. Therefore, by the argument of domination just above, all those kernels satisfy

Lemma 2. *Let $1 \leq j,k \leq N$ and assume the function $K : \mathbb{R}^N \mapsto \mathbb{C}$ is in $\mathcal{K}(\mathbb{R}^N)$ and its Fourier transform is a rational fraction which is only singular at the origin and belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. We consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N \cap M^\infty_{2N+1}(\mathbb{R}^N)^N$,

and we denote $g = K * f$. Then, we have for every $\sigma \in \mathbb{S}^{N-1}$,

- $R^N g(R\sigma) \underset{R\to+\infty}{\to} K_\infty(\sigma) \int_{\mathbb{R}^N} f(x)dx$,
- $R^{N+1} \partial_j g(R\sigma) \underset{R\to+\infty}{\to} K^j_\infty(\sigma) \int_{\mathbb{R}^N} f(x)dx$.

- $R^{N+2}\partial^2_{j,k}g(R\sigma) \underset{R\to+\infty}{\to} K^{j,k}_\infty(\sigma)\int_{\mathbb{R}^N} f(x)dx.$

Remarks. 1. We do not need to assume (ii) to prove the assertions on the pointwise convergence of the functions g and $\partial_j g$: we just need to suppose (ii) in the case of the functions $\partial^2_{j,k}g$.

2. The notations K_∞, K^j_∞ and $K^{j,k}_\infty$ denote the limits at infinity of the kernels K, $\partial_j K$ and $\partial^2_{j,k}K$ given by Theorem 5. In particular, we prove the pointwise convergence at infinity of some derivatives of g towards those limits. It will be very useful to compute some partial differential equations like equation (10). However, it introduces some technical difficulties on which we will come back in subsections 3.5 and 3.6.

3. For other equations, we can obtain the domination very differently. In particular, the algebraic decay conditions appearing in (i) and (ii) are suitable for our equations, but they can be modified in another context. In the article of J.L. Bona and Yi A. Li [8], domination for a different class of equations in dimension one is obtained using a different type of argument.

The following lemma yields another illustration of the above argument for the composed Riesz kernels. It will also be useful to prove Theorem 1.

Lemma 3. *Let $1 \leq j,k,l \leq N$ and $\sigma \in \mathbb{S}^{N-1}$. We consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N+1}(\mathbb{R}^N)$,

(iii) $d^2 f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N+2}(\mathbb{R}^N)$,

and we denote $g = R_{j,k} * f$. Then, we have

- $R^N g(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k)\int_{\mathbb{R}^N} f(x)dx.$

- $R^{N+1}\partial_l g(R\sigma) \underset{R\to+\infty}{\to} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(-(\delta_{j,k}\sigma_l + \delta_{j,l}\sigma_k + \delta_{k,l}\sigma_j) + (N+2)\sigma_j\sigma_k\sigma_l)\int_{\mathbb{R}^N} f(x)dx.$

Remarks. 1. We do not need to assume (iii) to show the existence of the pointwise limit of the function g. Moreover, the algebraic decay conditions appearing in (i), (ii) and (iii) should be fixed appropriately for different equations.

2. In Lemma 3 like in Lemma 2, we also prove the pointwise convergence at infinity of the gradient of g. It also introduces some technical difficulties on which we will come back in subsections 3.5 and 3.6.

Finally, by convolution equations (21) and (24), Lemmas 2 and 3 yield the pointwise convergence at infinity of the functions η and θ.

Proposition 4. *Let $\sigma \in \mathbb{S}^{N-1}$ and $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$. Then, there exist some bounded measurable functions η^α_∞ and θ^α_∞ on $\mathbb{S}^{N-1}$ such that*

$$\begin{cases} R^{N+|\alpha|}\partial^\alpha\eta(R\sigma) \underset{R\to+\infty}{\to} \eta^\alpha_\infty(\sigma), \\ R^{N-1+|\alpha|}\partial^\alpha\theta(R\sigma) \underset{R\to+\infty}{\to} \theta^\alpha_\infty(\sigma). \end{cases}$$

Remark. In particular, we prove the pointwise convergence at infinity of some derivatives of η and θ. Though it introduces some technical difficulties on which we will come back in subsections 3.5 and 3.6, it is a decisive step to derive equation (10).

On the other hand, in Theorem 1, we would like rather more than the pointwise convergence of the function

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

towards its limit v_∞. We would like to prove its uniform convergence, i.e. whether the function

$$\sigma \mapsto R^{N-1}(v(R\sigma) - 1)$$

converges to v_∞ in $L^\infty(\mathbb{S}^{N-1})$ when R tends to $+\infty$. Coming back to our model problem (27), it means that we must prove whether the function

$$\sigma \mapsto R^p K * f(R\sigma) = R^p \int_{\mathbb{R}^N} K(R\sigma - y)f(y)dy$$

converges in $L^\infty(\mathbb{S}^{N-1})$ when R tends to $+\infty$.

3.4 Uniformity of the convergence.

To solve this difficulty, our argument relies on Ascoli-Arzela's theorem. Indeed, we already know the existence of a pointwise limit at infinity, so, it will give the uniformity of the convergence. However, Ascoli-Arzela's theorem requires some compactness: we deduce it from the algebraic decay of the gradient of the function $K * f$. For instance, the sequence of functions

$$\sigma \mapsto R^p \nabla^{\mathbb{S}^{N-1}}(K * f)(R\sigma)$$

is uniformly bounded on $\mathbb{S}^{N-1}$, which yields the desired compactness.

Thus, in the context of the Gross-Pitaevskii equation, we convert the pointwise convergence of Proposition 4 in a uniform one.

Proposition 5. *There exist some functions* $(\eta_\infty, v_\infty) \in C^1(\mathbb{S}^{N-1})^2$ *and* $\theta_\infty \in C^2(\mathbb{S}^{N-1})$ *such that*

- $R^N \eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty(\sigma)$ in $C^1(\mathbb{S}^{N-1})$,
- $R^{N-1}\theta(R\sigma) \underset{R\to+\infty}{\to} \theta_\infty(\sigma)$ in $C^2(\mathbb{S}^{N-1})$,
- $R^{N-1}(v(R\sigma) - 1) \underset{R\to+\infty}{\to} v_\infty(\sigma)$ in $C^1(\mathbb{S}^{N-1})$.

Remark. Actually, we prove the convergence at infinity of η, θ and v in some spaces $C^1(\mathbb{S}^{N-1})$ or $C^2(\mathbb{S}^{N-1})$, better than $L^\infty(\mathbb{S}^{N-1})$. It will be fruitful to derive equation (10).

The main difficulty here is to compute the gradient of the function $K * f$. Indeed, the gradient of such a convolution is not always the convolution $(\nabla K) * f$, in particular if the kernel K is not sufficiently smooth. We will see how to overcome such a difficulty in the next subsection.

3.5 Derivation of equation (10).

In the previous subsections, we obtained a uniform limit at infinity, denoted $L_\infty : \mathbb{S}^{N-1} \to \mathbb{C}$, for the function

$$x \mapsto |x|^p K * f(x).$$

An ultimate goal for this equation and similar ones would be to obtain an explicit formula for L_∞. However, this seems rather difficult, though presumably not completely out of reach (see Conjecture 1 for the Gross-Pitaevskii equation). Instead, we compute an elliptic partial differential equation satisfied by L_∞, namely equation (10) in our context. In some cases, for instance assuming L_∞ is axisymmetric, this equation may lead to the explicit form of L_∞ (see Theorems 2 and 3).

In order to derive such an equation, we take the limit at infinity of the partial differential equation satisfied by the function $K * f$ on $\mathbb{R}^N$ (equation (2) in our case). The implementation of this argument requires some precise knowledge of the convergence at infinity of some derivatives of the convolution $K * f$ to the corresponding derivatives of L_∞. In order to obtain it, we face a new difficulty related to the singularity at the origin of the kernels. Indeed, many of the derivatives of our kernels present a non-integrable singularity at the origin, and therefore, we are not allowed to differentiate the convolution equation without additional care. The method to overcome this difficulty is reminiscent of some classical arguments in distribution theory, using integral formulae. More precisely, consider a kernel K which belongs to $\mathcal{K}(\mathbb{R}^N)$. Its gradient K is in $L^1(\mathbb{R}^N)$, which yields

$$\nabla(K * f) = (\nabla K) * f,$$

provided that f belongs for instance to some space $L^p(\mathbb{R}^N)$. However, we cannot write

$$d^2(K * f) = (d^2 K) * f,$$

mainly since we do not know enough integrability for the second derivative of K. Yet, we can find an explicit expression for the second derivative of $K * f$, provided that f is sufficiently smooth.

Lemma 4. *Let $1 \leq j, k \leq N$ and $K \in \mathcal{K}(\mathbb{R}^N)$. Consider a function $f \in C^1(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N$,

and denote $g = K * f$. Then, the second order partial derivative $\partial^2_{j,k} g$ of g is continuous on $\mathbb{R}^N$ and satisfies

$$\begin{aligned}\forall x \in \mathbb{R}^N, \partial^2_{j,k} g(x) = &\int_{B(0,1)^c} \partial^2_{j,k} K(y) f(x-y) dy + \int_{B(0,1)} \partial^2_{j,k} K(y)(f(x-y) - f(x)) dy \\ &+ \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy \right) f(x).\end{aligned} \tag{31}$$

Remarks. 1. Conditions (i) and (ii) are suitable in our context, since the functions F and G previously defined in equations (19) and (20) satisfy such conditions. However, they can be chosen differently for other equations.

2. Formula (31) is quite similar to the expected expression $(\partial^2_{j,k}K) * f$, which cannot hold since the function $\partial^2_{j,k}K$ presents a singularity at the origin. Indeed, the function K has a double partial derivative $D^2_{j,k}K$ in the sense of distributions, which is equal to

$$D^2_{j,k}K = \partial^2_{j,k}K 1_{B(0,1)^c} + PV(\partial^2_{j,k}K 1_{B(0,1)}) + \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy \right) \delta_0,$$

where $PV(\partial^2_{j,k}K 1_{B(0,1)})$ is the principal value at the origin of the function $\partial^2_{j,k}K$, given by

$$\forall \phi \in C^\infty_c(B(0,1)), < PV(\partial^2_{j,k}K 1_{B(0,1)}), \phi >= \int_{B(0,1)} \partial^2_{j,k}K(x)(\phi(x) - \phi(0))dx.$$

Then, the double partial derivative in the sense of distribution of $K * f$ is equal to the distribution $D^2_{j,k}K * f$, which yields formula (31).

Likewise, we can compute explicit formulae for the first and second order derivatives of the composed Riesz kernels.

Lemma 5. *Let $1 \leq j, k, l, m \leq N$ and denote*

$$\forall y \in \mathbb{R}^N \setminus \{0\}, R_{j,k}(y) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}}.$$

We consider a function $f \in C^2(\mathbb{R}^N)$ such that

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(iii) $d^2 f \in L^\infty(\mathbb{R}^N)$,

and we set $g = R_{j,k} * f$. Then, g is C^1 on $\mathbb{R}^N$ and satisfies for every $x \in \mathbb{R}^N$,

$$\begin{aligned}\partial_l g(x) = & \int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y) dy + \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x)) dy \\ & + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x) - y.\nabla f(x)) dy.\end{aligned} \tag{32}$$

Moreover, if f belongs to $C^3(\mathbb{R}^N)$ and verifies

(iv) $d^3 f \in L^\infty(\mathbb{R}^N)$,

g is C^2 on $\mathbb{R}^N$ and verifies for every $x \in \mathbb{R}^N$,

$$\begin{aligned}\partial^2_{l,m} g(x) = & \int_{B(0,1)^c} \partial^2_{l,m} R_{j,k}(y) f(x-y) dy + \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) \\ & - \frac{1}{2} d^2 f(x)(y,y)) dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (\partial_m f(x) - y.\nabla \partial_m f(x)) dy + \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) \\ & (f(x) - y.\nabla f(x) + \frac{1}{2} d^2 f(x)(y,y)) y_m dy.\end{aligned} \tag{33}$$

Remarks. 1. The algebraic decay conditions appearing in (i) and (ii) should be adapted for various other kernels.

2. The derivatives and double derivatives of the composed Riesz kernels present singularities at the origin, which are finite parts of the functions $\partial_l R_{j,k}$ and $\partial^2_{l,m} R_{j,k}$, and some derivatives of the Dirac mass δ_0. They both appear in formulae (32) and (33) as they previously appeared in formula (31).

Formulae (31), (32) and (33) suitably replace convolution equations to prove the convergence at infinity of some derivatives of the convolution $K * f$. Indeed, instead of computing the pointwise limit at infinity of (27), we now compute the limit at infinity of functions such as

$$x \mapsto |x|^p \int_{B(x,1)} \partial^2_{j,k} K(x-y)(f(y)-f(x))dy.$$

However, the argument is the same as in subsection 3.3. We first use Theorem 5 to prove the convergence at infinity of the derivatives of the kernel K, and then, Propositions 2 and 3 to dominate the convergence and get its uniformity. It yields the convergence at infinity of some derivatives of the convolution $K * f$, which was yet mentioned in Lemmas 2 and 3. Finally, by the above argument, we obtain some partial differential equation for the function L_∞, which completes the study of the asymptotics at infinity of a function given by a convolution equation. In particular, in our context, by equations (21) and (24), it yields a system of linear partial differential equations on the sphere $\mathbb{S}^{N-1}$ for the functions η_∞ and θ_∞, from which we can deduce equation (10).

Proposition 6. *The functions η_∞ and θ_∞ are in $C^\infty(\mathbb{S}^{N-1})$ and satisfy for every $\sigma \in \mathbb{S}^{N-1}$,*

$$\eta_\infty(\sigma) = c(\partial_1^{\mathbb{S}^{N-1}} \theta_\infty(\sigma) - (N-1)\sigma_1 \theta_\infty(\sigma)), \tag{34}$$

$$\Delta^{\mathbb{S}^{N-1}} \theta_\infty(\sigma) + (N-1)\theta_\infty(\sigma) = \frac{c}{2}(\partial_1^{\mathbb{S}^{N-1}} \eta_\infty(\sigma) - N\sigma_1 \eta_\infty(\sigma)). \tag{35}$$

3.6 Completing the proof of Theorem 1.

Theorem 1 is a consequence of Proposition 5, which yields the uniform convergence of the function

$$x \mapsto |x|^{N-1}(v(x)-1)$$

towards v_∞, and of Proposition 6, which specifies the partial differential equation (10) satisfied by v_∞.

However, in order to complete its proof, we must mention some technical difficulties. In the case of the travelling waves for the Gross-Pitaevskii equation, the decay estimates obtained in Proposition 2 for the functions η, $\psi\theta$ and v are not sufficient to dominate the convergence at infinity of the functions $d^2\eta$, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$ and to prove the uniformity of the convergences of $\nabla\eta$, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$. They are neither sufficient to apply Lemmas 2 and 3, nor to prove Proposition 5.

Thus, we improve Proposition 2 for the functions $d^2\eta$, $d^2(\psi\theta)$, d^2v and $d^3(\psi\theta)$ in the following theorem.

Theorem 6. *Let v, a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$. Then, we have*

- $(d^2(\psi\theta), d^2v) \in M^\infty_{N+1}(\mathbb{R}^N)^2$,

- $(d^2\eta, d^3(\psi\theta)) \in M^\infty_{N+2}(\mathbb{R}^N)^2$.

This improvement relies on the method introduced by J.L. Bona and Yi A. Li [8], A. de Bouard and J.C Saut [14] and M. Maris [40, 41]. To get a feeling for the idea of this method, let us compute for instance the algebraic decay of the function $d^2\eta$. By equation (21), we must estimate the algebraic decay of the function $d^2(K_0 * F)$, which reduces by equation (31) to prove in particular that the function

$$x \mapsto \int_{B(0,1)^c} \partial^2_{j,k} K_0(y) F(x-y) dy$$

belongs to $M^\infty_{N+2}(\mathbb{R}^N)$. The method just mentioned above now consists in writing for every $x \in \mathbb{R}^N$,

$$\begin{aligned}|x|^{N+2} \left| \int_{B(0,1)^c} \partial^2_{j,k} K_0(y) F(x-y) dy \right| \leq & A \left(\int_{B(0,1)^c} |\partial^2_{j,k} K_0(y)| |y|^{N+2} |F(x-y)| dy \right. \\ & \left. + \int_{B(0,1)^c} |\partial^2_{j,k} K_0(y)| |x-y|^{N+2} |F(x-y)| dy \right) \\ \leq & A (\|\partial^2_{j,k} K_0\|_{M^\infty_{N+2}(\mathbb{R}^N)} \|F\|_{L^1(\mathbb{R}^N)} \\ & + \|\partial^2_{j,k} K_0\|_{L^1(B(0,1)^c)} \|F\|^{\frac{N+2}{2N}}_{M^\infty_{2N}(\mathbb{R}^N)} \|F\|^{\frac{N-2}{2N}}_{L^\infty(\mathbb{R}^N)}),\end{aligned}$$

and verifying that those norms are finite. Thus, this method connects the algebraic decay of the function $d^2\eta$ for instance, to the decay of the kernels d^2K_0 or d^2K_j. The main point is that in the case of superlinear nonlinearities (such as the almost quadratic nonlinearities F and G), the decay of the function is equal to the decay of the kernels. Applying this argument to each integral appearing in equations (21) and (31), we can obtain the optimal algebraic decay of the function $d^2\eta$, which is equal to the decay of the kernels d^2K_0 and d^2K_j. This yields Theorem 6 [3], from which we deduce the useful following corollary concerning the nonlinear functions F and G.

Corollary 2. *The functions F and G belong to $M^\infty_{2N}(\mathbb{R}^N)$, their gradients to $M^\infty_{2N+1}(\mathbb{R}^N)$, and the second order derivatives of G, to $M^\infty_{2N+2}(\mathbb{R}^N)$.*

Finally, it completes the sketch of the proof of Theorem 1. Indeed, by Corollary 2, we now have sufficient decay rates for the nonlinear functions F and G to apply Lemmas 2 and 3 and prove the convergence at infinity of the functions $d^2\eta$, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$. Likewise, by Theorem 6, we also have sufficient decay rates for the functions $d^2\eta$, $d^2(\psi\theta)$ and $d^3(\psi\theta)$ to prove the uniformity of the convergences mentioned in Proposition 5.

4 Sketch of the proofs of Theorems 2 and 3.

Theorems 2 and 3 both rely on the same argument: the explicit integration of the system of equations (34) and (35). Indeed, this system presents the striking property to be integrable in dimension two and in the axisymmetric case. In both cases, it reduces to a system of linear ordinary differential equations of second order, which is entirely integrable in spherical coordinates, i.e.

$$\sigma = (\cos(\beta_1), \cos(\beta_2)\sin(\beta_1), \ldots, \sin(\beta_1)\ldots\sin(\beta_{N-1})).$$

In particular, the integration of this system yields formulae (11) and (13).

[3] Theorem 6 is supposed to be optimal. Indeed, it is commonly conjectured that the functions $\partial^\alpha\eta$, $\partial^\alpha\nabla(\psi\theta)$ and $\partial^\alpha\nabla v$ are in $M^\infty_{N+|\alpha|}(\mathbb{R}^N)$, at least in the case where $|\alpha| \leq N$.

Proposition 7. *In the axisymmetric case, there is some constant α such that for every $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{S}^{N-1}$,*

$$\eta_\infty(\sigma) = \alpha c \left(\frac{1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - N \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}} \right), \tag{36}$$

$$\theta_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}. \tag{37}$$

Likewise, in dimension two, there are some constants α and β such that for every $\sigma = (\sigma_1, \sigma_2) \in \mathbb{S}^1$,

$$\eta_\infty(\sigma) = \alpha c \left(\frac{1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} - \frac{2\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^2} \right) - 2\beta c \frac{\sigma_1\sigma_2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^2}, \tag{38}$$

$$\theta_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}}. \tag{39}$$

Remark. The result above in dimension two holds for every subsonic travelling wave of finite energy, and not only for the axisymmetric ones.

The only remaining difficulty is now to compute the values of the coefficients α and β. We link them with the energy $E(v)$ and the momentum $\vec{P}(v)$ by some integral relations obtained by standard integrations by parts.

Lemma 6. *Let v, a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$. Then, we have*

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2c\Big(1 - \frac{2}{c^2}\Big)p(v) = c\Big(\frac{2N}{c^2} - 1\Big) \int_{\mathbb{S}^{N-1}} \sigma_1 \theta_\infty(\sigma) d\sigma + \int_{\mathbb{S}^{N-1}} \sigma_1^2 \eta_\infty(\sigma) d\sigma, \tag{40}$$

$$\forall 2 \leq j \leq N, P_j(v) = \frac{c}{4} \int_{\mathbb{S}^{N-1}} \sigma_j \sigma_1 \eta_\infty(\sigma) d\sigma + \frac{N}{2} \int_{\mathbb{S}^{N-1}} \sigma_j \theta_\infty(\sigma) d\sigma. \tag{41}$$

Remark. Lemma 6 holds even if the travelling waves are not axisymmetric.

Theorems 2 and 3 then follow from equations (36), (37), (38), (39), (40) and (41), and from the standard Pohozaev identities, which were derived in [23].

Lemma 7 ([23]). *Let $0 < c < \sqrt{2}$. A finite energy solution v to equation* (2) *satisfies the two identities*

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2 \tag{42}$$

$$\forall 2 \leq j \leq N, E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + cp(v). \tag{43}$$

Remark. Lemma 7 holds even if the travelling waves are not axisymmetric and if the speed c is not subsonic ($c = 0$ or $c \geq \sqrt{2}$).

5 Plan of the paper.

The paper is divided in three parts. In the first part, we derive the improved decay estimates for the travelling waves in the Gross-Pitaevskii equation stated in Theorem 6.

In a first section, we prove Lemmas 4 and 5 to obtain explicit integral expressions for some derivatives of the functions η and $\psi\theta$, on which the proof of Theorem 6 relies. In the second section, we compute the algebraic decay of those derivatives by the argument mentioned above of J.L. Bona and Yi A. Li [8], A. de Bouard and J.C Saut [14] and M. Maris [40, 41]. Finally, we complete this section by inferring Corollary 2.

The proof of Theorem 1 forms the core of the second part. The first ingredient is the pointwise convergence at infinity of the kernels K_0, K_j and $L_{j,k}$: it follows from the proofs of Lemma 1 and Theorem 5 in the first section. The second and third sections are devoted to the proof of the pointwise convergence at infinity of the functions η, $\psi\theta$ and of some of their derivatives summed up in Proposition 4. It relies on Lemmas 2 and 3. In the fourth section, we deduce from Ascoli-Arzela's theorem and the improved decay estimates of the first part, the uniformity of the convergence described by Proposition 5. Finally, the last section is devoted to the proof of Proposition 6. Then, Theorem 1 follows from the remark that

$$v_\infty = \theta_\infty,$$

and the derivation of equation (10) from equations (34) and (35).

The third part is mainly concerned with the proofs of Theorems 2 and 3. In the first section, we integrate the system of equations (34) and (35) to deduce Proposition 7. In the second section, we show Lemma 6 to compute the values of the coefficients α and β in function of the energy $E(v)$ and the momentum $\vec{P}(v)$. Finally, we end the paper by deducing Corollary 1 from Lemma 7.

1 Sharp decay of some derivatives of a travelling wave.

We first improve the asymptotic decay estimates given in [24] by proving Theorem 6. We state integral representations of the functions $d^2\eta$, $d^2(\psi\theta)$ and $d^3(\psi\theta)$ and estimate their algebraic decay by the standard argument mentioned in the introduction.

1.1 Integral forms of the functions $d^2\eta$, $d^2\theta$ and $d^3\theta$.

As mentioned above, the functions $d^2\eta$, $d^2(\psi\theta)$ and $d^3(\psi\theta)$ express as linear combinations of convolution integrals.

Proposition 8. *Let $1 \leq j,k,l \leq N$ and $x \in \mathbb{R}^N$. Then,*

$$\begin{aligned} \partial^2_{j,k}\eta(x) = & \int_{B(0,1)^c} \partial^2_{j,k}K_0(y)F(x-y)dy + \int_{B(0,1)} \partial^2_{j,k}K_0(y)(F(x-y)-F(x))dy \\ & + \left(\int_{\mathbb{S}^{N-1}} \partial_j K_0(y)y_k dy\right) F(x) + 2c\sum_{i=1}^N \left(\int_{B(0,1)^c} \partial^2_{j,k}K_i(y)G_i(x-y)dy\right. \\ & \left. + \int_{B(0,1)} \partial^2_{j,k}K_i(y)(G_i(x-y)-G_i(x))dy + \left(\int_{\mathbb{S}^{N-1}} \partial_j K_i(y)y_k dy\right) G_i(x)\right), \end{aligned} \tag{44}$$

$$\partial^2_{j,k}(\psi\theta)(x) = \frac{c}{2}\partial_k K_j * F(x) + c^2 \sum_{i=1}^N \partial_k L_{i,j} * G_i(x) + \sum_{i=1}^N \Bigg(\int_{B(0,1)^c} \partial_k R_{i,j}(y) G_i(x-y) dy$$
$$+ \int_{B(0,1)} \partial_k R_{i,j}(y)(G_i(x-y) - G_i(x) + y.\nabla G_i(x)) dy + \int_{\mathbb{S}^{N-1}} R_{i,j}(y) y_k$$
$$(G_i(x) - y.\nabla G_i(x)) dy \Bigg), \tag{45}$$

$$\partial^3_{j,k,l}(\psi\theta)(x) = \frac{c}{2} \Bigg(\int_{B(0,1)^c} \partial^2_{k,l} K_j(y) F(x-y) dy + \int_{B(0,1)} \partial^2_{k,l} K_j(y)(F(x-y) - F(x)) dy$$
$$+ \left(\int_{\mathbb{S}^{N-1}} \partial_l K_j(y) y_k dy \right) F(x) \Bigg) + c^2 \sum_{i=1}^N \Bigg(\int_{B(0,1)^c} \partial^2_{k,l} L_{i,j}(y) G_i(x-y) dy$$
$$+ \int_{B(0,1)} \partial^2_{k,l} L_{i,j}(y)(G_i(x-y) - G_i(x)) dy + \left(\int_{\mathbb{S}^{N-1}} \partial_l L_{i,j}(y) y_k dy \right) G_i(x) \Bigg)$$
$$+ \sum_{i=1}^N \Bigg(\int_{B(0,1)^c} \partial^2_{k,l} R_{i,j}(y) G_i(x-y) dy + \int_{B(0,1)} \partial^2_{k,l} R_{i,j}(y)(G_i(x-y)$$
$$- G_i(x) + y.\nabla G_i(x) - \frac{1}{2} d^2 G_i(x)(y,y)) dy + \int_{\mathbb{S}^{N-1}} R_{i,j}(y) y_k (\partial_l G_i(x)$$
$$- y.\nabla \partial_l G_i(x)) dy + \int_{\mathbb{S}^{N-1}} \partial_k R_{i,j}(y) y_l (G_i(x) - y.\nabla G_i(x)$$
$$+ \frac{1}{2} d^2 G_i(x)(y,y)) dy \Bigg). \tag{46}$$

Proposition 8 is a straightforward consequence of Lemmas 4 and 5, so we postpone its proof after their proofs.

Proof of Lemma 4. Consider $t \in]-\frac{1}{2}, \frac{1}{2}[\setminus\{0\}$. On one hand, K is in $\mathcal{K}(\mathbb{R}^N)$, so, the function $\partial_k K$ belongs to $L^1(\mathbb{R}^N)$. On the other hand, f satisfies assumption (i), so, it is a continuous, bounded function on $\mathbb{R}^N$. Therefore, by standard convolution theory, the distribution $\partial_k g$ is actually a continuous function on $\mathbb{R}^N$, which writes

$$\forall x \in \mathbb{R}^N, \partial_k g(x) = \int_{\mathbb{R}^N} \partial_k K(y) f(x-y) dy.$$

Hence, we can compute

$$\frac{\partial_k g(x + te_j) - \partial_k g(x)}{t} = \int_{\mathbb{R}^N} \partial_k K(y) \frac{f(x + te_j - y) - f(x-y)}{t} dy$$
$$= \int_{\mathbb{R}^N} \frac{\partial_k K(y + te_j) - \partial_k K(y)}{t} f(x-y) dy,$$

and therefore,

$$\begin{aligned}\frac{\partial_k g(x+te_j)-\partial_k g(x)}{t} &= \int_{B(0,1)^c}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}f(x-y)dy\\ &+\left(\int_{B(0,1)}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}dy\right)f(x)\\ &+\int_{B(0,1)}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}(f(x-y)-f(x))dy.\end{aligned}\tag{47}$$

For the first term, we state

$$\forall y\in B(0,1)^c,\ \frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}f(x-y)\underset{t\to 0}{\to}\partial^2_{j,k}K(y)f(x-y),$$

while, by assumption (i) and since $K\in\mathcal{K}(\mathbb{R}^N)$,

$$\begin{aligned}\forall y\in B(0,1)^c,\left|\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}f(x-y)\right| &\le \frac{A}{t(1+|x-y|^{2N})}\int_0^t|\partial^2_{j,k}K(y+se_j)|ds\\ &\le\frac{A}{(1+|x-y|^{2N})(|y|-\frac{1}{2})^{N+2}},\end{aligned}$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)^c}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}f(x-y)dy\underset{t\to 0}{\to}\int_{B(0,1)^c}\partial^2_{j,k}K(y)f(x-y)dy.$$

For the second term, we compute by integration by parts since $K\in\mathcal{K}(\mathbb{R}^N)$,

$$\int_{B(0,1)}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}dy=\int_{\mathbb{S}^{N-1}}\frac{K(y+te_j)-K(y)}{t}y_k dy.$$

K being in $\mathcal{K}(\mathbb{R}^N)$ once more, we get

$$\forall y\in\mathbb{S}^{N-1},\left|\frac{K(y+te_j)-K(y)}{t}y_k\right|\le\frac{A}{t}\int_0^t|\partial_j K(y+se_j)|ds\le A,$$

hence, by the dominated convergence theorem,

$$\int_{B(0,1)}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}dy\underset{t\to 0}{\to}\int_{\mathbb{S}^{N-1}}\partial_j K(y)y_k dy.$$

For the last term, we find

$$\begin{aligned}&\int_{B(0,1)}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}(f(x-y)-f(x))dy\\ &=\int_{|y|<2|t|}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}(f(x-y)-f(x))dy\\ &+\int_{2|t|<|y|<1}\frac{\partial_k K(y+te_j)-\partial_k K(y)}{t}(f(x-y)-f(x))dy.\end{aligned}$$

On one hand, by assumption (ii) and since $K \in \mathcal{K}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\left| \int_{|y|<2|t|} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) dy \right| \\
&\leq \frac{A}{|t|} \int_{|y|<2|t|} \left(\frac{1}{|y+te_j|^{N-\frac{1}{2}}} + \frac{1}{|y|^{N-\frac{1}{2}}} \right) |y| dy \\
&\leq \frac{A}{|t|} \left(\int_{|y|<2|t|} \frac{dy}{|y|^{N-\frac{3}{2}}} + \int_{|y|<2|t|} \frac{dy}{|y+te_j|^{N-\frac{3}{2}}} + \int_{|y|<2|t|} \frac{|t|dy}{|y+te_j|^{N-\frac{1}{2}}} \right) \\
&\leq A\sqrt{|t|} \underset{t\to 0}{\to} 0.
\end{aligned}$$

0n the other hand, we obtain likewise for $2|t| < |y| < 1$,

$$\begin{aligned}
\left| \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) \right| &\leq \frac{A|y|}{t} \int_0^t |\partial^2_{j,k} K(y+se_j)| dy \\
&\leq \frac{A|y|}{(|y|-|t|)^{N+\frac{1}{2}}} \\
&\leq \frac{A}{|y|^{N-\frac{1}{2}}},
\end{aligned}$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) dy \underset{t\to 0}{\to} \int_{B(0,1)} \partial^2_{j,k} K(y)(f(x-y) - f(x)) dy.$$

Finally, the function $\partial_k g$ is differentiable in direction x_j and, by equation (47), its partial derivative $\partial^2_{j,k} g$ is given by formula (31). Moreover, the function $\partial_k g$ is actually of class C^1 on $\mathbb{R}^N$. Indeed, by formula (31), $\partial^2_{j,k} g$ is continuous on $\mathbb{R}^N$. It follows from the continuity of f, assumptions (i) and (ii), the fact that K belongs to $\mathcal{K}(\mathbb{R}^N)$ and a standard application of the dominated convergence theorem. □

We now turn to the proof of Lemma 5, which is similar.

Proof of Lemma 5. We first show formula (32). Since f is a smooth function on $\mathbb{R}^N$ which satisfies assumptions (i) and (ii), we can state by standard Riesz operator theory,

$$\forall x \in \mathbb{R}^N, g(x) = \int_{B(0,1)^c} R_{j,k}(y) f(x-y) dy + \int_{B(0,1)} R_{j,k}(y)(f(x-y) - f(x)) dy.$$

In particular, g is a continuous function on $\mathbb{R}^N$ (which can also be deduced from a standard application of the dominated convergence theorem thanks to the continuity of f and assumptions (i) and (ii)). Therefore, assuming $t \in]-\frac{1}{2}, \frac{1}{2}[\setminus\{0\}$, we compute

$$\begin{aligned}
\frac{g(x+te_l) - g(x)}{t} = &\int_{B(0,1)^c} R_{j,k}(y) \frac{f(x+te_l-y) - f(x-y)}{t} dy + \int_{B(0,1)} R_{j,k}(y) \\
&\left(\frac{f(x+te_l-y) - f(x-y)}{t} - \frac{f(x+te_l) - f(x)}{t} \right) dy.
\end{aligned} \tag{48}$$

On one hand, by assumption (ii),

$$\forall y \in B(0,1)^c, \left| R_{j,k}(y) \frac{f(x+te_l-y)-f(x-y)}{t} \right| \le \frac{A}{t|y|^N} \int_0^t |\partial_l f(x+se_l-y)| ds$$
$$\le \frac{A}{|y|^N(1+|x-y|^{2N})},$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)^c} R_{j,k}(y) \frac{f(x+te_l-y)-f(x-y)}{t} dy \underset{t\to 0}{\to} \int_{B(0,1)^c} R_{j,k}(y)\partial_l f(x-y)dy.$$

On the other hand, by assumption (iii),

$$\forall y \in B(0,1), \left| R_{j,k}(y) \left(\frac{f(x+te_l-y)-f(x-y)}{t} - \frac{f(x+te_l)-f(x)}{t} \right) \right|$$
$$\le \frac{A}{t|y|^N} \int_0^t |\partial_l f(x+se_l-y) - \partial_l f(x+se_l)| \, ds$$
$$\le \frac{A}{|y|^{N-1}} \sup_{z\in\mathbb{R}^N} |d^2 f(z)|.$$

Therefore, by the dominated convergence theorem,

$$\int_{B(0,1)} R_{j,k}(y) \left(\frac{f(x+te_l-y)-f(x-y)}{t} - \frac{f(x+te_l)-f(x)}{t} \right) dy$$
$$\underset{t\to 0}{\to} \int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy.$$

Thus, the function g is differentiable in direction x_l and, by equation (48), its partial derivative $\partial_l g$ is given by

$$\forall x \in \mathbb{R}^N, \partial_l g(x) = \int_{B(0,1)^c} R_{j,k}(y)\partial_l f(x-y)dy + \int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy. \tag{49}$$

Now, we integrate by parts the first term of the right member:

$$\int_{B(0,1)^c} R_{j,k}(y)\partial_l f(x-y)dy = \int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y)dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l f(x-y)dy. \tag{50}$$

It can be made rigorously by integrating by parts on $B(0,R) \setminus B(0,1)$ for some large R and taking the limit $R \to +\infty$, using assumptions (i) and (ii). Likewise, assumption (iii) yields for the second term

$$\int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy = \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy.$$

However, we find by integrating by parts,

$$\begin{aligned}
&\int_{\varepsilon<|y|<1} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy \\
&= \int_{\varepsilon<|y|<1} R_{j,k}(y)\partial_l f(x-y)dy \\
&= \int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y) f(x-y)dy + \int_{S(0,\varepsilon)} R_{j,k}(y)\frac{y_l}{\varepsilon} f(x-y)dy - \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l f(x-y)dy \\
&= \int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy + \int_{S(0,\varepsilon)} R_{j,k}(y)\frac{y_l}{\varepsilon}(f(x-y) - f(x) \\
&\quad + y.\nabla f(x))dy - \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x-y) - f(x) + y.\nabla f(x))dy.
\end{aligned}$$

Now, we remark by assumption (iii),

$$\forall y \in B(0,1), |\partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))| \leq \frac{A}{|y|^{N-1}} \sup_{z\in\mathbb{R}^N} |d^2 f(z)|,$$

so,

$$\begin{aligned}
&\int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy \\
&\underset{\varepsilon\to 0}{\to} \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy.
\end{aligned}$$

We also notice by assumption (iii),

$$\forall y \in S(0,\varepsilon), |R_{j,k}(y) y_l (f(x-y) - f(x) + y.\nabla f(x))| \leq \frac{A}{\varepsilon^{N-3}} \sup_{z\in\mathbb{R}^N} |d^2 f(z)|,$$

therefore,

$$\frac{1}{\varepsilon} \int_{S(0,\varepsilon)} R_{j,k}(y) y_l (f(x-y) - f(x) + y.\nabla f(x))dy \underset{\varepsilon\to 0}{\to} 0.$$

Finally, it leads to

$$\begin{aligned}
\int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy = &\int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy \\
&- \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x-y) - f(x) + y.\nabla f(x))dy.
\end{aligned} \tag{51}$$

Finally, by combining equations (49), (50) and (51), the partial derivative $\partial_l g$ is given by formula (32). Thus, the function g is actually of class C^1 on $\mathbb{R}^N$. Indeed, by formula (32), $\partial_l g$ is continuous on $\mathbb{R}^N$. By a standard application of the dominated convergence theorem, it follows from the smoothness of f and assumptions (i), (ii) and (iii).

We now turn to formula (33) and we assume again that $t \in]-\frac{1}{2}, \frac{1}{2}[\setminus\{0\}$. Since f satisfies assumptions (i), (ii) and (iii), $\partial_l g$ is continuous on $\mathbb{R}^N$ and satisfies formula (32),

$$\begin{aligned}
\forall x \in \mathbb{R}^N, \partial_l g(x) = &\int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y)dy + \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) \\
&+ y.\nabla f(x))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x) - y.\nabla f(x))dy.
\end{aligned}$$

Hence,

$$\frac{\partial_l g(x+te_m)-\partial_l g(x)}{t} = \int_{B(0,1)^c} \partial_l R_{j,k}(y) \frac{f(x+te_m-y)-f(x-y)}{t} dy + \int_{B(0,1)} \partial_l R_{j,k}(y) \Big(\frac{f(x+te_m-y)-f(x-y)}{t} - \frac{f(x+te_m)-f(x)}{t} + y. \frac{\nabla f(x+te_m)-\nabla f(x)}{t} \Big) dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) \Big(\frac{f(x+te_m)-f(x)}{t} - y. \frac{\nabla f(x+te_m)-\nabla f(x)}{t} \Big) y_l dy. \tag{52}$$

On one hand, by assumption (ii),

$$\forall y \in B(0,1)^c, \Big| \partial_l R_{j,k}(y) \frac{f(x+te_m-y)-f(x-y)}{t} \Big| \leq \frac{A}{t|y|^{N+1}} \int_0^t |\partial_m f(x+se_m-y)| ds \leq \frac{A}{|y|^{N+1}(1+|x-y|^{2N})},$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)^c} \partial_l R_{j,k}(y) \frac{f(x+te_m-y)-f(x-y)}{t} dy \underset{t\to 0}{\to} \int_{B(0,1)^c} \partial_l R_{j,k}(y) \partial_m f(x-y) dy.$$

On the other hand, assumption (iv) yields for every $y \in B(0,1)$,

$$\Big| \frac{\partial_l R_{j,k}(y)}{t} \big(f(x+te_m-y)-f(x-y)-f(x+te_m)+f(x)+y.(\nabla f(x+te_m)-\nabla f(x)) \big) \Big|$$
$$\leq \frac{A}{t|y|^{N+1}} \int_0^t |\partial_m f(x+se_m-y) - \partial_m f(x+se_m) + y.\nabla \partial_m f(x+se_m)| ds$$
$$\leq \frac{A}{|y|^{N-1}} \sup_{z\in\mathbb{R}^N} |d^3 f(z)|,$$

hence, by the dominated convergence theorem,

$$\int_{B(0,1)} \frac{\partial_l R_{j,k}(y)}{t} \Big(f(x+te_m-y)-f(x-y)-f(x+te_m)+f(x)+y.(\nabla f(x+te_m) - \nabla f(x)) \Big) dy \underset{t\to 0}{\to} \int_{B(0,1)} \partial_l R_{j,k}(y) (\partial_m f(x-y) - \partial_m f(x) + y.\nabla \partial_m f(x)) dy.$$

Finally, f is in $C^\infty(\mathbb{R}^N)$, which gives

$$\int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l \Big(\frac{f(x+te_m)-f(x)}{t} - y. \frac{\nabla f(x+te_m)-\nabla f(x)}{t} \Big) dy \underset{t\to 0}{\to} \int_{\mathbb{S}^{N-1}} y_l R_{j,k}(y) (\partial_m f(x) - y.\nabla \partial_m f(x)) dy.$$

Thus, the function $\partial_l g$ is differentiable in direction x_m and, by equation (52), its partial derivative $\partial^2_{l,m} g$ is given by

$$\forall x \in \mathbb{R}^N, \partial^2_{l,m} g(x) = \int_{B(0,1)^c} \partial_l R_{j,k}(y) \partial_m f(x-y) dy + \int_{B(0,1)} \partial_l R_{j,k}(y) (\partial_m f(x-y) - \partial_m f(x) + y.\nabla \partial_m f(x)) dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (\partial_m f(x) - y.\nabla \partial_m f(x)) dy. \tag{53}$$

We now integrate by parts the first term of the right member,

$$\int_{B(0,1)^c} \partial_l R_{j,k}(y)\partial_m f(x-y)dy = \int_{B(0,1)^c} \partial^2_{l,m} R_{j,k}(y) f(x-y)dy + \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m f(x-y)dy. \tag{54}$$

Similarly to equation (50), it can be made rigorously by integrating by parts on $B(0,R)\setminus B(0,1)$ for some large R and taking the limit $R\to+\infty$, using assumptions (i) and (ii). Likewise, assumption (iv) yields

$$\begin{aligned}&\int_{B(0,1)} \partial_l R_{j,k}(y)(\partial_m f(x-y) - \partial_m f(x) + y.\nabla\partial_m f(x))dy\\ &= \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y)(\partial_m f(x-y) - \partial_m f(x) + y.\nabla\partial_m f(x))dy.\end{aligned}$$

However, we compute by integrating by parts

$$\begin{aligned}&\int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y)(\partial_m f(x-y) - \partial_m f(x) + y.\nabla\partial_m f(x))dy\\ &= \int_{\varepsilon<|y|<1} \partial^2_{l,m} R_{j,k}(y) f(x-y)dy - \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m f(x-y)dy + \int_{S(0,\varepsilon)} \partial_l R_{j,k}(y)\frac{y_m}{\varepsilon}\\ &\quad f(x-y)dy - \int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y)dy\ \partial_m f(x) + \int_{\varepsilon<|y|<1} \partial_l R_{j,k}(y) y.\nabla\partial_m f(x)dy\\ &= \int_{\varepsilon<|y|<1} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy\\ &- \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy\\ &+ \int_{S(0,\varepsilon)} \partial_l R_{j,k}(y)\frac{y_m}{\varepsilon}(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy.\end{aligned}$$

We then notice by assumption (iv) for every $y\in B(0,1)$,

$$|\partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))| \leq \frac{A}{|y|^{N-1}}\sup_{z\in\mathbb{R}^N}|d^3 f(z)|,$$

therefore,

$$\begin{aligned}&\int_{\varepsilon<|y|<1} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy\\ &\underset{\varepsilon\to 0}{\to} \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy.\end{aligned}$$

We also remark by assumption (iv) for every $y\in S(0,\varepsilon)$,

$$|\partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))| \leq \frac{A}{\varepsilon^{N-3}}\sup_{z\in\mathbb{R}^N}|d^3 f(z)|,$$

which gives

$$\frac{1}{\varepsilon}\int_{S(0,\varepsilon)} \partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy \underset{\varepsilon\to 0}{\to} 0.$$

Thus, we find

$$\begin{aligned}
&\int_{B(0,1)} \partial_l R_{j,k}(y)(\partial_m f(x-y) - \partial_m f(x) + y.\nabla \partial_m f(x))dy \\
=& \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2} d^2 f(x)(y,y))dy \\
-& \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2} d^2 f(x)(y,y))dy.
\end{aligned} \tag{55}$$

Finally, by equations (53), (54) and (55), the partial derivative $\partial^2_{l,m} g$ is given by formula (33). Thus, the function g is actually of class C^2 on $\mathbb{R}^N$. Indeed, by formula (33), $\partial^2_{l,m} g$ is continuous on $\mathbb{R}^N$: it follows from the smoothness of f, assumptions (i), (ii), (iii) and (iv), and a standard application of the dominated convergence theorem. □

We then complete the proof of Proposition 8.

Proof of Proposition 8. By formulae (19) and (20), and Proposition 1, the functions F and G are C^∞ on $\mathbb{R}^N$ and equal to

$$\begin{cases} F = \frac{|\nabla \eta|^2}{2(1-\eta)} + 2(1-\eta)|\nabla(\psi\theta)|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta), \\ G = \eta \nabla(\psi\theta), \end{cases}$$

on a neighbourhood of infinity, so, by Proposition 2, they satisfy all the assumptions of Lemmas 4 and 5.

Likewise, by Proposition 3, the kernels K_0, K_j and $L_{j,k}$ are in $\mathcal{K}(\mathbb{R}^N)$. Formula (44) is then a consequence of equation (21) and Lemma 4, while formulae (45) and (46) follow from invoking equation (24) and Lemmas 4 and 5. □

Remark. It seems possible to compute similar formulae for higher derivatives of the functions η and $\psi\theta$: since it is useless here, we are not going to investigate this point any further. However, it is probably a good way to prove the sharp decay of higher derivatives, i.e. to show that the functions $\partial^\alpha \eta$, $\partial^\alpha \nabla(\psi\theta)$ and $\partial^\alpha \nabla v$ are in $M^\infty_{N+|\alpha|}(\mathbb{R}^N)$, at least in the case where $|\alpha| \leq N$.

1.2 Sharp decay of the functions $d^2\eta$, $d^2\theta$ and $d^3\theta$.

We now infer Theorem 6 from Proposition 8. We improve the asymptotic decay rate of the functions $d^2\eta$, $d^2\theta$, d^2v and $d^3\theta$ by the argument mentioned in the introduction. We first apply it in the following lemma.

Lemma 8. *Let $1 \leq j,k \leq N$ and $K \in \mathcal{K}(\mathbb{R}^N)$. Consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N \cap M^\infty_{2N}(\mathbb{R}^N)^N$.

Then,

$$\partial^2_{j,k}(K * f) \in M^\infty_{N+2}(\mathbb{R}^N).$$

Proof. Let $g = K * f$. By assumptions (i) and (ii), Lemma 4 yields

$$\forall x \in \mathbb{R}^N, \partial^2_{j,k} g(x) = \int_{B(0,1)^c} \partial^2_{j,k} K(y) f(x-y) dy + \int_{B(0,1)} \partial^2_{j,k} K(y)(f(x-y) - f(x)) dy + \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy \right) f(x).$$

By assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$, the first term satisfies

$$\begin{aligned} |x|^{N+2} \left| \int_{B(0,1)^c} \partial^2_{j,k} K(y) f(x-y) dy \right| &\leq A \left(\int_{B(0,1)^c} |y|^{N+2} |\partial^2_{j,k} K(y)| |f(x-y)| dy \right. \\ &\quad \left. + \int_{B(0,1)^c} |\partial^2_{j,k} K(y)| |x-y|^{N+2} |f(x-y)| dy \right) \\ &\leq A(\|\partial^2_{j,k} K\|_{M^\infty_{N+2}(\mathbb{R}^N)} \|f\|_{L^1(\mathbb{R}^N)} \\ &\quad + \|\partial^2_{j,k} K\|_{L^1(B(0,1)^c)} \|f\|^{\frac{N+2}{2N}}_{M^\infty_{2N}(\mathbb{R}^N)} \|f\|^{\frac{N-2}{2N}}_{L^\infty(\mathbb{R}^N)}) \leq A. \end{aligned}$$

By assumption (ii) and since $K \in \mathcal{K}(\mathbb{R}^N)$, the second term verifies

$$\begin{aligned} |x|^{N+2} \left| \int_{B(0,1)} \partial^2_{j,k} K(y)(f(x-y) - f(x)) dy \right| &\leq A |x|^{N+2} \int_{B(0,1)} |y| |\partial^2_{j,k} K(y)| dy \\ &\quad \sup_{z \in B(x,1)} |\nabla f(z)| \\ &\leq A \frac{|x|^{N+2}}{1 + |x|^{2N}} \leq A, \end{aligned}$$

and likewise, by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$|x|^{N+2} \left| \int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy \right| |f(x)| \leq A \frac{|x|^{N+2}}{1 + |x|^{2N}} \leq A.$$

Thus, the function g belongs to $M^\infty_{N+2}(\mathbb{R}^N)$. □

We next prove a similar lemma for the composed Riesz kernels $R_{j,k}$.

Lemma 9. *Let $1 \leq j, k, l, m \leq N$ and consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(iii) $d^2 f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

Then,

$$\partial_l (R_{j,k} * f) \in M^\infty_{N+1}(\mathbb{R}^N).$$

Moreover, if f also satisfies

(iv) $d^3 f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

then,

$$\partial^2_{l,m} (R_{j,k} * f) \in M^\infty_{N+2}(\mathbb{R}^N).$$

Proof. Let $g = R_{j,k} * f$. On one hand, by assumptions (i), (ii) and (iii), Lemma 5 leads to

$$\forall x \in \mathbb{R}^N, \partial_l g(x) = \int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y)dy + \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x)$$
$$+ y.\nabla f(x))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x) - y.\nabla f(x))dy.$$

By assumption (i), the first term verifies

$$|x|^{N+1} \left| \int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y)dy \right| \leq A \int_{B(0,1)^c} (|y|^{N+1} |\partial_l R_{j,k}(y)||f(x-y)|$$
$$+ |\partial_l R_{j,k}(y)||x-y|^{N+1}|f(x-y)|)dy$$
$$\leq A \left(\int_{\mathbb{R}^N} |f(t)|dt + \int_{B(0,1)^c} |\partial_l R_{j,k}(y)|dy \right) \leq A.$$

By assumption (iii), the second term satisfies

$$|x|^{N+1} \left| \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy \right|$$
$$\leq A|x|^{N+1} \int_{B(0,1)} |y|^2 |\partial_l R_{j,k}(y)|dy \sup_{z \in B(x,1)} |d^2 f(z)|$$
$$\leq A \frac{|x|^{N+1}}{1+|x|^{2N}} \leq A,$$

and likewise, by assumptions (i) and (ii),

$$|x|^{N+1} \left| \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x) - y.\nabla f(x))dy \right| \leq A \frac{|x|^{N+1}}{1+|x|^{2N}} \leq A.$$

Hence, the derivative $\partial_l (R_{j,k} * f)$ is in $M^\infty_{N+1}(\mathbb{R}^N)$.
On the other hand, by assumptions (i), (ii), (iii) and (iv), Lemma 5 also gives

$$\forall x \in \mathbb{R}^N, \partial^2_{l,m} g(x) = \int_{B(0,1)^c} \partial^2_{l,m} R_{j,k}(y) f(x-y)dy + \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x)$$
$$+ y.\nabla f(x) - \frac{1}{2} d^2 f(x)(y,y))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (\partial_m f(x)$$
$$- y.\nabla \partial_m f(x))dy + \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m (f(x) - y.\nabla f(x)$$
$$+ \frac{1}{2} d^2 f(x)(y,y))dy.$$

Likewise, by assumption (i), the first term satisfies

$$|x|^{N+2} \left| \int_{B(0,1)^c} \partial^2_{l,m} R_{j,k}(y) f(x-y)dy \right| \leq A \int_{B(0,1)^c} (|y|^{N+2} |\partial^2_{l,m} R_{j,k}(y)||f(x-y)|$$
$$+ |\partial^2_{l,m} R_{j,k}(y)||x-y|^{N+2}|f(x-y)|)dy$$
$$\leq A \left(\int_{\mathbb{R}^N} |f(t)|dt + \int_{B(0,1)^c} |\partial^2_{l,m} R_{j,k}(y)|dy \right) \leq A.$$

For the second term, assumption (iv) yields

$$\begin{aligned}
&|x|^{N+2}\left|\int_{B(0,1)}\partial^2_{l,m}R_{j,k}(y)(f(x-y)-f(x)+y.\nabla f(x)-\frac{1}{2}d^2f(x)(y,y))dy\right|\\
\leq &A|x|^{N+2}\int_{B(0,1)}|y|^3|\partial^2_{l,m}R_{j,k}(y)|dy\sup_{z\in B(x,1)}|d^3f(z)|\\
\leq &A\frac{|x|^{N+2}}{1+|x|^{2N}}\leq A,
\end{aligned}$$

while for the third term, assumptions (ii) and (iii) give

$$|x|^{N+2}\left|\int_{\mathbb{S}^{N-1}}R_{j,k}(y)y_l(\partial_m f(x)-y.\nabla\partial_m f(x))dy\right|\leq A\frac{|x|^{N+2}}{1+|x|^{2N}}\leq A,$$

and likewise, for the last term, by assumptions (i), (ii) and (iii),

$$|x|^{N+2}\left|\int_{\mathbb{S}^{N-1}}\partial_l R_{j,k}(y)y_m(f(x)-y.\nabla f(x)+\frac{1}{2}d^2f(x)(y,y))dy\right|\leq A\frac{|x|^{N+2}}{1+|x|^{2N}}\leq A.$$

Thus, the function $\partial^2_{l,m}(R_{j,k}*f)$ belongs to $M^\infty_{N+2}(\mathbb{R}^N)$. □

Finally, Theorem 6 follows from Lemmas 8 and 9.

Proof of Theorem 6. Equation (21) writes

$$\eta=K_0*F+2c\sum_{j=1}^N K_j*G_j.$$

However, by Proposition 3, the kernels K_0 and K_j are in $\mathcal{K}(\mathbb{R}^N)$, whereas by formulae (19) and (20), and Propositions 1 and 2, the functions F and G satisfy all the assumptions of Lemma 8. Thus, the function $d^2\eta$ belongs to $M^\infty_{N+2}(\mathbb{R}^N)$ by Lemma 8.

Likewise, equation (24) states

$$\partial_j(\psi\theta)=\frac{c}{2}K_j*F+c^2\sum_{k=1}^N L_{j,k}*G_k+\sum_{k=1}^N R_{j,k}*G_k.$$

Then, Propositions 1, 2 and 3, and formulae (19) and (20) yield for every $l\in\{1,\dots,N\}$ and $x\in\mathbb{R}^N$,

$$\begin{aligned}
|x|^{N+1}|\partial_l(K_j*F)(x)|=&|x|^{N+1}|(\partial_l K_j)*F(x)|\\
\leq &A\int_{\mathbb{R}^N}(|y|^{N+1}|\partial_l K_j(y)||F(x-y)|+|\partial_l K_j(y)|\\
&|x-y|^{N+1}|F(x-y)|)dy\\
\leq &A\left(\int_{\mathbb{R}^N}|F(t)|dt+\int_{\mathbb{R}^N}|\partial_l K_j(y)|dy\right)\leq A.
\end{aligned}$$

Therefore, the function $\partial_l(K_j*F)$ is in $M^\infty_{N+1}(\mathbb{R}^N)$. Likewise, the functions $\partial_l(L_{j,k}*G_k)$ belong to $M^\infty_{N+1}(\mathbb{R}^N)$, so, since the functions G_k satisfy all the assumptions of Lemma 9, it follows from this lemma that the function $d^2(\psi\theta)$ also belongs to $M^\infty_{N+1}(\mathbb{R}^N)$.

The proof is identical for the function $d^3(\psi\theta)$ by Lemmas 8 and 9, and formula (24), so, we omit it.

Finally, by Proposition 1, the function d^2v is C^∞ on $\mathbb{R}^N$ and equal to

$$\partial^2_{j,k}v = \left(\sqrt{1-\eta}(i\partial^2_{j,k}\theta - \partial_j\theta\partial_k\theta) - \frac{\partial^2_{j,k}\eta + i(\partial_j\theta\partial_k\eta + \partial_k\theta\partial_j\eta)}{2\sqrt{1-\eta}} - \frac{\partial_j\eta\partial_k\eta}{4(1-\eta)^{\frac{3}{2}}}\right)e^{i\theta}$$

on a neighbourhood of infinity. Since the functions $\nabla\eta$, $\nabla(\psi\theta)$, $d^2\eta$ and $d^2(\psi\theta)$ are bounded and respectively belong to $M^\infty_{N+1}(\mathbb{R}^N)$, $M^\infty_N(\mathbb{R}^N)$, $M^\infty_{N+2}(\mathbb{R}^N)$ and $M^\infty_{N+1}(\mathbb{R}^N)$, and since η converges to 0 at infinity by Proposition 2, d^2v belongs to $M^\infty_{N+1}(\mathbb{R}^N)$. □

Before turning to the first order development at infinity of the function v, we establish Corollary 2.

Proof of Corollary 2. Corollary 2 is a consequence of the superlinear nature of F and G. By formulae (19) and (20), and Proposition 1, the functions F and G are C^∞ on $\mathbb{R}^N$ and equal to

$$\begin{cases} F = \frac{|\nabla\eta|^2}{2(1-\eta)} + 2(1-\eta)|\nabla\theta|^2 + 2\eta^2 - 2c\eta\partial_1\theta, \\ G = \eta\nabla\theta \end{cases}$$

on $B(0,3R_0)^c$. Thus, we compute for every $x \in B(0,3R_0)^c$,

$$|x|^{2N}(|F(x)| + |G(x)|) \leq A|x|^{2N}(|\nabla\eta(x)|^2 + |\nabla\theta(x)|^2 + \eta(x)^2 + |\eta(x)||\nabla\theta(x)|),$$

$$\begin{aligned} |x|^{2N+1}(|\nabla F(x)| + |\nabla G(x)|) \leq & A|x|^{2N+1}(|d^2\eta(x)||\nabla\eta(x)| + |\nabla\eta(x)|^3 + |\eta(x)||\nabla\eta(x)| \\ & + |\nabla\eta(x)||\nabla\theta(x)|^2 + |\nabla\theta(x)||d^2\theta(x)| + |\nabla\eta(x)||\nabla\theta(x)| \\ & + |\eta(x)||d^2\theta(x)|), \end{aligned}$$

$$\begin{aligned} |x|^{2N+2}|d^2G(x)| \leq & A|x|^{2N+2}(|d^2\eta(x)||\nabla\theta(x)| + |\nabla\eta(x)||d^2\theta(x)| \\ & + |\eta(x)||d^3\theta(x)|). \end{aligned}$$

Corollary 2 then follows from Proposition 2 and Theorem 6. □

2 Asymptotic development at first order.

Now, we consider the existence of a first order asymptotic expansion for the subsonic travelling waves of finite energy. By the method mentioned in the introduction, we first deduce the pointwise convergence of the Gross-Pitaevskii kernels, then, the pointwise convergence of all the convolution integrals which appear in formulae (21) and (24). We finish the proof of Theorem 1 by showing the convergences above are actually uniform on the sphere $\mathbb{S}^{N-1}$ and by computing a partial differential equation for the first order terms of this asymptotic expansion.

2.1 Pointwise convergence of Gross-Pitaevskii kernels.

We first prove Theorem 5, i.e. the pointwise convergence of the Gross-Pitaevskii kernels K_0, K_j and $L_{j,k}$. As claimed in the introduction, it follows from the form of their Fourier transforms through Lemma 1, whose proof is mentioned below.

Proof of Lemma 1. Consider some integer $j \in \{1, \dots, N\}$. The Fourier transform of K belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Therefore, the function f given by

$$\forall x \in \mathbb{R}^N, f(x) = (-ix_j)^{N-1} K(x),$$

is continuous on $\mathbb{R}^N$. Indeed, its Fourier transform

$$\widehat{f} = \partial_j^{N-1} \widehat{K}$$

belongs to $L^1(\mathbb{R}^N)$. Moreover, if $g \in S(\mathbb{R}^N)$, we compute

$$< x_j f, \widehat{g} >=< f, x_j \widehat{g} >= -i < f, \widehat{\partial_j g} >= -i < \widehat{f}, \partial_j g >,$$

so, $\widehat{f}$ being in $L^1(\mathbb{R}^N)$, we can write

$$< x_j f, \widehat{g} >= -i \int_{\mathbb{R}^N} \widehat{f}(\xi) \partial_j g(\xi) d\xi.$$

We then deduce from an integration by parts that for every $\lambda > 0$,

$$\begin{aligned} < x_j f, \widehat{g} > = i \int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) g(\xi) d\xi + i \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(g(\xi) - g(0)) d\xi \\ + \frac{ig(0)}{\lambda} \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi. \end{aligned}$$

However, g is in $S(\mathbb{R}^N)$, therefore,

$$g(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{g}(x) e^{ix.\xi} dx,$$

which yields

$$\begin{aligned} < x_j f, \widehat{g} > = \frac{i}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{g}(x) \Bigg(\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi \\ + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi \Bigg) dx. \end{aligned}$$

Therefore, by standard duality, the tempered distribution $x_j f$ is equal to the tempered distribution Ψ given for every $x \in \mathbb{R}^N$ by

$$\begin{aligned} \Psi(x) &= \frac{i}{(2\pi)^N} \Bigg(\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi \Bigg) \\ &= \frac{i}{(2\pi)^N} \Bigg(\int_{B(0,\lambda)^c} \partial_j^N \widehat{K}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{K}(\xi)(e^{ix.\xi} - 1) d\xi \\ &+ \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{N-1} \widehat{K}(\xi) d\xi \Bigg). \end{aligned}$$

Indeed, Ψ is a tempered distribution because, since $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, Ψ belongs to $L^1_{loc}(\mathbb{R}^N)$ and satisfies

$$\forall x \in \mathbb{R}^N, |\Psi(x)| \le A(1 + |x|).$$

Moreover, since $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$ once more, by a standard application of the dominated convergence theorem, Ψ is also continuous on $\mathbb{R}^N$. Thus, the function $x \mapsto x_j f(x) = x_j(-ix_j)^{N-1}K(x)$ is continuous on $\mathbb{R}^N$ and verifies for every $x \in \mathbb{R}^N$,

$$x_j(-ix_j)^{N-1}K(x) = \frac{i}{(2\pi)^N}\left(\int_{B(0,\lambda)^c} \partial_j^N \widehat{K}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{K}(\xi)(e^{ix.\xi}-1)d\xi \right.$$
$$\left. + \frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j \partial_j^{N-1}\widehat{K}(\xi)d\xi\right).$$

It then only remains to choose $\lambda = \frac{1}{R}$ and $x = R\sigma - y$ to get formula (29). □

Theorem 5 is then a consequence of Lemma 1.

Proof of Theorem 5. Let $1 \leq j \leq N$ and let us first make the additional assumption

$$\alpha = 0.$$

We will remove it later. The function $\widehat{K}$ is a rational fraction only singular at the origin, so, all its derivatives are also rational fractions only singular at the origin. Thus, we can state for every $i \in \{0, 1, 2\}$,

$$\partial_j^{N+i-1}\widehat{K} = \frac{\sum\limits_{k=0}^{d_i} P_{k,i}}{\sum\limits_{k=0}^{d'_i} Q_{k,i}} \tag{56}$$

where

- the functions $P_{k,i}$ and $Q_{k,i}$ are homogeneous polynomial functions either equal to 0 or of degree k.
- the polynomial functions $P_i = \sum\limits_{k=0}^{d_i} P_{k,i}$ and $Q_i = \sum\limits_{k=0}^{d'_i} Q_{k,i}$ are relatively prime.
- the polynomial function Q_i does not vanish on $\mathbb{R}^N \setminus \{0\}$.

Moreover, consider $\xi \in \mathbb{R}^N \setminus \{0\}$ and denote

- $l(\xi) = \begin{cases} \min\{k \in \{0, \dots, d_i\}, P_{k,i}(\xi) \neq 0\}, \text{ if } \exists k \in \{0, \dots, d_i\}, P_{k,i}(\xi) \neq 0, \\ +\infty, \text{ otherwise,} \end{cases}$
- $l'(\xi) = \min\{k \in \{0, \dots, d'_i\}, Q_{k,i}(\xi) \neq 0\}$.

The functions l and l' are well-defined on $\mathbb{R}^N \setminus \{0\}$, and we can set

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, R_i(\xi) = \begin{cases} \delta_{l'(\xi), l(\xi)+N-1+i} \frac{P_{l(\xi),i}(\xi)}{Q_{l'(\xi),i}(\xi)}, \text{ if } l(\xi) \neq +\infty, \\ 0, \text{ otherwise.} \end{cases}$$

Now, we claim

Claim 1. *The function R_i belongs to $M^\infty_{N+i-1}(\mathbb{R}^N)$ and satisfies*

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \frac{1}{R^{N+i-1}}\partial_j^{N+i-1}\widehat{K}\left(\frac{\xi}{R}\right) \underset{R \to +\infty}{\to} R_i(\xi). \tag{57}$$

Proof of Claim 1. The case $l(\xi) = +\infty$ being straightforward since

$$\frac{1}{R^{N+i-1}}\partial_j^{N+i-1}\widehat{K}\left(\frac{\xi}{R}\right) = \frac{\sum\limits_{k=0}^{d_i} R^{-k}P_{k,i}(\xi)}{\sum\limits_{k=0}^{d'_i} R^{N+i-1-k}Q_{k,i}(\xi)} = 0 = R_i(\xi),$$

consider $R > 0$ and $\xi \in \mathbb{R}^N \setminus \{0\}$ such that

$$l(\xi) \neq +\infty.$$

Formula (56) becomes

$$\frac{1}{R^{N+i-1}}\partial_j^{N+i-1}\widehat{K}\left(\frac{\xi}{R}\right) = \frac{\sum\limits_{k=0}^{d_i} R^{-k}P_{k,i}(\xi)}{\sum\limits_{k=0}^{d'_i} R^{N+i-1-k}Q_{k,i}(\xi)} \underset{R\to+\infty}{\sim} \frac{P_{l(\xi),i}(\xi)}{R^{N+i-1-l'(\xi)+l(\xi)}Q_{l'(\xi),i}(\xi)}.$$

However, the function $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, which means in particular that

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \left|\frac{1}{R^{N+i-1}}\partial_j^{N+i-1}\widehat{K}\left(\frac{\xi}{R}\right)\right| \leq \frac{A}{|\xi|^{N+i-1}}. \tag{58}$$

Thus, we first deduce

$$\frac{1}{R^{N+i-1}}\partial_j^{N+i-1}\widehat{K}\left(\frac{\xi}{R}\right) \underset{R\to+\infty}{\to} \delta_{N+i-1+l(\xi),l'(\xi)}\frac{P_{l(\xi),i}(\xi)}{Q_{l'(\xi),i}(\xi)} = R_i(\xi),$$

and secondly, by taking the limit $R \to +\infty$ in inequality (58),

$$|R_i(\xi)| \leq \frac{A}{|\xi|^{N+i-1}},$$

i.e. the function R_i belongs to $M^\infty_{N+i-1}(\mathbb{R}^N)$. □

Now, we turn back to the proof of Theorem 5. Consider $(\sigma, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$ such that

$$\sigma_j \neq 0$$

and remark once again that the function $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$. By Lemma 1, we can state for every positive number R sufficiently large

$$\begin{aligned} R^N K(R\sigma - y) =& \frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N}\left(\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi + \int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi)\right. \\ & \left.(e^{i\xi.(R\sigma-y)} - 1)d\xi + R\int_{S(0,\frac{1}{R})} \xi_j \partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi\right). \end{aligned} \tag{59}$$

Our goal is to prove the convergence of each term of the right member towards a bounded measurable function independent of y.

Step 1. *The first term of the right member of equation* (59) *satisfies*

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi \underset{R\to+\infty}{\to} -\frac{1}{i\sigma_j}\left(\int_{B(0,1)^c} R_2(\xi)e^{i\xi.\sigma}d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi)e^{i\xi.\sigma}d\xi\right).$$

Indeed, for every $\lambda > \frac{1}{R}$,

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \lim_{\lambda\to+\infty} \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi.$$

Moreover, by integrating by parts,

$$\begin{aligned}\int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi =& \frac{1}{i(R\sigma_j - y_j)} \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^N \widehat{K}(\xi) \partial_j (e^{i\xi.(R\sigma-y)}) d\xi \\ =& \frac{1}{i(R\sigma_j - y_j)} \Bigg(- \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \\ &+ \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi - R \int_{S(0,\frac{1}{R})} \xi_j \\ &\partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \Bigg).\end{aligned}$$

However, $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, therefore,

$$\int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{\lambda\to+\infty}{\to} \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi,$$

while

$$\left| \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right| \leq \frac{A\lambda^{N-1}}{\lambda^{N+2}} \underset{\lambda\to+\infty}{\to} 0.$$

Thus, we obtain

$$\begin{aligned}\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \frac{1}{i(R\sigma_j - y_j)} \Bigg(&- \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \\ &- R \int_{S(0,\frac{1}{R})} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \Bigg).\end{aligned} \tag{60}$$

On one hand, the first term verifies

$$\frac{1}{R} \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \frac{1}{R^{N+1}} \int_{B(0,1)^c} \partial_j^{N+1} \widehat{K}\left(\frac{\xi}{R}\right) e^{i\xi.(\sigma-\frac{y}{R})} d\xi.$$

However, by assertion (57),

$$\frac{1}{R^{N+1}} \partial_j^{N+1} \widehat{K}\left(\frac{\xi}{R}\right) \underset{R\to+\infty}{\to} R_2(\xi),$$

and, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$\forall \xi \in B(0,1)^c, \left| \frac{1}{R^{N+1}} \partial_j^{N+1} \widehat{K}\left(\frac{\xi}{R}\right) e^{i\xi.(\sigma-\frac{y}{R})} \right| \leq \frac{A}{|\xi|^{N+1}},$$

hence, by the dominated convergence theorem,

$$\frac{1}{R} \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{R\to+\infty}{\to} \int_{B(0,1)^c} R_2(\xi) e^{i\xi.\sigma} d\xi.$$

On the other hand, the second terms writes

$$\int_{S(0,\frac{1}{R})} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma - y)} d\xi = \frac{1}{R^N} \int_{\mathbb{S}^{N-1}} \xi_j \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) e^{i\xi.(\sigma - \frac{y}{R})} d\xi.$$

Likewise, by assertion (57),

$$\frac{1}{R^N} \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) \underset{R\to+\infty}{\to} R_1(\xi).$$

and, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$\forall \xi \in \mathbb{S}^{N-1}, \left| \frac{\xi_j}{R^N} \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) e^{i\xi.(\sigma - \frac{y}{R})} \right| \leq A,$$

which gives by the dominated convergence theorem,

$$\int_{S(0,\frac{1}{R})} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma - y)} d\xi \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi) e^{i\xi.\sigma} d\xi.$$

In conclusion, equation (60) yields

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma - y)} d\xi \underset{R\to+\infty}{\to} -\frac{1}{i\sigma_j} \left(\int_{B(0,1)^c} R_2(\xi) e^{i\xi.\sigma} d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi) e^{i\xi.\sigma} d\xi \right),$$

which ends the proof of Step 1.

Step 2. *The second term of the right member of equation* (59) *satisfies*

$$\int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi) (e^{i\xi.(R\sigma - y)} - 1) d\xi \underset{R\to+\infty}{\to} \int_{B(0,1)} R_1(\xi) (e^{i\xi.\sigma} - 1) d\xi.$$

Indeed, we have

$$\int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi) (e^{i\xi.(R\sigma - y)} - 1) d\xi = \frac{1}{R^N} \int_{B(0,1)} \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) (e^{i\xi.(\sigma - \frac{y}{R})} - 1) d\xi.$$

Likewise, by assertion (57),

$$\frac{1}{R^N} \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) \underset{R\to+\infty}{\to} R_1(\xi),$$

and, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$, we have for every $R > 2|y|$,

$$\forall \xi \in B(0,1), \left| \frac{1}{R^N} \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) (e^{i\xi.(\sigma - \frac{y}{R})} - 1) \right| \leq \frac{A}{|\xi|^N} \left| \xi.(\sigma - \frac{y}{R}) \right| \leq \frac{A}{|\xi|^{N-1}}.$$

Hence, the dominated convergence theorem gives

$$\int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi) (e^{i\xi.(R\sigma - y)} - 1) d\xi \underset{R\to+\infty}{\to} \int_{B(0,1)} R_1(\xi) (e^{i\xi.\sigma} - 1) d\xi,$$

which is the desired result.

Step 3. *The last term of the right member of equation* (59) *verifies*

$$R\int_{S(0,\frac{1}{R})}\xi_j\partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi\underset{R\to+\infty}{\to}\int_{\mathbb{S}^{N-1}}\xi_jR_0(\xi)e^{i\xi.\sigma}d\xi.$$

Indeed, we compute

$$R\int_{S(0,\frac{1}{R})}\xi_j\partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi=\frac{1}{R^{N-1}}\int_{\mathbb{S}^{N-1}}\xi_j\partial_j^{N-1}\widehat{K}\left(\frac{\xi}{R}\right)e^{i\xi.(\sigma-\frac{y}{R})}d\xi.$$

However, by assertion (57),

$$\frac{1}{R^{N-1}}\partial_j^{N-1}\widehat{K}\left(\frac{\xi}{R}\right)\underset{R\to+\infty}{\to}R_0(\xi),$$

and, since $\widehat{K}\in\widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$\forall\xi\in\mathbb{S}^{N-1},\left|\frac{1}{R^{N-1}}\xi_j\partial_j^{N-1}\widehat{K}\left(\frac{\xi}{R}\right)e^{i\xi.(\sigma-\frac{y}{R})}\right|\leq A,$$

which yields by the dominated convergence theorem,

$$R\int_{S(0,\frac{1}{R})}\xi_j\partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi\underset{R\to+\infty}{\to}\int_{\mathbb{S}^{N-1}}\xi_jR_0(\xi)e^{i\xi.\sigma}d\xi.$$

Finally, by equation (59), and Steps 1, 2 and 3, we conclude

$$R^NK(R\sigma-y)\underset{R\to+\infty}{\to}K_\infty(\sigma),$$

where K_∞ is given by

$$\begin{aligned}K_\infty(\sigma)=&\frac{i^N}{(2\pi\sigma_j)^N}\Bigg(\int_{B(0,1)}R_1(\xi)(e^{i\xi.\sigma}-1)d\xi+\int_{\mathbb{S}^{N-1}}\xi_jR_0(\xi)e^{i\xi.\sigma}d\xi\\&-\frac{1}{i\sigma_j}\bigg(\int_{B(0,1)^c}R_2(\xi)e^{i\xi.\sigma}d\xi+\int_{\mathbb{S}^{N-1}}\xi_jR_1(\xi)e^{i\xi.\sigma}d\xi\bigg)\Bigg).\end{aligned}\tag{61}$$

It then only remains to show that the function K_∞ is uniformly bounded on the sphere $\mathbb{S}^{N-1}$. Indeed, up to choose another integer $j\in\{1,\ldots,N\}$, we can suppose that

$$\frac{1}{\sqrt{N}}\leq\sigma_j\leq1.$$

We then deduce from Claim 1 and from this additional assumption that

$$|K_\infty(\sigma)|\leq A_N\left(\int_{B(0,1)}\frac{d\xi}{|\xi|^{N-1}}+\int_{\mathbb{S}^{N-1}}\frac{d\xi}{|\xi|^{N-2}}+\int_{B(0,1)^c}\frac{d\xi}{|\xi|^{N+1}}+\int_{\mathbb{S}^{N-1}}\frac{d\xi}{|\xi|^{N-1}}\right),$$

so, the function K_∞ is uniformly bounded on $\mathbb{S}^{N-1}$.

Now, we complete the proof of Theorem 5 by considering the case

$$\alpha\neq0.$$

We first compute

$$\widehat{\partial^\alpha K}(\xi)=i^{|\alpha|}\xi^\alpha\widehat{K}(\xi).$$

We then consider $\beta \in \mathbb{N}^N$ such that $|\beta| = |\alpha|$ and denote L_β, the tempered distribution of Fourier transform

$$\widehat{L_\beta} = \partial^\beta \widehat{\partial^\alpha K}.$$

We claim that the function $\widehat{L_\beta}$ belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Indeed, by Leibnitz's formula,

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \widehat{L_\beta}(\xi) = \partial^\beta(i^{|\alpha|}\xi^\alpha \widehat{K}(\xi)) = \sum_{0 \leq \gamma \leq \beta} A_{\gamma,\beta} \partial^\gamma(\xi^\alpha)\partial^{\beta-\gamma}\widehat{K}(\xi),$$

so, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$(1+|\xi|^2)|\widehat{L_\beta}(\xi)| \leq A(1+|\xi|^2) \sum_{0 \leq \gamma \leq \beta} \frac{|\xi|^{|\alpha|-|\gamma|}}{(1+|\xi|^2)|\xi|^{|\beta|-|\gamma|}} \leq A.$$

Therefore, the function $\widehat{L_\beta}$ is in $L^\infty(\mathbb{R}^N) \cap M_2^\infty(\mathbb{R}^N)$. Likewise, a straightforward inductive argument for the derivatives of $\widehat{L_\beta}$ yields that $\widehat{L_\beta}$ is a rational fraction which is only singular at the origin and belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Thus, by the proof ahead for the case $\alpha = 0$, there exists a bounded measurable function $L_{\beta,\infty}$ such that

$$R^N L_\beta(R\sigma - y) \underset{R \to +\infty}{\to} L_{\beta,\infty}(\sigma).$$

Moreover, we compute

$$L_\beta(x) = (-i)^{|\beta|} x^\beta \partial^\alpha K(x),$$

so,

$$R^N(-i)^{|\beta|}(R\sigma - y)^\beta \partial^\alpha K(R\sigma - y) \underset{R \to +\infty}{\to} L_{\beta,\infty}(\sigma),$$

and

$$R^{N+|\alpha|}\sigma^\beta \partial^\alpha K(R\sigma - y) \underset{R \to +\infty}{\to} i^{|\alpha|} L_{\beta,\infty}(\sigma).$$

However, we can always choose β such that

$$|\sigma^\beta| \geq \frac{1}{N^{\frac{|\alpha|}{2}}},$$

so,

$$\left| \frac{i^{|\alpha|}}{\sigma^\beta} L_{\beta,\infty}(\sigma) \right| \leq N^{\frac{|\alpha|}{2}} \max_{|\beta|=|\alpha|} \|L_{\beta,\infty}\|_{L^\infty(\mathbb{S}^{N-1})}.$$

Thus, there is a bounded measurable function K_∞^α on the sphere $\mathbb{S}^{N-1}$ such that

$$R^{N+|\alpha|}\partial^\alpha K(R\sigma - y) \underset{R \to +\infty}{\to} K_\infty^\alpha(\sigma),$$

which completes the proof of Theorem 5. □

One application of Theorem 5 is given by the next corollary.

Corollary 3. *Let $1 \leq j,k \leq N$, $\alpha \in \mathbb{N}^N$ and $\sigma \in \mathbb{S}^{N-1}$. There exist bounded measurable functions $K_{0,\infty}^\alpha$, $K_{j,\infty}^\alpha$ and $L_{j,k,\infty}^\alpha$ on the sphere $\mathbb{S}^{N-1}$ such that*

$$\forall y \in \mathbb{R}^N, \begin{cases} R^{N+|\alpha|}\partial^\alpha K_0(R\sigma - y) \underset{R \to +\infty}{\to} K_{0,\infty}^\alpha(\sigma), \\ R^{N+|\alpha|}\partial^\alpha K_j(R\sigma - y) \underset{R \to +\infty}{\to} K_{j,\infty}^\alpha(\sigma), \\ R^{N+|\alpha|}\partial^\alpha L_{j,k}(R\sigma - y) \underset{R \to +\infty}{\to} L_{j,k,\infty}^\alpha(\sigma). \end{cases}$$

Proof. We infer from formulae (22), (23) and (25) that $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ are rational fractions which are only singular at the origin and belong to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Corollary 3 is then a consequence of Theorem 5. □

Remark. Formula (61) gives an expression of the limit K_∞ in function of the kernel K. It is quite involved to compute explicitly such an expression. However, we can conjecture the limit of the non-isotropic kernels K_0, K_j and $L_{j,k}$. Indeed, consider for instance the kernel K_0. By formula (22), its Fourier transform writes

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}.$$

Turning back to the proof of Theorem 5, we remark that the limit at infinity of K_0 is formally identical to the limit at infinity of the kernel R_0 whose Fourier transform is

$$\widehat{R_0}(\xi) = \frac{|\xi|^2}{2|\xi|^2 - c^2\xi_1^2}.$$

Indeed, the only terms which appear in the limit at infinity of the kernel K_0 are the homogeneous terms of lowest degree of the numerator and denominator of $\widehat{K_0}$. Moreover, up to a change of variables, the kernel R_0 is related to the composed Riesz kernels. Indeed, it is equal to

$$\widehat{R_0}(\xi) = \sum_{j=1}^{N} \frac{1}{2(1-\frac{c^2}{2})^{\delta_{j,1}}} \widehat{R_{j,j}}\Big(\sqrt{1-\frac{c^2}{2}}\xi_1, \ldots, \xi_N\Big).$$

Since we know the limit at infinity of the composed Riesz kernels by formula (30), we deduce that

$$K_{0,\infty}(\sigma) = R_{0,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-3}{2}}c^2}{8\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\left(1 - \frac{N\sigma_1^2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\right). \tag{62}$$

Likewise, by formulae (23) and (25), we can compute formally the limit at infinity of the kernel K_j,

$$K_{j,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-1}{2}}}{4\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\left(\delta_{j,1}\Big(1-\frac{c^2}{2}\Big)^{-\frac{\delta_{j,1}+1}{2}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}}\sigma_1\sigma_j}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\right), \tag{63}$$

and of the kernel $L_{j,k}$,

$$L_{j,k,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})}{2c^2\pi^{\frac{N}{2}}}\left(\Big(1-\frac{c^2}{2}\Big)^{\frac{N}{2}}\left(\frac{\delta_{j,k}(1-\frac{c^2}{2})^{-\frac{\delta_{j,1}+\delta_{k,1}+1}{2}}}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}-\delta_{k,1}+\frac{1}{2}}\sigma_j\sigma_k}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\right) - \delta_{j,k} + N\sigma_j\sigma_k\right). \tag{64}$$

Formulae (62), (63) and (64) lead to Conjecture 1 as mentioned in Section 2.3.

2.2 Pointwise convergence of convolution integrals involving the Gross-Pitaevskii kernels.

Now, we turn to the pointwise convergence of all the convolution integrals involving the Gross-Pitaevskii kernels K_0, K_j and $L_{j,k}$.

Proposition 9. *Let $\sigma \in \mathbb{S}^{N-1}$, $1 \leq j,k \leq N$ and $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$. Then, the following assertion holds*

$$R^{N+|\alpha|}\partial^\alpha(K * f)(R\sigma) \underset{R\to+\infty}{\to} K_\infty^\alpha(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

for K, either equal to K_0, K_j or $L_{j,k}$, and f, either equal to F, G_j or G_k.

The proof of Proposition 9 is a straightforward consequence of Corollaries 2 and 3, and Lemma 2, so we postpone its proof after the proof of Lemma 2.

Proof of Lemma 2. We divide the proof in three steps which correspond to each desired assertion.

Step 1. *The next assertion holds*

$$R^N g(R\sigma) \underset{R\to+\infty}{\to} K_\infty(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

where K_∞ denotes the bounded measurable function given by Theorem 5.

Indeed, consider $R > 0$ and write the expression of the function g

$$\begin{aligned} R^N g(R\sigma) &= \int_{\mathbb{R}^N} R^N K(R\sigma - y)f(y)dy \\ &= \int_{|R\sigma-y|\leq\frac{R}{2}} R^N K(R\sigma - y)f(y)dy + \int_{|R\sigma-y|>\frac{R}{2}} R^N K(R\sigma - y)f(y)dy. \end{aligned} \tag{65}$$

On one hand, by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\begin{aligned} \left|\int_{|R\sigma-y|\leq\frac{R}{2}} R^N K(R\sigma - y)f(y)dy\right| &= \left|\int_{|\sigma-z|\leq\frac{1}{2}} R^{2N} K(R(\sigma - z))f(Rz)dz\right| \\ &\leq A\int_{|\sigma-z|\leq\frac{1}{2}} \frac{R^{2N}}{(1+R^{2N}|z|^{2N})(R^{N-1}|\sigma-z|^{N-1})}dz \\ &\leq \frac{A}{R^{N-1}}\int_{|\sigma-z|\leq\frac{1}{2}} \frac{dz}{|z|^{2N}|\sigma-z|^{N-1}} \underset{R\to+\infty}{\to} 0. \end{aligned}$$

On the other hand, by Theorem 5,

$$R^N 1_{|R\sigma-y|>\frac{R}{2}} K(R\sigma - y)f(y) \underset{R\to+\infty}{\to} K_\infty(\sigma)f(y),$$

while by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\forall y \in B\left(R\sigma, \frac{R}{2}\right)^c, |R^N K(R\sigma - y)f(y)| \leq \frac{AR^N}{|R\sigma-y|^N(1+|y|^{2N})} \leq \frac{A}{1+|y|^{2N}},$$

hence, by the dominated convergence theorem,

$$\int_{|R\sigma-y|>\frac{R}{2}} R^N K(R\sigma - y)f(y)dy \underset{R\to+\infty}{\to} K_\infty(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

which gives the desired result by equation (65).

Step 2. *The following assertion is valid*

$$R^{N+1}\partial_j g(R\sigma) \underset{R\to+\infty}{\to} K_\infty^j(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

where K_∞^j denotes the bounded measurable function given by Theorem 5.

The proof is quite similar to the proof of Step 1. Indeed, consider $R>0$ and state likewise

$$R^{N+1}\partial_j g(R\sigma) = \int_{|R\sigma-y|\leq\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y)f(y)dy + \int_{|R\sigma-y|>\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y) f(y)dy. \tag{66}$$

On one hand, by assumption (i) and since $K\in\mathcal{K}(\mathbb{R}^N)$,

$$\left|\int_{|R\sigma-y|\leq\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y)f(y)dy\right| \leq A\int_{|\sigma-z|\leq\frac{1}{2}} \frac{R^{2N+1}}{(1+R^{2N}|z|^{2N})(R^{N-\frac{1}{2}}|\sigma-z|^{N-\frac{1}{2}})}dz$$
$$\leq \frac{A}{R^{N-\frac{3}{2}}} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by Theorem 5,

$$R^{N+1}1_{|R\sigma-y|>\frac{R}{2}}\partial_j K(R\sigma-y)f(y) \underset{R\to+\infty}{\to} K_\infty^j(\sigma)f(y),$$

while by assumption (i) and since $K\in\mathcal{K}(\mathbb{R}^N)$,

$$\forall y\in B\left(R\sigma,\frac{R}{2}\right)^c, |R^{N+1}\partial_j K(R\sigma-y)f(y)| \leq \frac{AR^{N+1}}{|R\sigma-y|^{N+1}(1+|y|^{2N})} \leq \frac{A}{1+|y|^{2N}},$$

hence, by the dominated convergence theorem,

$$\int_{|R\sigma-y|>\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y)f(y)dy \underset{R\to+\infty}{\to} K_\infty^j(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

which ends the proof of Step 2 by equation (66).

Step 3. *The assertion*

$$R^{N+2}\partial_{j,k}^2 g(R\sigma) \underset{R\to+\infty}{\to} K_\infty^{j,k}(\sigma)\int_{\mathbb{R}^N} f(x)dx$$

holds if $K_\infty^{j,k}$ denotes the bounded measurable function defined in Theorem 5.

Indeed, Lemma 4 gives

$$\partial_{j,k}^2 g(R\sigma) = \int_{B(0,1)^c} \partial_{j,k}^2 K(y)f(R\sigma-y)dy + \int_{B(0,1)} \partial_{j,k}^2 K(y)(f(R\sigma-y)-f(R\sigma))dy$$
$$+\left(\int_{\mathbb{S}^{N-1}} \partial_j K(y)y_k dy\right) f(R\sigma),$$

which yields by an integration by parts and the change of variables $z = R\sigma - y$,

$$R^{N+2}\partial^2_{j,k}g(R\sigma) = R^{N+2}\int_{B(R\sigma,\frac{R}{2})^c}\partial^2_{j,k}K(R\sigma - z)f(z)dz + R^{N+2}\int_{B(R\sigma,\frac{R}{2})}\partial^2_{j,k}K(R\sigma - z)$$
$$(f(z) - f(R\sigma))dz + 2R^{N+1}\left(\int_{S(0,\frac{R}{2})}\partial_j K(y)y_k dy\right)f(R\sigma). \tag{67}$$

On one hand, we compute by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$R^{N+1}\left|\int_{S(0,\frac{R}{2})}\partial_j K(y)y_k dy\right||f(R\sigma)| \leq \frac{AR^{N+1}}{1+R^{2N}}\int_{S(0,\frac{R}{2})}\frac{dy}{|y|^N} \leq \frac{A}{R^N} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by assumption (ii) and since $K \in \mathcal{K}(\mathbb{R}^N)$, we find

$$\begin{aligned}
&R^{N+2}\left|\int_{B(R\sigma,\frac{R}{2})}\partial^2_{j,k}K(R\sigma - z)(f(z) - f(R\sigma))dz\right| \\
\leq &AR^{N+2}\Bigg(\int_{B(R\sigma,1)}\frac{dz}{|R\sigma - z|^{N-\frac{1}{2}}}\sup_{y\in B(R\sigma,1)}|\nabla f(y)| + \int_{1\leq|R\sigma-z|\leq\frac{R}{2}}\frac{dz}{|R\sigma - z|^{N+1}} \\
&\sup_{y\in B(R\sigma,\frac{R}{2})}|\nabla f(y)|\Bigg) \\
\leq &\frac{A}{R^{N-1}} \underset{R\to+\infty}{\to} 0.
\end{aligned}$$

Finally, Theorem 5 gives

$$R^{N+2}1_{|R\sigma-z|>\frac{R}{2}}\partial^2_{j,k}K(R\sigma - z)f(z) \underset{R\to+\infty}{\to} K^{j,k}_\infty(\sigma)f(z),$$

while by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\forall z \in B\left(R\sigma, \frac{R}{2}\right)^c, |R^{N+2}\partial^2_{j,k}K(R\sigma - z)f(z)| \leq \frac{AR^{N+2}}{|R\sigma - z|^{N+2}(1+|z|^{2N})} \leq \frac{A}{1+|z|^{2N}},$$

hence, by the dominated convergence theorem,

$$\int_{B(R\sigma,\frac{R}{2})^c}\partial^2_{j,k}K(R\sigma - z)f(z)dz \underset{R\to+\infty}{\to} K^{j,k}_\infty(\sigma)\int_{\mathbb{R}^N}f(x)dx,$$

which ends the proofs of Step 3 and Lemma 2 by equation (67). □

Before investigating the pointwise convergence of the convolution integrals involving the composed Riesz kernels, we complete the proof of Proposition 9.

Proof of Proposition 9. By Corollary 2, the functions F and G satisfy assumptions (i) and (ii) of Lemma 2. Moreover, the functions K_0, K_j and $L_{j,k}$ belong to $\mathcal{K}(\mathbb{R}^N)$ by Proposition 3 and their Fourier transforms are rational fractions in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, only singular at the origin by formulae (22), (23) and (25). Thus, Proposition 9 follows from Lemma 2 applied to the kernels K_0, K_j and $L_{j,k}$, and to the functions F and G. □

2.3 Pointwise convergence of convolution integrals involving the composed Riesz kernels.

We now establish Proposition 4 by studying the pointwise convergence of the convolution integrals involving the composed Riesz kernels $R_{j,k}$.

Proposition 10. *Let $1 \leq j,k,l \leq N$ and $\sigma \in \mathbb{S}^{N-1}$. Then, we have*

$$\begin{cases} R^N R_{j,k} * G_k(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k)\int_{\mathbb{R}^N} G_k(x)dx, \\ R^{N+1}\partial_l R_{j,k} * G_k(R\sigma) \underset{R\to+\infty}{\to} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}((N+2)\sigma_j\sigma_k\sigma_l - \delta_{j,k}\sigma_l - \delta_{j,l}\sigma_k - \delta_{k,l}\sigma_j)\int_{\mathbb{R}^N} G_k. \end{cases}$$

Proof. By Corollary 2, the functions G_k verify assumptions (i), (ii) and (iii) of Lemma 3. Thus, Proposition 10 follows from Lemma 3 and it only remains to prove this lemma. □

Proof of Lemma 3. We split the proof in two steps which correspond to each desired assertion.

Step 1. *We have*

$$R^N g(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k)\int_{\mathbb{R}^N} f(x)dx.$$

Indeed, equation (30) yields for every $R > 0$,

$$R^N g(R\sigma) = \frac{\Gamma(\frac{N}{2})R^N}{2\pi^{\frac{N}{2}}}\left(\int_{|y|>\frac{R}{2}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} f(R\sigma - y)dy + \int_{|y|\leq\frac{R}{2}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} (f(R\sigma - y) - f(R\sigma))dy\right),$$

so, by the change of variable $z = R\sigma - y$,

$$\begin{aligned} R^N g(R\sigma) = \frac{\Gamma(\frac{N}{2})R^N}{2\pi^{\frac{N}{2}}}\Bigg(&\int_{|R\sigma - z|>\frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z)dz \\ &+ \int_{|R\sigma - z|\leq\frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} (f(z) - f(R\sigma))dz\Bigg). \end{aligned} \tag{68}$$

However, on one hand, we compute

$$\begin{aligned} &R^N \left|\int_{|R\sigma - z|\leq\frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} (f(z) - f(R\sigma))dz\right| \\ \leq& AR^N \int_{|R\sigma - z|\leq\frac{R}{2}} \frac{dz}{|R\sigma - z|^{N-1}} \sup_{x\in B(R\sigma,\frac{R}{2})} |\nabla f(x)| \\ \leq& \frac{AR^{N+1}}{1 + R^{2N+1}} \underset{R\to+\infty}{\to} 0. \end{aligned}$$

On the other hand, we find

$$R^N 1_{|R\sigma - z|>\frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) \underset{R\to+\infty}{\to} (\delta_{j,k} - N\sigma_j\sigma_k)f(z).$$

Moreover, assumption (i) yields

$$\forall z \in B\left(R\sigma, \frac{R}{2}\right)^c, R^N \left| \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) \right| \leq \frac{A}{1 + |z|^{2N}},$$

so, by the dominated convergence theorem,

$$R^N \int_{|R\sigma - z| > \frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) dz \underset{R\to+\infty}{\to} (\delta_{j,k} - N\sigma_j\sigma_k) \int_{\mathbb{R}^N} f,$$

which leads to the desired result by equation (68).

Now, we show the second assertion, which relies on equation (32).

Step 2. *We have*

$$R^{N+1}\partial_l g(R\sigma) \underset{R\to+\infty}{\to} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} (-(\delta_{j,k}\sigma_l + \delta_{j,l}\sigma_k + \delta_{k,l}\sigma_j) + (N+2)\sigma_j\sigma_k\sigma_l) \int_{\mathbb{R}^N} f(x)dx.$$

The proof is rather similar to the previous one. Indeed, consider $R > 0$ and integrate equation (32) by parts:

$$\partial_l g(R\sigma) = \int_{B(0,\frac{R}{2})^c} \partial_l R_{j,k}(y) f(R\sigma - y) dy + \int_{B(0,\frac{R}{2})} \partial_l R_{j,k}(y)(f(R\sigma - y) - f(R\sigma) + y.\nabla f(R\sigma))dy + \frac{2}{R} \int_{S(0,\frac{R}{2})} R_{j,k}(y) y_l (f(R\sigma) - y.\nabla f(R\sigma))dy.$$

By the change of variable $z = R\sigma - y$, it becomes

$$R^{N+1}\partial_l g(R\sigma) = R^{N+1} \int_{B(R\sigma,\frac{R}{2})^c} \partial_l R_{j,k}(R\sigma - z) f(z) dz + R^{N+1} \int_{B(R\sigma,\frac{R}{2})} \partial_l R_{j,k}(R\sigma - z) (f(z) - f(R\sigma) + (R\sigma - z).\nabla f(R\sigma))dz + 2R^N \int_{S(0,\frac{R}{2})} R_{j,k}(y) y_l (f(R\sigma) - y.\nabla f(R\sigma))dy. \tag{69}$$

Now, by assumptions (i) and (ii),

$$R^N \left| \int_{S(0,\frac{R}{2})} R_{j,k}(y) y_l (f(R\sigma) + y.\nabla f(R\sigma))dy \right| \leq AR^N \left(\frac{1}{1 + R^{2N}} + \frac{R}{1 + R^{2N+1}} \right) \underset{R\to+\infty}{\to} 0,$$

while by assumptions (iii),

$$\begin{aligned} &R^{N+1} \left| \int_{B(R\sigma,\frac{R}{2})} \partial_l R_{j,k}(R\sigma - z)(f(z) - f(R\sigma) + (R\sigma - z).\nabla f(R\sigma))dz \right| \\ \leq &AR^{N+1} \int_{B(R\sigma,\frac{R}{2})} \frac{dz}{|R\sigma - z|^{N-1}} \sup_{x \in B(R\sigma,\frac{R}{2})} |d^2 f(x)| \\ \leq &A \frac{R^{N+2}}{1 + R^{2N+2}} \underset{R\to+\infty}{\to} 0. \end{aligned}$$

However, we compute

$$R^{N+2} 1_{|R\sigma - z| > \frac{R}{2}} \partial_l R_{j,k}(R\sigma - z) f(z) \underset{R\to+\infty}{\to} ((N+2)\sigma_j\sigma_k\sigma_l - (\delta_{j,k}\sigma_l + \delta_{j,l}\sigma_k + \delta_{k,l}\sigma_j)) f(z),$$

and by assumption (i),

$$\forall z \in B\left(R\sigma, \frac{R}{2}\right)^c, R^{N+1}\left|\partial_l R_{j,k}(R\sigma - z) f(z)\right| \leq \frac{A}{1+|z|^{2N}},$$

so, by the dominated convergence theorem,

$$R^{N+1}\int_{|R\sigma - z| > \frac{R}{2}} \partial_l R_{j,k}(R\sigma - z) f(z) dz$$
$$\underset{R\to+\infty}{\to} ((N+2)\sigma_j\sigma_k\sigma_l - (\delta_{j,k}\sigma_l + \delta_{j,l}\sigma_k + \delta_{k,l}\sigma_j)) \int_{\mathbb{R}^N} f(x)dx,$$

which completes the proofs of Step 2 and of Lemma 3 by equation (69). □

We are now in position to show Proposition 4.

Proof of Proposition 4. It follows from equations (21) and (24), and from Propositions 9 and 10 that there exist bounded measurable functions η_∞, η_∞^j, θ_∞^j, $\eta_\infty^{j,k}$ and $\theta_\infty^{j,k}$ such that for every $\sigma \in \mathbb{S}^{N-1}$,

$$\begin{cases} R^N \eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty(\sigma), \\ R^{N+1}\partial_j \eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty^j(\sigma), \\ R^N \partial_j \theta(R\sigma) \underset{R\to+\infty}{\to} \theta_\infty^j(\sigma), \\ R^{N+2}\partial_{j,k}^2 \eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty^{j,k}(\sigma), \\ R^{N+1}\partial_{j,k}^2 \theta(R\sigma) \underset{R\to+\infty}{\to} \theta_\infty^{j,k}(\sigma). \end{cases}$$

In particular, we can compute for every $\sigma \in \mathbb{S}^{N-1}$,

$$\eta_\infty(\sigma) = K_{0,\infty}(\sigma)\int_{\mathbb{R}^N} F(x)dx + 2c\sum_{j=1}^N K_{j,\infty}(\sigma)\int_{\mathbb{R}^N} G_j(x)dx \tag{70}$$

$$\theta_\infty^j(\sigma) = \frac{c}{2}K_{j,\infty}(\sigma)\int_{\mathbb{R}^N} F(x)dx + \sum_{k=1}^N (c^2 L_{j,k,\infty}(\sigma) + \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k))\int_{\mathbb{R}^N} G_k(x)dx. \tag{71}$$

Thus, it only remains to consider the existence of the function θ_∞. It follows from the next lemma.

Lemma 10. *Let $f \in C^1(\mathbb{R}^N, \mathbb{C})$ and $M > 1$. Assume that for every $j \in \{1, \ldots, N\}$, there is a bounded function f_∞^j defined on the sphere $\mathbb{S}^{N-1}$ such that*

$$\forall \sigma \in \mathbb{S}^{N-1}, R^M \partial_j f(R\sigma) \underset{R\to+\infty}{\to} f_\infty^j(\sigma),$$

and that

$$f(x) \underset{|x|\to+\infty}{\to} \lambda_\infty \in \mathbb{C}.$$

Then,

$$\forall \sigma \in \mathbb{S}^{N-1}, R^{M-1}(f(R\sigma) - \lambda_\infty) \underset{R\to+\infty}{\to} f_\infty(\sigma) = -\frac{1}{M-1}\sum_{j=1}^N \sigma_j f_\infty^j(\sigma).$$

Proof of Lemma 10. Indeed, f belongs to $C^1(\mathbb{R}^N,\mathbb{C})$ and converges to λ_∞ at infinity, so, since $M>1$, we can state

$$\forall R>1, f(R\sigma)-\lambda_\infty=-\int_R^{+\infty}\sum_{j=1}^N \partial_j f(r\sigma)\sigma_j dr.$$

Moreover, we have

$$\sum_{j=1}^N \partial_j f(r\sigma)\sigma_j=\frac{1}{r^M}\sum_{j=1}^N f_\infty^j(\sigma)\sigma_j+\underset{r\to+\infty}{o}\left(\frac{1}{r^M}\right),$$

therefore,

$$\int_R^{+\infty}\sum_{j=1}^N \partial_j f(r\sigma)\sigma_j dr=\frac{1}{(M-1)R^{M-1}}\sum_{j=1}^N f_\infty^j(\sigma)\sigma_j+\underset{R\to+\infty}{o}\left(\frac{1}{R^{M-1}}\right),$$

which yields

$$R^{M-1}(f(R\sigma)-\lambda_\infty)\underset{R\to+\infty}{\to}-\frac{1}{M-1}\sum_{j=1}^N f_\infty^j(\sigma)\sigma_j=f_\infty(\sigma).$$

□

At this stage, we notice that the function $\psi\theta$ satisfies all the assumptions of Lemma 10 with $M=N$ and $\lambda_\infty=0$. Thus, there is a bounded measurable function θ_∞ such that

$$R^{N-1}\theta(R\sigma)\underset{R\to+\infty}{\to}\theta_\infty(\sigma)=-\frac{1}{N-1}\sum_{j=1}^N\sigma_j\theta_\infty^j(\sigma).$$

Moreover, by equation (71), we compute the next more explicit form of θ_∞,

$$\begin{aligned}\theta_\infty(\sigma)=&-\frac{1}{N-1}\Bigg(\frac{c}{2}\Big(\sum_{j=1}^N\sigma_jK_{j,\infty}(\sigma)\Big)\int_{\mathbb{R}^N}F(x)dx+\sum_{k=1}^N\Big(c^2\sum_{j=1}^N\sigma_jL_{j,k,\infty}(\sigma)\\&-\frac{(N-1)\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\sigma_k\Big)\int_{\mathbb{R}^N}G_k(x)dx.\Bigg)\end{aligned}\tag{72}$$

□

Remark. Conjecture 1 follows from formulae (70) and (72). Indeed, in the first section of the second part, we computed formally the values of $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ (see formulae (62), (63) and (64)). By equations (70) and (72), it only remains to compute the values of $\int_{\mathbb{R}^N}F(x)dx$ and $\int_{\mathbb{R}^N}G_k(x)dx$ to get explicit expressions of the limits η_∞ and θ_∞. In the third part, we will compute such integrals and we will obtain that

$$\int_{\mathbb{R}^N}F(x)dx=2((4-N)E(v)+c(N-3)p(v)),$$

and

$$\int_{\mathbb{R}^N}G_k(x)dx=2P_k(v).$$

Finally, by equations (62), (63), (64), (70) and (72), it yields the value of the functions η_∞,

$$\eta_\infty(\sigma) = \frac{c\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Bigg(\Big(\frac{4-N}{2}cE(v)+\Big(2+\frac{N-3}{2}c^2\Big)p(v)\Big)\Big(\frac{1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}$$
$$-\frac{N\sigma_1^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big)-2\Big(1-\frac{c^2}{2}\Big)\sum_{j=2}^{N}P_j(v)\frac{N\sigma_1\sigma_j}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Bigg),$$

and θ_∞,

$$\theta_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Bigg(\Big(\frac{4-N}{2}cE(v)+\Big(2+\frac{N-3}{2}c^2\Big)p(v)\Big)\frac{\sigma_1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}$$
$$+2\Big(1-\frac{c^2}{2}\Big)\sum_{j=2}^{N}P_j(v)\frac{\sigma_j}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Bigg).$$

Since v_∞ is equal to θ_∞, it leads formally to Conjecture 1.

2.4 Uniformity of the convergence.

Now, we show the uniformity of the previous pointwise convergence. Actually, Proposition 5 even yields a little more. Indeed, the functions $\sigma \mapsto R^N\eta(R\sigma)$ and $\sigma \mapsto R^{N-1}\theta(R\sigma)$ converge to η_∞, respectively θ_∞, in $C^1(\mathbb{S}^{N-1})$, respectively $C^2(\mathbb{S}^{N-1})$, when R tends to $+\infty$. As claimed in the introduction, it follows from the decay estimates of Theorem 6 and Ascoli-Arzela's theorem.

Proof of Proposition 5. Consider the functions $(\eta_R)_{R>0}$ and $(\theta_R)_{R>0}$ defined by

$$\forall \sigma\in\mathbb{S}^{N-1}, \begin{cases} \eta_R(\sigma) = R^N\eta(R\sigma), \\ \theta_R(\sigma) = R^{N-1}(\psi\theta)(R\sigma), \\ v_R(\sigma) = R^{N-1}(v(R\sigma)-1). \end{cases}$$

Step 1. *Computation of some derivatives of the functions η_R and θ_R and of their limits at infinity.*

We first compute some explicit expressions of some derivatives of η_R and θ_R and of their limits when $R\to+\infty$. It will be fruitful to prove the uniformity of the convergence and to deduce Proposition 6. By Proposition 4, we first get for every $\sigma\in\mathbb{S}^{N-1}$,

$$\begin{cases} \eta_R(\sigma) \underset{R\to+\infty}{\to} \eta_\infty(\sigma), \\ \theta_R(\sigma) \underset{R\to+\infty}{\to} \theta_\infty(\sigma). \end{cases}$$

By definition, we then have for every $j\in\{1,\ldots,N\}$ and for every function $f\in C^1(\mathbb{S}^{N-1})$,

$$\partial_j^{\mathbb{S}^{N-1}} f(\sigma) = \lim_{t\to 0}\frac{f(\frac{\sigma+te_j}{|\sigma+te_j|})-f(\sigma)}{t}.$$

Therefore, considering a function $f\in C^1(\mathbb{R}^N)$ and denoting for every $R>0$ and $\sigma\in\mathbb{S}^{N-1}$,

$$f_R(\sigma) = f(R\sigma),$$

we compute

$$\partial_j^{\mathbb{S}^{N-1}} f_R(\sigma) = R(\partial_j f(R\sigma) - \sigma_j \sum_{i=1}^{N} \sigma_i \partial_i f(R\sigma)). \tag{73}$$

Likewise, we find for every $k \in \{1, \ldots, N\}$ and $\sigma \in \mathbb{S}^{N-1}$,

$$\partial_j^{\mathbb{S}^{N-1}} \sigma_k = \delta_{j,k} - \sigma_j \sigma_k. \tag{74}$$

Thus, it follows from formula (73) that

$$\begin{cases} \partial_j^{\mathbb{S}^{N-1}} \eta_R(\sigma) = R^{N+1}(\partial_j \eta(R\sigma) - \sigma_j \sum\limits_{k=1}^{N} \sigma_k \partial_k \eta(R\sigma)), \\ \partial_j^{\mathbb{S}^{N-1}} \theta_R(\sigma) = R^{N}(\partial_j (\psi\theta)(R\sigma) - \sigma_j \sum\limits_{k=1}^{N} \sigma_k \partial_k (\psi\theta)(R\sigma)). \end{cases} \tag{75}$$

By Proposition 4, it gives

$$\begin{cases} \partial_j^{\mathbb{S}^{N-1}} \eta_R(\sigma) \underset{R \to +\infty}{\to} \eta_\infty^j(\sigma) - \sigma_j \sum\limits_{k=1}^{N} \sigma_k \eta_\infty^k(\sigma), \\ \partial_j^{\mathbb{S}^{N-1}} \theta_R(\sigma) \underset{R \to +\infty}{\to} \theta_\infty^j(\sigma) - \sigma_j \sum\limits_{k=1}^{N} \sigma_k \theta_\infty^k(\sigma). \end{cases}$$

Moreover, the functions η and $\psi\theta$ satisfy all the assumptions of Lemma 10 with $M = N+1$, respectively $M = N$, and $\lambda_\infty = 0$. Therefore, Lemma 10 leads to

$$\begin{cases} \sum\limits_{k=1}^{N} \sigma_k \eta_\infty^k(\sigma) = -N \eta_\infty(\sigma), \\ \sum\limits_{k=1}^{N} \sigma_k \theta_\infty^k(\sigma) = -(N-1) \theta_\infty(\sigma), \end{cases} \tag{76}$$

and finally,

$$\begin{cases} \partial_j^{\mathbb{S}^{N-1}} \eta_R(\sigma) \underset{R \to +\infty}{\to} \eta_\infty^j(\sigma) + N \sigma_j \eta_\infty(\sigma), \\ \partial_j^{\mathbb{S}^{N-1}} \theta_R(\sigma) \underset{R \to +\infty}{\to} \theta_\infty^j(\sigma) + (N-1) \sigma_j \theta_\infty(\sigma). \end{cases} \tag{77}$$

Likewise, formulae (73) and (74) yield for every $(j,k) \in \{1, \ldots, N\}^2$,

$$\begin{aligned} \partial_k^{\mathbb{S}^{N-1}} (\partial_j^{\mathbb{S}^{N-1}} \theta_R)(\sigma) = & R^{N+1} \Bigg(\partial_{j,k}^2 \theta(R\sigma) - \sum_{l=1}^{N} \sigma_l \Big(\sigma_k \partial_{j,l}^2 \theta(R\sigma) + \sigma_j \partial_{k,l}^2 \theta(R\sigma) - \sigma_k \sigma_j \\ & \sum_{m=1}^{N} \sigma_m \partial_{l,m}^2 \theta(R\sigma) \Big) \Bigg) - R^N \sum_{l=1}^{N} \Big((\delta_{j,k} - \sigma_j \sigma_k) \sigma_l + (\delta_{k,l} - \sigma_k \sigma_l) \sigma_j \Big) \\ & \partial_l \theta(R\sigma), \end{aligned} \tag{78}$$

so, by Proposition 4,

$$\begin{aligned} \partial_k^{\mathbb{S}^{N-1}} (\partial_j^{\mathbb{S}^{N-1}} \theta_R)(\sigma) \underset{R \to +\infty}{\to} & \theta_\infty^{j,k}(\sigma) - \sum_{l=1}^{N} \sigma_l \Big(\sigma_k \theta_\infty^{j,l}(\sigma) + \sigma_j \theta_\infty^{k,l}(\sigma) - \sigma_k \sigma_j \sum_{m=1}^{N} \sigma_m \\ & \theta_\infty^{l,m}(\sigma) \Big) - \sum_{l=1}^{N} \Big((\delta_{j,k} - \sigma_j \sigma_k) \sigma_l + (\delta_{k,l} - \sigma_k \sigma_l) \sigma_j \Big) \theta_\infty^l(\sigma). \end{aligned}$$

However, the function $\partial_j \theta$ also satisfies the assumptions of Lemma 10 with $M = N + 1$ and $\lambda_\infty = 0$. Thus, we obtain likewise

$$\sum_{l=1}^{N} \sigma_l \theta_\infty^{j,l}(\sigma) = -N \theta_\infty^j(\sigma), \tag{79}$$

and

$$\partial_k^{\mathbb{S}^{N-1}} (\partial_j^{\mathbb{S}^{N-1}} \theta_R)(\sigma) \underset{R \to +\infty}{\to} \theta_\infty^{j,k}(\sigma) + N\sigma_k \theta_\infty^j(\sigma) + (N-1)\sigma_j \theta_\infty^k(\sigma) + (N-1)(\delta_{j,k} + (N-2)\sigma_j \sigma_k)\theta_\infty(\sigma). \tag{80}$$

Step 2. *Uniformity of the convergence.*

Now, assume for the sake of contradiction that $(\eta_R)_{R>0}$ does not converge to η_∞ in $C^1(\mathbb{S}^{N-1})$. There are then some real number $\varepsilon > 0$, and a sequence of positive real numbers $(R_n)_{n\in\mathbb{N}}$ tending to $+\infty$, such that

$$\forall n \in \mathbb{N}, \|\eta_{R_n} - \eta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \|\nabla^{\mathbb{S}^{N-1}} \eta_{R_n} - \nabla^{\mathbb{S}^{N-1}} \eta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} > \varepsilon.$$

However, on one hand, by Proposition 2 and equation (75), there is some real number A such that

$$\forall n \in \mathbb{N}, \begin{cases} \|\eta_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq A, \\ \|\nabla^{\mathbb{S}^{N-1}} \eta_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq A R_n^{N+1} \|\nabla \eta(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})} \leq A. \end{cases}$$

On the other hand, formulae (73), (74) and (75), Proposition 2 and Theorem 6 yield that

$$\|d^{2,\mathbb{S}^{N-1}} \eta_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq A(R_n^{N+1} \|\nabla \eta(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})} + R_n^{N+2} \|d^2 \eta(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})}) \leq A.$$

Therefore, by Ascoli-Arzela's theorem, up to a subsequence, $(\eta_{R_n})_{n\in\mathbb{N}}$ converges in the space $C^1(\mathbb{S}^{N-1})$. By Proposition 4, its limit is necessarily equal to η_∞, which yields a contradiction. Thus, $(\eta_R)_{R>0}$ converges to η_∞ in $C^1(\mathbb{S}^{N-1})$. In particular, η_∞ is of class C^1 on $\mathbb{S}^{N-1}$ and satisfies by equations (77) for every $j \in \{1, \ldots, N\}$,

$$\partial_j^{\mathbb{S}^{N-1}} \eta_\infty(\sigma) = \eta_\infty^j(\sigma) + N\sigma_j \eta_\infty(\sigma). \tag{81}$$

Likewise, by Proposition 2, Theorem 6 and equations (75) and (78), there is some real number A such that

$$\begin{cases} \|\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A, \\ \|\nabla^{\mathbb{S}^{N-1}} \theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A R^N \|\nabla(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} \leq A, \\ \|d^{2,\mathbb{S}^{N-1}} \theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A R^N (\|\nabla(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} + R\|d^2(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})}) \leq A. \end{cases}$$

Formulae (73), (74) and (78) then give

$$\|d^{3,\mathbb{S}^{N-1}} \theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A(R^N \|\nabla(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} + R^{N+1} \|d^2(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} + R^{N+2} \|d^3(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})}),$$

so, by Proposition 2 and Theorem 6,

$$\|d^{3,\mathbb{S}^{N-1}} \theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A.$$

Thus, up to the argument by contradiction above, the functions $(\theta_R)_{R>0}$ converge to θ_∞ in $C^2(\mathbb{S}^{N-1})$. In particular, θ_∞ is in $C^2(\mathbb{S}^{N-1})$ and satisfies by equations (77) and (80) for every $(j,k) \in \{1,\ldots,N\}^2$,

$$\partial_j^{\mathbb{S}^{N-1}} \theta_\infty(\sigma) = \theta_\infty^j(\sigma) + (N-1)\sigma_j \theta_\infty(\sigma), \tag{82}$$

and

$$\begin{aligned} \partial_k^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}}\theta_\infty)(\sigma) = \theta_\infty^{j,k}(\sigma) + N\sigma_k\theta_\infty^j(\sigma) + (N-1)\sigma_j\theta_\infty^k(\sigma) + (N-1)(\delta_{j,k} \\ + (N-2)\sigma_j\sigma_k)\theta_\infty(\sigma). \end{aligned} \tag{83}$$

Finally, we consider the uniform convergence of the function v_R. By definition, we have for every $\sigma \in \mathbb{S}^{N-1}$ and $R > 3R_0$,

$$v_R(\sigma) = R^{N-1}(\sqrt{1-\eta(R\sigma)}e^{i\theta(R\sigma)} - 1),$$

so, by Proposition 2 and the proof of the uniform convergences of η_R and θ_R just above,

$$\begin{aligned} &\|v_R - i\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \\ \leq& R^{N-1}\|\sqrt{1-\eta(R.)} - 1\|_{L^\infty(\mathbb{S}^{N-1})} + \|R^{N-1}(e^{i\theta(R.)} - 1) - i\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \\ \leq& A\bigg(\frac{1}{R}\|\eta_R\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R^{N-1}}\|\theta_R^2\|_{L^\infty(\mathbb{S}^{N-1})} + \|\theta_R - \theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})}\bigg) \\ &\underset{R\to+\infty}{\to} 0. \end{aligned}$$

Likewise, we compute for every $j \in \{1,\ldots,N\}$ by equation (73),

$$\begin{aligned} \partial_j^{\mathbb{S}^{N-1}} v_R(\sigma) = R^N\Bigg(& i\sqrt{1-\eta(R\sigma)}\partial_j\theta(R\sigma) - \frac{\partial_j\eta(R\sigma)}{2\sqrt{1-\eta(R\sigma)}} - \sigma_j\sum_{k=1}^N \sigma_k\bigg(-\frac{\partial_k\eta(R\sigma)}{2\sqrt{1-\eta(R\sigma)}} \\ &+ i\sqrt{1-\eta(R\sigma)}\partial_k\theta(R\sigma)\bigg)\Bigg)e^{i\theta(R\sigma)} \\ =& \Bigg(i\sqrt{1-\eta(R\sigma)}\partial_j^{\mathbb{S}^{N-1}}\theta_R(\sigma) - \frac{\partial_j^{\mathbb{S}^{N-1}}\eta_R(\sigma)}{2R\sqrt{1-\eta(R\sigma)}}\Bigg)e^{i\theta(R\sigma)}. \end{aligned}$$

Therefore, by Proposition 2 and the proof of the convergences in $C^1(\mathbb{S}^{N-1})$ of η_R and θ_R just above,

$$\begin{aligned} \|\partial_j^{\mathbb{S}^{N-1}} v_R - i\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leq& A\Big(\|\partial_j^{\mathbb{S}^{N-1}}\theta_R - i\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \|(\sqrt{1-\eta(R.)} \\ & e^{i\theta(R.)} - 1)\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R}\|\partial_j^{\mathbb{S}^{N-1}}\eta_R\|_{L^\infty(\mathbb{S}^{N-1})}\Big) \\ \leq& A\Big(\|\partial_j^{\mathbb{S}^{N-1}}\theta_R - i\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R^N}\|\eta_R\|_{L^\infty(\mathbb{S}^{N-1})} \\ &+ \frac{1}{R^{N-1}}\|\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R}\|\partial_j^{\mathbb{S}^{N-1}}\eta_R\|_{L^\infty(\mathbb{S}^{N-1})}\Big) \\ &\underset{R\to+\infty}{\to} 0. \end{aligned}$$

Thus, denoting $v_\infty = \theta_\infty$, v_∞ is a smooth function on $\mathbb{S}^{N-1}$, which satisfies

$$\|v_R - v_\infty\|_{C^1(\mathbb{S}^{N-1})} \underset{R\to+\infty}{\to} 0.$$

This concludes the proof of Proposition 5. □

2.5 Partial differential equations satisfied by η_∞, θ_∞ and v_∞.

Finally, we deduce from the proof of Proposition 5 just above the partial differential equations satisfied by η_∞ and θ_∞.

Proof of Proposition 6. Let $\sigma \in \mathbb{S}^{N-1}$. On one hand, we compute from equation (2) on a neighbourhood of infinity

$$\Delta\eta + 2|\nabla v|^2 + 2c\partial_1\theta - 2\eta - 2c\eta\partial_1\theta + 2\eta^2 = 0,$$

so, for every $R > 0$,

$$R^N(\Delta\eta(R\sigma) + 2|\nabla v(R\sigma)|^2 - 2c\partial_1\theta(R\sigma) - 2\eta(R\sigma) + 2c\eta(R\sigma)\partial_1\theta(R\sigma) + 2\eta(R\sigma)^2) = 0.$$

Taking the limit $R \to +\infty$, it gives by Propositions 2 and 4, and Theorem 6,

$$\eta_\infty(\sigma) = c\theta^1_\infty(\sigma),$$

which reduces to equation (34) by equation (82).
On the other hand, equation (18) yields on a neighbourhood of infinity

$$R^{N+1}(\Delta\theta(R\sigma) - \frac{c}{2}\partial_1\eta(R\sigma) - \nabla\eta(R\sigma).\nabla\theta(R\sigma) - \eta\Delta\theta(R\sigma)) = 0.$$

Therefore, Propositions 2 and 4, and Theorem 6 yield once again at the limit $R \to +\infty$,

$$\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) = \frac{c}{2}\eta^1_\infty(\sigma),$$

which gives by equation (81),

$$\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) = \frac{c}{2}(\partial_1^{\mathbb{S}^{N-1}}\eta_\infty(\sigma) - N\sigma_1\eta_\infty(\sigma)).$$

However, by equations (82) and (83),

$$\begin{aligned}\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) &= \sum_{j=1}^N \partial_j^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}}\theta_\infty)(\sigma) - (2N-1)\sum_{j=1}^N \sigma_j\theta^j_\infty(\sigma)\\ &- (N-1)\sum_{j=1}^N(1 + (N-2)\sigma_j^2)\theta_\infty(\sigma)\\ &= \Delta^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) - (2N-1)\sum_{j=1}^N \sigma_j\theta^j_\infty(\sigma) - (N-1)(2N-2)\theta_\infty(\sigma).\end{aligned}$$

Equation (76) then states

$$\sum_{j=1}^N \sigma_j\theta^j_\infty(\sigma) = -(N-1)\theta_\infty(\sigma),$$

so,

$$\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) = \Delta^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) + (N-1)\theta_\infty(\sigma).$$

Thus, we finally find equation (35),

$$\Delta^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) + (N-1)\theta_\infty(\sigma) = \frac{c}{2}(\partial_1^{\mathbb{S}^{N-1}}\eta_\infty(\sigma) - N\sigma_1\eta_\infty(\sigma)).$$

Now, it only remains to prove that the functions θ_∞ and η_∞ are smooth on $\mathbb{S}^{N-1}$. Indeed, equations (34) and (35) give

$$\Delta^{\mathbb{S}^{N-1}}\theta_\infty - \frac{c^2}{2}\partial_1^{\mathbb{S}^{N-1}}(\partial_1^{\mathbb{S}^{N-1}}\theta_\infty) + c^2(N-1)\sigma_1\partial_1^{\mathbb{S}^{N-1}}\theta_\infty + (N-1)(1+\frac{c^2}{2} - (N+1)\frac{c^2}{2}\sigma_1^2)\theta_\infty = 0. \tag{84}$$

Thus, θ_∞ is solution on $\mathbb{S}^{N-1}$ of an elliptic partial differential system with smooth coefficients. By standard elliptic theory, it is of class C^∞ on $\mathbb{S}^{N-1}$. By equation (34), η_∞ is also smooth on $\mathbb{S}^{N-1}$. □

We conclude the second part by the proof of Theorem 1, which follows from Proposition 5 and equation (84).

Proof of Theorem 1. By Proposition 5, there exists a smooth function $v_\infty = \theta_\infty$ on $\mathbb{S}^{N-1}$ such that

$$|x|^{N-1}(v(x)-1) - iv_\infty\left(\frac{x}{|x|}\right) \underset{|x|\to+\infty}{\to} 0 \text{ uniformly}.$$

Moreover, by equation (84), v_∞ satisfies the linear partial differential equation (10). □

3 Asymptotics in dimension two and in the axisymmetric case.

In the last part, we focus on the axisymmetric case and on the case of dimension two. In both cases, the system of equations (34) and (35) reduces to an entirely integrable system of linear ordinary differential equations of second order. In Proposition 7, we compute explicitly its solutions up to undetermined constants α and β. Lemma 6 in connection with the Pohozaev identities of Lemma 7 links the value of α and β with the energy $E(v)$ and the momentum $\vec{P}(v)$, which completes the proof of Theorems 2 and 3. Finally, we deduce Corollary 1 from Lemma 7.

3.1 Explicit expression for the first order term.

This section is devoted to the integration of the system of equations (34) and (35) in dimension two and in the axisymmetric case. It relies on the use of spherical coordinates. That is the reason why we first recall some of their properties.

Indeed, let $\Phi_N : \Omega = \mathbb{R}_+ \times [0,\pi]^{N-2} \times [0,2\pi] \mapsto \mathbb{R}^N$, the function defined by

$$\Phi_N(r,\beta_1,\ldots,\beta_{N-1}) = (r\cos(\beta_1), r\sin(\beta_1)\cos(\beta_2),\ldots, r\prod_{i=1}^{N-1}\sin(\beta_i)).$$

The function Φ_N is smooth on Ω and its Jacobian matrix is

$$J(\Phi_N)(r,\beta_1,\ldots,\beta_{N-1}) = (J_{i,j})_{1\leq i,j\leq N},$$

where

$$\begin{cases} J_{1,j} = \prod_{k=1}^{j-1} \sin(\beta_k)\cos(\beta_j), \\ J_{i,j} = 0, \text{ if } i \geq 2 \text{ and } j \leq i-2, \\ J_{i,i-1} = -r \prod_{k=1}^{i-1} \sin(\beta_k), \\ J_{i,j} = r \prod_{k=1}^{j-1} \sin(\beta_k)\cos(\beta_j)\cos(\beta_{i-1}), \text{ otherwise.} \end{cases}$$

Thus, $J(\Phi_N)$ is invertible if and only if $r \neq 0$ and $\beta_j \neq 0$ modulo π for every $j \in \{1, \ldots, N-2\}$. Moreover, its inverse is

$$J(\Phi_N)^{-1}(r, \beta_1, \ldots, \beta_{N-1}) = (J^{-1}_{i,j})_{1\leq i,j\leq N},$$

where

$$\begin{cases} J^{-1}_{i,1} = \prod_{k=1}^{i-1} \sin(\beta_k)\cos(\beta_i), \\ J^{-1}_{i,j} = 0, \text{ if } j \geq 2 \text{ and } i \leq j-2, \\ J^{-1}_{j-1,j} = -\frac{\sin(\beta_{j-1})}{r \prod_{k=1}^{j-2} \sin(\beta_k)}, \\ J^{-1}_{i,j} = \frac{\prod_{k=j}^{i-1} \sin(\beta_k)}{r \prod_{k=1}^{j-2} \sin(\beta_k)} \cos(\beta_{j-1})\cos(\beta_i), \text{ otherwise.} \end{cases}$$

Therefore, if we consider a smooth function $f \in C^\infty(\mathbb{R}^N)$ and denote

$$g = f \circ \Phi_N,$$

the chain rule theorem yields

$$\forall y \in \Omega, J(\Phi_N)(y) \begin{pmatrix} \partial_1 f(\Phi_N(y)) \\ \vdots \\ \partial_N f(\Phi_N(y)) \end{pmatrix} = \begin{pmatrix} \partial_r g(y) \\ \vdots \\ \partial_{\beta_{N-1}} g(y) \end{pmatrix}.$$

Moreover, assuming f is axisymmetric around axis x_1 or the dimension N is equal to two, the function g is independent on the variables β_2, ... and β_N, which yields for every $j \in \{2, \ldots, N\}$,

$$\begin{aligned} \partial_1 f(\Phi_N(y)) &= \cos(\beta_1)\partial_r g(y) - \frac{\sin(\beta_1)}{r}\partial_{\beta_1} g(y), \\ \partial_j f(\Phi_N(y)) &= \prod_{k=1}^{j-1} \sin(\beta_k)\cos(\beta_j)\partial_r g(y) + \frac{\cos(\beta_1)\cos(\beta_j)}{r}\prod_{k=2}^{j-1} \sin(\beta_k)\partial_{\beta_1} g(y), \\ \partial^2_{1,1} f(\Phi_N(y)) &= \cos^2(\beta_1)\partial^2_{r,r} g(y) + \frac{2\sin(\beta_1)\cos(\beta_1)}{r^2}\partial_{\beta_1} g(y) - \frac{2\sin(\beta_1)\cos(\beta_1)}{r}\partial^2_{r,\beta_1} g(y) \\ &\quad + \frac{\sin^2(\beta_1)}{r}\partial_r g(y) + \frac{\sin^2(\beta_1)}{r^2}\partial^2_{\beta_1,\beta_1} g(y), \\ \partial^2_{j,j} f(\Phi_N(y)) &= \prod_{k=2}^{j-1} \sin^2(\beta_k)\cos^2(\beta_j)\Big(\sin^2(\beta_1)\partial^2_{r,r} g(y) + \frac{2\sin(\beta_1)\cos(\beta_1)}{r}\partial^2_{r,\beta_1} g(y) \\ &\quad - \frac{2\sin(\beta_1)\cos(\beta_1)}{r^2}\partial_{\beta_1} g(y) + \frac{\cos^2(\beta_1)}{r}\partial_r g(y) + \frac{\cos^2(\beta_1)}{r^2}\partial^2_{\beta_1,\beta_1} g(y) \\ &\quad - \frac{1}{r}\partial_r g(y) - \frac{\cos(\beta_1)}{r^2\sin(\beta_1)}\partial_{\beta_1} g(y)\Big) + \frac{1}{r}\partial_r g(y) + \frac{\cos(\beta_1)}{r^2\sin(\beta_1)}\partial_{\beta_1} g(y), \\ \Delta f(\Phi_N(y)) &= \partial^2_{r,r} g(y) + \frac{N-1}{r}\partial_r g(y) + \frac{1}{r^2}(\partial^2_{\beta_1,\beta_1} g(y) + (N-2)\mathrm{cotan}(\beta_1)\partial_{\beta_1} g(y)), \end{aligned}$$

provided that $r \neq 0$ and $\sin(\beta_1) \neq 0$. Finally, consider now a smooth function $f \in C^\infty(\mathbb{S}^{N-1})$ and denote

$$g(\beta_1, \ldots, \beta_{N-1}) = f(\Phi_N(1, \beta_1, \ldots, \beta_{N-1})).$$

Assuming f is axisymmetric around axis x_1 or the dimension N is two, we deduce that for every $y = (1, \beta_1, \ldots, \beta_{N-1})$ such that $\sin(\beta_1) \neq 0$,

$$\begin{aligned} \partial_1^{\mathbb{S}^{N-1}} f(\Phi_N(y)) &= -\sin(\beta_1)\partial_{\beta_1} g(y), \\ \partial_{1,1}^{2,\mathbb{S}^{N-1}} f(\Phi_N(y)) &= \sin^2(\beta_1)\partial^2_{\beta_1,\beta_1} g(y) + 2\sin(\beta_1)\cos(\beta_1)\partial_{\beta_1} g(y), \\ \Delta_{\mathbb{S}^{N-1}} f(\Phi_N(y)) &= \partial^2_{\beta_1,\beta_1} g(y) + (N-2)\mathrm{cotan}(\beta_1)\partial_{\beta_1} g(y). \end{aligned} \tag{85}$$

Proposition 7 is then a consequence of formulae (85), and equations (34) and (35).

Proof of Proposition 7. In this proof, the dimension N is assumed to be two, or the travelling wave v is supposed to be axisymmetric around axis x_1. Thus, the functions η_∞ and θ_∞ only depend on the variable β_1 in spherical coordinates. Up to a misuse of notations, we will consider them as functions of β_1.

However, by Proposition 6, θ_∞ is smooth on $\mathbb{S}^{N-1}$ and satisfies equation (84). Therefore, in the new variables, it is smooth on $[0, \pi]$ in dimension $N \geq 3$, respectively $[0, 2\pi]$ in dimension two. Moreover, by equation (84) and formulae (85), it verifies the second order ordinary differential equation

$$\begin{aligned} &(1 - \frac{c^2}{2}\sin^2(\beta_1))\theta''_\infty(\beta_1) + ((N-2)\mathrm{cotan}(\beta_1) - Nc^2\cos(\beta_1)\sin(\beta_1))\theta'_\infty(\beta_1) + (N-1) \\ &(1 + \frac{c^2}{2} - (N+1)\frac{c^2}{2}\cos^2(\beta_1))\theta_\infty(\beta_1) = 0. \end{aligned} \tag{86}$$

The articles of C.A. Jones, S.J. Putterman and P.H. Roberts [29, 30] yield one particular solution of equation (86) in dimensions two and three. Generalising its form to every dimension, we find a first solution equal to

$$Sol_1(\beta_1) = \frac{\cos(\beta_1)}{(1 - \frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}}}.$$

However, the set of solutions on $]0, \pi[$ in dimension $N \geq 3$, respectively $]0, \pi[$ and $]\pi, 2\pi[$ in dimension two, is a vectorial space of dimension two. In order to find another independent solution, we let

$$u(\beta_1) = \frac{\theta_\infty(\beta_1)}{Sol_1(\beta_1)},$$

for every $\beta_1 \in]0, \pi[\setminus\{\frac{\pi}{2}\}$ in dimension $N \geq 3$, respectively $\beta_1 \in]0, \pi[\setminus\{\frac{\pi}{2}\} \cup]\pi, 2\pi[\setminus\{\frac{3\pi}{2}\}$ in dimension two. We then compute the next ordinary differential equation for the function u,

$$\sin(\beta_1)\cos(\beta_1)(1 - \frac{c^2}{2}\sin^2(\beta_1))u''(\beta_1) + (N - 2 - N\sin^2(\beta_1) + c^2\sin^4(\beta_1))u'(\beta_1) = 0.$$

After a first integration, we deduce that there is some real constant A such that

$$u'(\beta_1) = A\frac{(1 - \frac{c^2}{2}\sin^2(\beta_1))^{\frac{N-2}{2}}}{\cos^2(\beta_1)\sin^{N-2}(\beta_1)},$$

and, after another integration, we infer that there is another real constant B such that

$$u(\beta_1) = B + A\sum_{k=0}^{p-1} \frac{1}{2(k-p)+3} C_{p-1}^k \left(1-\frac{c^2}{2}\right)^{k+p-\frac{3}{2}} \tan^{2(k-p)+3}(\beta_1)$$

if $N = 2p$, and if $N = 2p+1$,

$$\begin{aligned} u(\beta_1) =& B + A\left(1-\frac{c^2}{2}\right)^{p-1}\left(\sqrt{1+\left(1-\frac{c^2}{2}\right)\tan^2(\beta_1)} + \sum_{k=1}^{p}\frac{C_p^k}{a_k}\right.\\ &\left.\left(\ln\left(\frac{\sqrt{1-\frac{c^2}{2}}\tan(\beta_1)}{1+\sqrt{1+(1-\frac{c^2}{2})\tan^2(\beta_1)}}\right) - \sum_{q=1}^{k-1}\frac{a_{q+1}}{2q}\frac{\sqrt{1+(1-\frac{c^2}{2})\tan^2(\beta_1)}}{(1-\frac{c^2}{2})^q\tan^{2q}(\beta_1)}\right)\right), \end{aligned}$$

where

$$\forall k \in \mathbb{N}^*, a_k = \frac{(-4)^{k-1}((k-1)!)^2}{(2(k-1))!}.$$

Thus, we find another particular solution equal to

$$Sol_2(\beta_1) = \frac{\sin(\beta_1)}{1-\frac{c^2}{2}\sin^2(\beta_1)}$$

if $N = 2$,

$$Sol_2(\beta_1) = \frac{\cos(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^p}\sum_{k=0}^{p-1}\frac{1}{2(k-p)+3}C_{p-1}^k\left(1-\frac{c^2}{2}\right)^{k+p-\frac{3}{2}}\tan^{2(k-p)+3}(\beta_1)$$

if $N = 2p$ and $p > 1$, and if $N = 2p+1$,

$$\begin{aligned} Sol_2(\beta_1) =& \frac{\cos(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{2p+1}{2}}}\left(1-\frac{c^2}{2}\right)^{p-1}\left(\sqrt{1+\left(1-\frac{c^2}{2}\right)\tan^2(\beta_1)} + \sum_{k=1}^{p}\frac{C_p^k}{a_k}\right.\\ &\left.\left(\ln\left(\frac{\sqrt{1-\frac{c^2}{2}}\tan(\beta_1)}{1+\sqrt{1+(1-\frac{c^2}{2})\tan^2(\beta_1)}}\right) - \sum_{q=1}^{k-1}\frac{a_{q+1}}{2q}\frac{\sqrt{1+(1-\frac{c^2}{2})\tan^2(\beta_1)}}{(1-\frac{c^2}{2})^q\tan^{2q}(\beta_1)}\right)\right). \end{aligned}$$

In particular, we remark that

$$Sol_2(\beta_1) \underset{\beta_1\to 0}{\sim} \frac{(1-\frac{c^2}{2})^{p-\frac{3}{2}}}{(3-2p)\beta_1^{2p-3}}, \tag{87}$$

if $N = 2p$ and $p > 1$,

$$Sol_2(\beta_1) \underset{\beta_1\to 0}{\sim} \ln(\beta_1) \tag{88}$$

if $N = 3$, and if $N = 2p+1$ with $p > 1$,

$$Sol_2(\beta_1) \underset{\beta_1\to 0}{\sim} \frac{1}{(2-2p)\beta_1^{2p-2}}. \tag{89}$$

Thus, every solution v of equation (86) writes as

$$v(\beta_1) = A Sol_1(\beta_1) + B Sol_2(\beta_1)$$

on $]0,\pi[\setminus\{\frac{\pi}{2}\}$ in dimension $N \geq 3$, respectively $]0,\pi[\setminus\{\frac{\pi}{2}\}$ and $]\pi,2\pi[\setminus\{\frac{3\pi}{2}\}$ in dimension two.

Actually, θ_∞ is a smooth, bounded solution of equation (86). By assertions (87), (88) and (89), the functions Sol_2 are not bounded at the point $\beta_1 = 0$ in dimension $N \geq 3$, so, there is some real constant α such that

$$\theta_\infty(\beta_1) = \alpha Sol_1(\beta_1) = \frac{\alpha\cos(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}}},$$

which yields formula (37) in the axisymmetric case. On the other hand, in dimension two, both solutions Sol_1 and Sol_2 are smooth and bounded on $\mathbb{S}^1$. Therefore, there are some real constants α and β such that

$$\theta_\infty(\beta_1) = \alpha\frac{\cos(\beta_1)}{1-\frac{c^2}{2}+\frac{c^2\cos^2(\beta_1)}{2}} + \beta\frac{\sin(\beta_1)}{1-\frac{c^2}{2}+\frac{c^2\cos^2(\beta_1)}{2}},$$

which is formula (39). Moreover, in dimension two, the axisymmetric travelling waves are even functions of β_1. Thus, if the travelling wave v is axisymmetric, the function θ_∞ is an even function of β_1, which means that the constant β vanishes and which leads to equation (37) in dimension two.

Now, equation (34) yields in spherical coordinates, up to a new misuse of notations,

$$\eta_\infty(\beta_1) = -c(\sin(\beta_1)\theta'_\infty(\beta_1) + (N-1)\cos(\beta_1)\theta_\infty(\beta_1)).$$

In dimension two, it gives equation (38),

$$\eta_\infty(\beta_1) = \alpha c\left(\frac{1}{1-\frac{c^2}{2}\sin^2(\beta_1)} - \frac{2\cos^2(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^2}\right) - 2\beta c\frac{\sin(\beta_1)\cos(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^2},$$

while in the axisymmetric case, it gives formula (36),

$$\eta_\infty(\beta_1) = \alpha c\left(\frac{1}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}}} - \frac{N\cos^2(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}+1}}\right).$$

This ends the proof of Proposition 7. □

3.2 Value of the stretched dipole coefficient.

Finally, we link the values of the coefficients α and β to the energy $E(v)$ and the momentum $\vec{P}(v)$. The proof essentially relies on integral equations which are summed up by Lemmas 6 and 7. In Lemma 7, we state Pohozaev's identities for equation (2). They follow from the multiplication of equation (2) by the standard Pohozaev multipliers $x_j\partial_j v(x)$ and several integrations by parts. They were already derived in [23], so, we omit their proof here. On the other hand, Lemma 6 provides integral equations (40) and (41). In particular, equation (40) is similar to the new integral relation of [23]. The main difference is that the speed c is now supposed to be subsonic, whereas it was supersonic in [23].

Proof of Lemma 6.

Step 1. *Proof of equation* (40).

The proof relies on the multiplication of equation (2) by the standard multipliers v and iv. Indeed, consider the function defined by

$$\forall R > 0, \Phi(R) = \int_{B(0,R)} \eta(x)dx.$$

The multiplication of equation (2) by the function v gives after some integrations by parts,

$$\int_{B(0,R)} (|\nabla v|^2 + \eta^2) = c \int_{B(0,R)} i\partial_1 v.v + \Phi(R) + \int_{S(0,R)} \partial_\nu v.v,$$

which also writes for R sufficiently large,

$$\int_{B(0,R)} (|\nabla v|^2 + \eta^2) = c \int_{B(0,R)} \Big(i\partial_1 v.v + \partial_1(\psi\theta)\Big) + \Phi(R) - \frac{1}{2} \int_{S(0,R)} \partial_\nu \eta - c \int_{S(0,R)} \nu_1 \theta. \quad (90)$$

By Proposition 1, we infer

$$\int_{B(0,R)} (|\nabla v|^2 + \eta^2) \underset{R \to +\infty}{\to} \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2),$$

while by definition,

$$\int_{B(0,R)} (i\partial_1 v.v + \partial_1(\psi\theta)) \underset{R \to +\infty}{\to} 2p(v). \quad (91)$$

However, Proposition 2 yields

$$\left| \int_{S(0,R)} \partial_\nu \eta \right| \leq \frac{AR^{N-1}}{R^{N+1}} \underset{R \to +\infty}{\to} 0,$$

while Proposition 5 gives

$$\int_{S(0,R)} \nu_1 \theta = R^{N-1} \int_{\mathbb{S}^{N-1}} \sigma_1 \theta(R\sigma) d\sigma \underset{R \to +\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1 \theta_\infty(\sigma) d\sigma. \quad (92)$$

Thus, equation (90) leads to

$$\Phi(R) \underset{R \to +\infty}{\to} \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2cp(v) + c \int_{\mathbb{S}^{N-1}} \sigma_1 \theta_\infty(\sigma) d\sigma. \quad (93)$$

On the other hand, we can also multiply equation (2) by the function iv to find

$$\frac{c}{2} \partial_1 \eta + \operatorname{div}(i\nabla v.v) = 0. \quad (94)$$

Now, we multiply this equation by the function x_1 and integrate by parts to obtain

$$\frac{c}{2} \Phi(R) + \int_{B(0,R)} i\partial_1 v.v = \int_{S(0,R)} \Big(\frac{c}{2} R\nu_1^2 \eta + R\nu_1 i\partial_\nu v.v\Big),$$

which also writes for R sufficiently large

$$\frac{c}{2} \Phi(R) = - \int_{B(0,R)} (\partial_1(\psi\theta) + i\partial_1 v.v) + \int_{S(0,R)} \Big(\frac{c}{2} R\nu_1^2 \eta + R\nu_1 i\partial_\nu v.v + \nu_1 \theta\Big). \quad (95)$$

By Proposition 5, we get

$$\int_{S(0,R)} R\nu_1^2 \eta = R^N \int_{\mathbb{S}^{N-1}} \sigma_1^2 \eta(R\sigma) d\sigma \underset{R \to +\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1^2 \eta_\infty(\sigma) d\sigma.$$

We then compute

$$\int_{S(0,R)} R\nu_1 i\partial_\nu v.v = -\int_{S(0,R)} R\nu_1\rho^2\partial_\nu\theta = \int_{S(0,R)} R\nu_1\eta\partial_\nu\theta - \int_{S(0,R)} R\nu_1\sum_{k=1}^N \nu_k\partial_k\theta.$$

However, on one hand, Proposition 2 gives

$$\left|\int_{S(0,R)} R\nu_1\eta\partial_\nu\theta\right| \leq \frac{AR^N}{R^{2N}} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by Propositions 2 and 4, equation (76) and the dominated convergence theorem, we compute

$$\int_{S(0,R)} R\nu_1\sum_{k=1}^N \nu_k\partial_k\theta = \int_{\mathbb{S}^{N-1}} R^N\sigma_1\sum_{k=1}^N \sigma_k\partial_k\theta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1\sum_{k=1}^N \sigma_k\theta_\infty^k(\sigma)d\sigma$$
$$= -(N-1)\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma.$$

Thus, it follows from equations (91), (92) and (95) that

$$\Phi(R) \underset{R\to+\infty}{\to} -\frac{4}{c}p(v) + \int_{\mathbb{S}^{N-1}} \sigma_1^2\eta_\infty(\sigma)d\sigma + \frac{2N}{c}\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma.$$

By equation (93) and by uniqueness of the limit of the function Φ in $+\infty$, we finally find

$$\int_{\mathbb{R}^N}(|\nabla v|^2+\eta^2)-2cp(v)+c\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma = -\frac{4}{c}p(v)+\int_{\mathbb{S}^{N-1}}(\sigma_1^2\eta_\infty(\sigma)+\frac{2N}{c}\sigma_1\theta_\infty(\sigma))d\sigma,$$

which yields immediately equation (40).

Step 2. *Proof of equation* (41)

The proof relies once more on equation (94) just above. Here, we multiply it by the function x_j for any $j \in \{2, \dots, N\}$ and integrate by parts on the ball $B(0,R)$ to obtain

$$\int_{B(0,R)} i\partial_j v.v = \int_{S(0,R)} \Big(\frac{c}{2}R\nu_1\nu_j\eta + R\nu_j i\partial_\nu v.v\Big),$$

which also writes for R sufficiently large

$$\int_{B(0,R)}(\partial_j(\psi\theta) + i\partial_j v.v) = \int_{S(0,R)} \Big(\frac{c}{2}R\nu_j\nu_1\eta + R\nu_j i\partial_\nu v.v + \nu_j\theta\Big). \tag{96}$$

By Proposition 5,

$$\int_{S(0,R)} R\nu_1\nu_j\eta = R^N\int_{\mathbb{S}^{N-1}} \sigma_1\sigma_j\eta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1\sigma_j\eta_\infty(\sigma)d\sigma,$$

and

$$\int_{S(0,R)} \nu_j\theta = R^{N-1}\int_{\mathbb{S}^{N-1}} \sigma_j\theta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_j\theta_\infty(\sigma)d\sigma.$$

Likewise, we compute

$$\int_{S(0,R)} R\nu_j i\partial_\nu v.v = -\int_{S(0,R)} R\nu_j\rho^2\partial_\nu\theta = \int_{S(0,R)} R\nu_j\eta\partial_\nu\theta - \int_{S(0,R)} R\nu_j\sum_{k=1}^N \nu_k\partial_k\theta.$$

However, on one hand, Proposition 2 gives

$$\left|\int_{S(0,R)} R\nu_j \eta \partial_\nu \theta\right| \leq \frac{AR^N}{R^{2N}} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by Propositions 2 and 4, equation (76) and the dominated convergence theorem, we get

$$\int_{S(0,R)} R\nu_j \sum_{k=1}^N \nu_k \partial_k \theta = \int_{\mathbb{S}^{N-1}} R^N \sigma_j \sum_{k=1}^N \sigma_k \partial_k \theta(R\sigma) d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_j \sum_{k=1}^N \sigma_k \theta_\infty^k(\sigma) d\sigma$$
$$= -(N-1) \int_{\mathbb{S}^{N-1}} \sigma_j \theta_\infty(\sigma) d\sigma.$$

Thus, it follows from the definition of the momentum and from equation (96) that

$$2\vec{P}_j(v) = \frac{c}{2} \int_{\mathbb{S}^{N-1}} \sigma_1 \sigma_j \eta_\infty(\sigma) d\sigma + N \int_{\mathbb{S}^{N-1}} \sigma_j \theta_\infty(\sigma) d\sigma,$$

which is equation (41). □

We now complete the proof of Theorem 2.

Proof of Theorem 2. By Proposition 7, we already know

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \theta_\infty(\sigma) = \frac{\alpha \sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}}}.$$

Thus, it only remains to deduce the value of the stretched dipole coefficient α from formula (40). Indeed, by Proposition 7, formula (40) writes

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2c\Big(1 - \frac{2}{c^2}\Big) p(v) = \alpha c \Bigg(\Big(\frac{2N}{c^2} - 1\Big) \int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}}} d\sigma + \int_{\mathbb{S}^{N-1}} \Bigg(\frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N\sigma_1^4}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}+1}} \Bigg) d\sigma \Bigg).$$

Denoting

$$J_1 = \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2c\Big(1 - \frac{2}{c^2}\Big) p(v),$$

and

$$J_2 = \frac{2N}{c} \int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}}} d\sigma - Nc \int_{\mathbb{S}^{N-1}} \frac{\sigma_1^4}{(1 - \frac{c^2}{2} + \frac{c^2 \sigma_1^2}{2})^{\frac{N}{2}+1}} d\sigma,$$

it also writes

$$J_1 = \alpha J_2. \tag{97}$$

Now, we express J_1 in function of the energy $E(v)$ and the momentum $p(v)$. Indeed, Lemma 7 yields

$$\int_{\mathbb{R}^N} |\partial_1 v|^2 = E(v),$$

and

$$\int_{\mathbb{R}^N} |\nabla_\perp v|^2 = (N-1)(E(v) - cp(v)),$$

where $\nabla_\perp v$ is defined by

$$\nabla_\perp v = (\partial_2 v, \dots, \partial_N v).$$

However, by definition,

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\partial_1 v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \eta^2,$$

so,

$$\int_{\mathbb{R}^N} \eta^2 = 2(N-1)cp(v) - 2(N-2)E(v).$$

Thus, we conclude that

$$J_1 = (4-N)E(v) + \Big((N-3)c + \frac{4}{c}\Big)p(v). \tag{98}$$

On the other hand, we can explicitly compute the value of J_2 in function of c and N. Indeed, we have

$$J_2 = \frac{2N}{c}\Big(1 - \frac{c^2}{2}\Big)\int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}} d\sigma. \tag{99}$$

Therefore, we are reduced to estimate the integral defined by

$$I(N,c) = \int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}} d\sigma. \tag{100}$$

In dimension two, we use the polar coordinates to compute such an integral,

$$\begin{aligned} I(N,c) = \int_0^{2\pi} \frac{\cos^2(\beta)}{(1 - \frac{c^2}{2}\sin^2(\beta))^2} d\beta &= 4\int_0^{+\infty} \frac{dt}{(1 + (1 - \frac{c^2}{2})t^2)^2} \\ &= \frac{4}{\sqrt{1 - \frac{c^2}{2}}}\int_0^{+\infty} \frac{du}{(1+u^2)^2} \\ &= \frac{\pi}{\sqrt{1 - \frac{c^2}{2}}}, \end{aligned}$$

where we made the successive changes of variables $t = \tan(\beta)$ and $u = \sqrt{1 - \frac{c^2}{2}}t$.

In dimension $N \geq 3$, we use the spherical coordinates:

$$I(N,c) = |\mathbb{S}^{N-2}| \int_0^{\pi} \frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1 - \frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}} d\beta. \tag{101}$$

At this stage, the computations depend on the parity of the dimension N. Assuming first that $N = 2p+2$ is even, we find

$$|\mathbb{S}^{2p}| = \frac{2^{2p+1}\pi^p p!}{(2p)!},$$

and

$$\begin{aligned} \int_0^{\pi} \frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1 - \frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}} d\beta &= 2\int_0^{+\infty} \frac{t^{2p}}{(1 + (1 - \frac{c^2}{2})t^2)^{2+p}} dt \\ &= \frac{2}{(1 - \frac{c^2}{2})^{p+\frac{1}{2}}}\int_0^{+\infty} \frac{u^{2p}}{(1+u^2)^{2+p}} du \\ &= \frac{2}{(1 - \frac{c^2}{2})^{p+\frac{1}{2}}}\int_0^{+\infty} \frac{\mathrm{th}(s)^{2p}}{\mathrm{ch}(s)^3} ds, \end{aligned}$$

where we made the changes of variables $t = \tan(\beta)$, $u = \sqrt{1 - \frac{c^2}{2}}t$ and $u = \text{sh}(s)$. Then, consider

$$\forall p \in \mathbb{N}, I_p = \int_0^{+\infty} \frac{\text{th}(s)^{2p}}{\text{ch}(s)} ds.$$

An integration by parts gives

$$I_p - I_{p+1} = \int_0^{+\infty} \frac{\text{th}(s)^{2p}}{\text{ch}(s)^3} ds = \frac{I_{p+1}}{2p+1}.$$

Since $I_0 = \frac{\pi}{2}$, the value of I_p is

$$I_p = \frac{(2p)!\pi}{2^{2p+1}(p!)^2},$$

and finally,

$$\int_0^{+\infty} \frac{\text{th}(s)^{2p}}{\text{ch}(s)^3} ds = \frac{(2(p+1))!\pi}{2^{2p+3}((p+1)!)^2(2p+1)}.$$

Thus, equation (101) writes

$$I(2p+2, c) = \frac{\pi^{p+1}}{(1 - \frac{c^2}{2})^{p+\frac{1}{2}}(p+1)!}. \tag{102}$$

In particular, formula (102) remains valid when $p = 0$.

On the other hand, assuming that $N = 2p+3$ is odd, we compute

$$|\mathbb{S}^{2p+1}| = \frac{2\pi^{p+1}}{p!},$$

and

$$\begin{aligned}
\int_0^{\pi} \frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1 - \frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}} d\beta &= 2\int_0^1 \frac{u^2(1-u^2)^p}{(1 + \frac{c^2}{2}(u^2-1))^{p+\frac{5}{2}}} du \\
&= \frac{4\sqrt{2}}{c^{2p+3}(1 - \frac{c^2}{2})^{p+1}} \int_0^{\frac{c}{\sqrt{2-c^2}}} \frac{v^2(c^2(1+v^2) - 2v^2)^p}{(1+v^2)^{p+\frac{5}{2}}} dv \\
&= \frac{4\sqrt{2}}{c^{2p+3}(1 - \frac{c^2}{2})^{p+1}} \int_0^{\frac{c}{\sqrt{2}}} (c^2 - 2w^2)^p w^2 dw \\
&= \frac{2}{(1 - \frac{c^2}{2})^{p+1}} \int_0^{\frac{\pi}{2}} (\sin^{2p+1}(\theta) - \sin^{2p+3}(\theta)) d\theta,
\end{aligned}$$

where we successively made the changes of variables $u = \cos(\beta)$, $v = \frac{cu}{\sqrt{2-c^2}}$, $w = \frac{v}{\sqrt{1+v^2}}$ and $w = \frac{c}{\sqrt{2}}\cos(\theta)$. Now, Wallis' formulae yield

$$\int_0^{\frac{\pi}{2}} (\sin^{2p+1}(\theta) - \sin^{2p+3}(\theta)) d\theta = \frac{4^p(p!)^2}{(2p+1)!(2p+3)},$$

which gives

$$\int_0^{\pi} \frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1 - \frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}} d\beta = \frac{2^{2p+1}(p!)^2}{(1 - \frac{c^2}{2})^{p+1}(2p+1)!(2p+3)},$$

and finally, by equation (101),

$$I(2p+3, c) = \frac{(4\pi)^{p+1}p!}{(1 - \frac{c^2}{2})^{p+1}(2p+1)!(2p+3)}. \tag{103}$$

In conclusion, if $N=2p+2$, we have by equations (97), (98), (99), (100) and (102),

$$\alpha=\frac{(1-\frac{c^2}{2})^{p-\frac{1}{2}}p!}{2\pi^{p+1}}\bigg((1-p)cE(v)+\Big(2+\frac{2p-1}{2}c^2\Big)p(v)\bigg),$$

and if $N=2p+3$, by equations (97), (98), (99), (100) and (103),

$$\alpha=\frac{(1-\frac{c^2}{2})^p(2p+1)!}{(4\pi)^{p+1}p!}\bigg(\frac{1-2p}{2}cE(v)+(2+pc^2)p(v)\bigg).$$

It yields immediately equation (12) by using the definition of the function Γ, and completes the proof of Theorem 2. □

By the same arguments, we complete the proof of Theorem 3.

Proof of Theorem 3. By Proposition 7, we already know that

$$\forall\sigma\in\mathbb{S}^1, v_\infty(\sigma)=\theta_\infty(\sigma)=\alpha\frac{\sigma_1}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}+\beta\frac{\sigma_2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}.$$

Thus, it only remains to deduce the values of the coefficients α and β from equations (40) and (41). Indeed, by Proposition 7, formula (40) writes in dimension two,

$$\int_{\mathbb{R}^2}(|\nabla v|^2+\eta^2)-2c\bigg(1-\frac{2}{c^2}\bigg)p(v)=\alpha\bigg(\frac{4}{c}\int_{\mathbb{S}^1}\frac{\sigma_1^2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}d\sigma-2c\int_{\mathbb{S}^1}\frac{\sigma_1^4}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}d\sigma\bigg).$$

Actually, we remark that we recover formula (97) in dimension two. Therefore, the value of α is exactly the same as in the proof of Theorem 2, i.e.

$$\alpha=\frac{1}{2\pi\sqrt{1-\frac{c^2}{2}}}\bigg(cE(v)+\Big(2-\frac{c^2}{2}\Big)p(v)\bigg).$$

Likewise, by Proposition 7, formula (41) writes in dimension two

$$P_2(v)=\frac{\beta}{2}\bigg(2\int_{\mathbb{S}^1}\frac{\sigma_2^2}{1-\frac{c^2\sigma_2^2}{2}}d\sigma-c^2\int_{\mathbb{S}^1}\frac{\sigma_1^2\sigma_2^2}{(1-\frac{c^2\sigma_2^2}{2})^2}d\sigma\bigg). \tag{104}$$

Denoting

$$J_3:=2\int_{\mathbb{S}^1}\frac{\sigma_2^2}{1-\frac{c^2\sigma_2^2}{2}}d\sigma-c^2\int_{\mathbb{S}^1}\frac{\sigma_1^2\sigma_2^2}{(1-\frac{c^2\sigma_2^2}{2})^2}d\sigma,$$

we compute

$$\begin{aligned}J_3=(2-c^2)\int_{\mathbb{S}^1}\frac{\sigma_2^2}{(1-\frac{c^2\sigma_2^2}{2})^2}d\sigma&=4(2-c^2)\int_0^{\frac{\pi}{2}}\frac{\sin^2(t)}{(1-\frac{c^2\sin^2(t)}{2})^2}dt\\&=\frac{8}{\sqrt{1-\frac{c^2}{2}}}\int_0^{+\infty}\frac{u^2}{(1+u^2)^2}du\\&=\frac{8}{\sqrt{1-\frac{c^2}{2}}}\int_0^{+\infty}\frac{\mathrm{sh}^2(v)}{\mathrm{ch}^3(v)}dv\\&=\frac{2\pi}{\sqrt{1-\frac{c^2}{2}}},\end{aligned}$$

where we successively made the changes of variables $u = \sqrt{1 - \frac{c^2}{2}} \tan(t)$ and $u = \text{sh}(v)$. Then, the computation of J_3 yields by equation (104),

$$\beta = \frac{\sqrt{1 - \frac{c^2}{2}}}{\pi} P_2(v),$$

which concludes the proof of Theorem 3. □

Finally, we conclude the paper by the proof of Corollary 1, which is an immediate consequence of Theorem 2 and Lemma 7.

Proof of Corollary 1. By equations (97), (99), (100), (102) and (103), there is some real number $A_{c,N} > 0$ such that

$$\alpha = A_{c,N} \left(\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2cp(v) + \frac{4}{c^2} p(v) \right) = A_{c,N} J_1. \tag{105}$$

However, Lemma 7 gives on one hand

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2.$$

On the other hand, by definition,

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\partial_1 v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{4} \int_{\mathbb{R}^N} \eta^2,$$

so,

$$E(v) = \int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \eta^2.$$

Thus, we compute

$$J_1 = 2(E(v) - cp(v)) + \frac{4}{c^2} p(v) + \frac{1}{2} \int_{\mathbb{R}^N} \eta^2. \tag{106}$$

Moreover, Lemma 7 once more yields

$$E(v) - cp(v) = \frac{1}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 \geq 0,$$

and likewise,

$$cp(v) = E(v) - \frac{1}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 = \frac{N-2}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \eta^2 \geq 0.$$

Therefore, J_1 is the sum of three non negative terms.

Now assume that α is equal to 0. $A_{c,N}$ being strictly positive, J_1 is equal to 0. By formula (106), it follows that

$$E(v) - cp(v) = p(v) = \int_{\mathbb{R}^N} \eta^2 = 0,$$

so, the energy $E(v)$ vanishes, and the travelling wave v is a complex constant of modulus one.

Reciprocally, if v is constant, the energy $E(v)$ and the momentum $p(v)$ vanish, and α is equal to 0 by equation (105), which ends the proof of Corollary 1. □

Remark. By the proof of Corollary 1, the stretched dipole coefficient α is always non negative.

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Chapitre VI

Asymptotics for solitary waves in the generalised Kadomtsev-Petviashvili equations.

Abstract.
We investigate the asymptotic behaviour of the localised solitary waves in the generalised Kadomtsev-Petviashvili equations. In particular, we give their first order development at infinity in every dimension $N \geq 2$.

Introduction.

1 Motivation and main results.

In this paper, we focus on the localised solitary waves in the generalised Kadomtsev-Petviashvili equations

$$\begin{cases} \partial_t u + u^p \partial_1 u + \partial_1^3 u - \sum_{j=2}^{N} \partial_j u_j = 0, \\ \forall j \in \{2, \ldots, N\}, \partial_1 u_j = \partial_j u. \end{cases} \tag{1}$$

We will assume here that p is a rational number, which writes $p = \frac{m}{n}$, where m and n are relatively prime, and n is odd. The function $u \mapsto u^p$ is then defined by the standard convention

$$\forall u \in \mathbb{R}, u^p = Sign(u)^m |u|^p.$$

Two cases at least are physically relevant. First the case $p = 1$ which corresponds to the standard Kadomtsev-Petviashvili equation, and which is a universal model for dispersive, weakly nonlinear long waves, essentially unidimensional in the direction of propagation x_1 (see the article of B.B. Kadomtsev and V.I. Petviashvili [31]). The case $p = 2$ appears as a model for the evolution of sound waves in antiferromagnetics (see the article of G.E. Falkovitch and S.K. Turitsyn [19]).

The generalised Kadomtsev-Petviashvili equations conserve at least formally two quantities: the L^2-norm of the function u

$$I(u) = \int_{\mathbb{R}^N} u^2(x) dx, \tag{2}$$

and the energy of u

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \Big(\partial_1 u(x)^2 + \sum_{j=2}^{N} u_j(x)^2 \Big) dx - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^N} u(x)^{p+2} dx. \tag{3}$$

The localised solitary waves of the generalised Kadomtsev-Petviashvili equations are the solutions u of (1) of the form

$$u(x,t) = v(x_1 - ct, x_\perp), \ x_\perp = (x_2, \ldots, x_N),$$

which belong to the space Y defined as the closure of the space $\partial_1 C_0^\infty(\mathbb{R}^N)$ for the norm

$$\forall \phi \in C_0^\infty(\mathbb{R}^N), \|\partial_1 \phi\|_Y = \Big(\|\nabla \phi\|_{L^2(\mathbb{R}^N)}^2 + \|\partial_{1,1}^2 \phi\|_{L^2(\mathbb{R}^N)}^2 \Big)^{\frac{1}{2}}. \tag{4}$$

They are formally critical points on Y of the action S defined by

$$\forall v \in Y, S(v) = E(v) + \frac{c}{2} I(v). \tag{5}$$

The parameter $c > 0$ represents the speed of the solitary wave, which moves in direction x_1. The time-independent equation for v writes

$$\begin{cases} -c\partial_1 v + v^p \partial_1 v + \partial_1^3 v - \sum\limits_{j=2}^{N} \partial_j v_j = 0, \\ \forall j \in \{2, \ldots, N\}, \partial_1 v_j = \partial_j v. \end{cases} \tag{6}$$

Actually, we can always make the additional assumption

$$c = 1.$$

Indeed, if v is a solitary wave with speed c, the function $\tilde{v}$, given by the scale change,

$$\forall x \in \mathbb{R}^N, \tilde{v}(x_1, x_\perp) = c^{-\frac{1}{p}} v\Big(\frac{x_1}{\sqrt{c}}, \frac{x_\perp}{c} \Big), \tag{7}$$

is a solitary wave with speed 1. In order to simplify the notations, we will assume from now on that the speed c of the solitary wave v is equal to 1. We will recover the arbitrary case by the scale change (7). In particular, with this additional hypothesis, the solitary wave v solves the equation

$$-\Delta v + \partial_1^4 v + \frac{1}{p+1} \partial_1^2 (v^{p+1}) = 0, \tag{8}$$

which is the starting point of our analysis.

A. de Bouard and J.C. Saut first studied mathematically the existence and the properties of the solitary waves in the generalised Kadomtsev-Petviashvili equations. In their first paper [13], they completely solved the problem of their existence in dimensions two and three: they proved there exist non-trivial solutions of equation (6) in Y if and only if

$$0 < p < \frac{4}{2N-3}.$$

Moreover, they gave some regularity properties of the solitary waves. In particular, any solution of (6) in Y actually belongs to $H^\infty(\mathbb{R}^N)$ when p is an integer. In their second paper [14], they focused on qualitative properties of the solitary waves. They proved the axisymmetry around axis x_1 of the ground states (the solitary waves which minimise the action S on the space Y). They also described the algebraic decay of any solitary wave in dimensions two and three.

Theorem ([14]). *In dimension two, any solitary wave v of equation (1) satisfies*

$$r^2 v \in L^\infty(\mathbb{R}^2), r^2 = x_1^2 + x_2^2.$$

In dimension three, any solitary wave v of equation (1) satisfies

$$\forall 0 \leq \delta < \frac{3}{2}, r^\delta v \in L^2(\mathbb{R}^3), r^2 = x_1^2 + x_2^2 + x_3^2.$$

Remark. Their theorem is sharp in dimension two. Indeed, we know an explicit solution of equation (6) in dimension two, the lump solution v_c given by

$$\forall (x_1, x_2) \in \mathbb{R}^2, v_c(x_1, x_2) = 24c \frac{3 - cx_1^2 + c^2 x_2^2}{(3 + cx_1^2 + c^2 x_2^2)^2}. \tag{9}$$

In particular, we cannot expect a decay rate better than r^{-2} in dimension two.

The goal of this article is to complement their description of the asymptotic behaviour of a solitary wave in every dimension $N \geq 2$.

Theorem 1. *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that*

$$0 < p < \frac{4}{2N-3},$$

and consider the function $v_\infty \in C^\infty(\mathbb{S}^{N-1})$ given by

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(p+1)} (1 - N\sigma_1^2) \int_{\mathbb{R}^N} v(x)^{p+1} dx. \tag{10}$$

Then, the function $x \mapsto |x|^N v(x)$ is bounded on $\mathbb{R}^N$, and

$$\forall \sigma \in \mathbb{S}^{N-1}, R^N v(R\sigma) \underset{R \to +\infty}{\to} v_\infty(\sigma). \tag{11}$$

Moreover, if $\frac{1}{N} \leq p < \frac{4}{2N-3}$, this convergence is uniform, which means that it holds in $L^\infty(\mathbb{S}^{N-1})$.

Remarks. 1. The function v_∞ is well-defined on the sphere $\mathbb{S}^{N-1}$. Indeed, the integral $\int_{\mathbb{R}^N} v(x)^{p+1} dx$ is finite. By Theorem 7, the function v belongs to all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q \leq +\infty$, in particular, to $L^{p+1}(\mathbb{R}^N)$.

2. In view of the existence results of A. de Bouard and J.C. Saut [13] in dimensions two and three, we conjecture [1] that, in dimension $N \geq 4$, there exist non-trivial solutions of equation (6) in Y if and only if

$$0 < p < \frac{4}{2N-3}.$$

[1] Indeed, their proof can probably be generalised to every dimension $N \geq 4$. Their argument relies on the concentration-compactness lemma of P.L. Lions [34] and on the embedding theorem for anisotropic Sobolev spaces, which states that the space Y embeds in $L^q(\mathbb{R}^N)$ for every $2 \leq q \leq \frac{4N+1}{2N-3}$ (see the book of O.V. Besov, V.P. Il'in and S.M. Nikolskii [3]). This embedding theorem holds in every dimension $N \geq 2$, so, the argument of A. de Bouard and J.C. Saut can certainly be adapted to every dimension $N \geq 2$. The aim of this article is not to prove such existence results, so, we will not consider this existence conjecture any further. However, in Corollary 2, we will derive from some integral identities (which are of independent interest in this article) that there are no non-trivial solutions of equation (6) in Y if $p \geq \frac{4}{2N-3}$.

In the case of the standard Kadomtsev-Petviashvili equation ($p = 1$), we can link the asymptotic behaviour of a solitary wave v to its energy $E(v)$ and its action $S(v)$.

Theorem 2. *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that $N = 2$ or $N = 3$, and $p = 1$. Then, the function v_∞ is equal to*

$$\begin{aligned} \forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) &= \frac{(7-2N)\Gamma(\frac{N}{2})}{2(2N-5)\pi^{\frac{N}{2}}}(1 - N\sigma_1^2)E(v) \\ &= \frac{(7-2N)\Gamma(\frac{N}{2})}{4\pi^{\frac{N}{2}}}(1 - N\sigma_1^2)S(v). \end{aligned} \tag{12}$$

Theorem 2 follows from the standard Pohozaev identities, which were already derived by A. de Bouard and J.C. Saut in [13]. Its main interest is to link the asymptotic behaviour of a solitary wave with some integral quantities which are conserved by equation (1). In the case of the standard Kadomtsev-Petviashvili equation, the asymptotic behaviour of a solitary wave depends on its energy, whereas this may not be the case if the exponent p is different from 1. We can expect more rigidity for the standard Kadomtsev-Petviashvili equation: the conjecture on the uniqueness of the solitary waves is more likely to hold for the standard Kadomtsev-Petviashvili equation.

When m is an odd number, Theorem 1 gives a sharp decay rate at infinity for any non-trivial solitary wave: the decay exponent N is the decay exponent of any non-trivial solitary waves. Indeed, assume m is odd and consider a non-trivial solitary wave v in Y such that its decay exponent α is strictly more than N. It means that there is some positive real number A such that

$$\forall x \in \mathbb{R}^N, |v(x)| \leq \frac{A}{|x|^\alpha}.$$

By Theorem 1, the function v_∞ is then identically equal to 0 on $\mathbb{S}^{N-1}$, which gives

$$\int_{\mathbb{R}^N} v(x)^{p+1} dx = 0.$$

Since m is odd, we conclude that v is trivial, which leads to a contradiction. Thus, Theorem 1 is optimal for any non-trivial solitary wave when m is an odd number (which holds in particular for the standard Kadomtsev-Petviashvili equation).

On the other hand, Theorem 1 may not be sharp if m is even. The decay rate at infinity given by Theorem 1 may not be optimal for any non-trivial solitary waves: there may be non-trivial solitary waves which decay faster at infinity. This could be the case if the function v_∞ was identically equal to 0, that is if

$$\int_{\mathbb{R}^N} v(x)^{p+1} dx = 0.$$

Actually, we do not know any non-trivial solitary waves which verify such assumption. However, L. Paumond considers in [43] an equation very similar to equation (1): it writes on $\mathbb{R}^5$

$$\begin{cases} \partial_t u + u^p \partial_1 u + \partial_1^7 u - \sum\limits_{j=2}^{5} \partial_j u_j = 0, \\ \forall j \in \{2, \ldots, 5\}, \partial_1 u_j = \partial_j u. \end{cases}$$

When m is even, he proves the existence of non-trivial solitary waves v for this equation which satisfies the symmetry

$$\forall x \in \mathbb{R}^5, v(x_1, x_2, x_3, x_4, x_5) = -v(x_1, x_4, x_5, x_2, x_3).$$

In particular, such solutions satisfy

$$\int_{\mathbb{R}^N} v(x)^{p+1} dx = 0.$$

Moreover, we conjecture that Theorem 1 remains valid for this equation. Therefore, the functions v_∞ associated to the solutions of L. Paumond are identically equal to 0: the decay rate given by Theorem 1 may not be optimal for such non-trivial solitary waves.

Now, in the hope of clarifying the proof of Theorem 1 and in order to specify general arguments which could prove fruitful for other equations, we are going to explain the main arguments of the proof of Theorem 1.

2 Sketch of the proof of Theorem 1.

Theorem 1 deals with the asymptotic behaviour of a solitary wave v: we compute its algebraic decay at infinity and then, its first order asymptotic expansion. Our proof is reminiscent of a series of articles of J.L. Bona and Yi A. Li [8], A. de Bouard and J.C. Saut [14], M. Maris [40, 41] and [22, 24, 26]. It relies on the use of convolution equations and, in particular, on a precise analysis of the kernels they involve.

2.1 Convolution equations.

By equation (8), the solitary wave v satisfies at least formally, two convolution equations on which the proof of Theorem 1 relies,

$$v = iH_0 * (v^p \partial_1 v), \tag{13}$$

and

$$v = \frac{1}{p+1} K_0 * v^{p+1}. \tag{14}$$

Here, H_0 and K_0 are the kernels of Fourier transform,

$$\widehat{H_0}(\xi) = \frac{\xi_1}{|\xi|^2 + \xi_1^4}, \tag{15}$$

and

$$\widehat{K_0}(\xi) = \frac{\xi_1^2}{|\xi|^2 + \xi_1^4}. \tag{16}$$

Equations (13) and (14) link the asymptotic properties of the solitary wave v to the behaviour at infinity of the kernels H_0 and K_0. This requires to derive them rigorously by a precise analysis of some properties of the kernels they involve. Moreover, this analysis is also useful to study the asymptotic behaviour of solutions of such equations.

2.2 Main properties of the kernels.

This section is devoted to the study of the kernels H_0, K_0 and $K_k = -i\partial_k K_0$, given by

$$\forall k \in \{1, \ldots, N\}, \widehat{K_k}(\xi) = \frac{\xi_k \xi_1^2}{|\xi|^2 + \xi_1^4}. \tag{17}$$

In view of the comment above, we first describe the asymptotic properties and the singularities near the origin of the kernels H_0, K_0 and K_k in order to compute later the asymptotic properties of the solitary waves.

Algebraic decay at infinity and singularities near the origin.

Consider the spaces of functions $M_\alpha^\infty(\Omega)$ defined by

$$M_\alpha^\infty(\Omega) = \{u : \Omega \mapsto \mathbb{C}, \|u\|_{M_\alpha^\infty(\Omega)} = \sup\{|x|^\alpha |u(x)|, x \in \Omega\} < +\infty\},$$

for every $\alpha > 0$ and every open subset Ω of $\mathbb{R}^N$. We will say that a function f presents some algebraic decay at infinity if it belongs to some space $M_\alpha^\infty(B(0,1)^c)$ for some positive real number α. Likewise, we will say that f presents some algebraic explosion near the origin if it belongs to some space $M_\alpha^\infty(B(0,1))$ for some positive real number α.

One of the goals of Theorem 1 is to derive the algebraic decay at infinity of the solitary waves v of the generalised Kadomtsev-Petviashvili equations. As mentioned above, the first step towards this aim is to study the algebraic decay at infinity and the algebraic explosion near the origin of the kernels H_0, K_0 and K_k. More precisely, we establish the algebraic decay at infinity of those kernels in the following theorem.

Theorem 3. *Let $k \in \{1, \ldots, N\}$. The kernels H_0, K_0 and K_k are continuous on $B(0,1)^c$ and respectively belong to $M_{N-1}^\infty(B(0,1)^c)$, $M_N^\infty(B(0,1)^c)$ and $M_{N+1}^\infty(B(0,1)^c)$.*

Remark. We conjecture that Theorem 3 is sharp in the sense that the kernels H_0, K_0 and K_k do not belong to $M_{\alpha-1}^\infty(B(0,1)^c)$, $M_\alpha^\infty(B(0,1)^c)$ and $M_{\alpha+1}^\infty(B(0,1)^c)$ for any $\alpha > N$.

Likewise, we describe their singularities near the origin in the next theorem.

Theorem 4. *Let $1 \leq k \leq N$. Then, there exists some positive real number A such that for every $x \in B(0,1)$,*

$$\begin{cases} (x_1^2 + |x_\perp|)^{N-2}\ |H_0(x)| \leq A(1 + \delta_{N,2}|\ln(|x|)|), \\ (x_1^2 + |x_\perp|)^{N-\frac{3}{2}}\ |K_0(x)| \leq A, \\ (x_1^2 + |x_\perp|)^{N-\frac{1+\delta_{k,1}}{2}}\ |K_k(x)| \leq A. \end{cases}$$

Remarks. 1. By Theorem 4, the kernels H_0, K_0 and K_k respectively belong to the spaces $M_{2N-4}^\infty(B(0,1))$, $M_{2N-3}^\infty(B(0,1))$ and $M_{2N-1-\delta_{k,1}}^\infty(B(0,1))$ in dimension $N > 2$. However, those spaces are not suitable to describe their singularities near the origin. Indeed, their singularities are non-isotropic: they are much more singular in direction x_1. This comes from formulae (15), (16) and (17): the Fourier transforms of the kernels are more integrable at infinity in the direction ξ_1, than in any other direction.

2. We conjecture that Theorem 4 is sharp in the sense that it gives the right exponents of the singularities near the origin of the kernels H_0, K_0 and K_k.

Moreover, we can also describe the singularities near the origin of the kernels H_0, K_0 and K_k in terms of L^p-spaces.

Corollary 1. *Let $1 \leq j, k \leq N$ and $q \in [1, +\infty[$. Then, the functions H_0, K_0 and $x_j^{1-\delta_{k,1}} K_k$ respectively belong to $L^q(B(0,1))$ if $q < \frac{2N-1}{2N-4}$, $q < \frac{2N-1}{2N-3}$ and $q < \frac{2N-1}{2N-2}$.*

The proofs of Theorems 3 and 4 both rely on a careful study of the Fourier transforms of the kernels H_0, K_0 and K_k. By formulae (15), (16) and (17), they are rational fractions which write

$$\forall \xi \in \mathbb{R}^N, R(\xi) = \frac{P(\xi)}{|\xi|^2 + \xi_1^4}, \tag{18}$$

where P is some polynomial function on $\mathbb{R}^N$ of the form

$$\forall \xi \in \mathbb{R}^N, P(\xi) = \prod_{j=1}^{N} \xi_j^{d_j}. \tag{19}$$

The function P is equal to ξ_1 for the kernel H_0, ξ_1^2 for the kernel K_0 and to $\xi_k \xi_1^2$ for the kernel K_k. In this section, we deduce Theorems 3 and 4, and Corollary 1 from some slightly more general results for some tempered distributions f whose Fourier transforms $\widehat{f} = R$ are rational fractions of form (18)-(19). Indeed, we can compute explicitly the algebraic decay of such distributions.

Proposition 1. *Let f be a tempered distribution on $\mathbb{R}^N$ whose Fourier transform writes on form* (18)-(19)*, and denote*

$$d = \sum_{j=1}^{N} d_j = d_1 + d_\perp.$$

Assume moreover that $d \neq 0$ if $N = 2$ and $d_1 + 2d_\perp \leq 4$. Then, f belongs to the space $M^\infty_{N-2+d}(B(0,1)^c)$.

Likewise, we can describe their singularities near the origin.

Proposition 2. *Let f be a tempered distribution on $\mathbb{R}^N$ whose Fourier transform writes on form* (18)-(19)*, and denote*

$$d = \sum_{j=1}^{N} d_j = d_1 + d_\perp.$$

Assume moreover that $d \neq 0$ if $N = 2$, and $d_1 + 2d_\perp \leq 4$. Then, there exists some positive real number A such that for every $x \in B(0,1) \setminus \{0\}$,

$$(x_1^2 + |x_\perp|)^{N - \frac{5}{2} + \frac{d_1}{2} + d_\perp} |f(x)| \leq A(1 + \delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0} |\ln(|x|)|). \tag{20}$$

In particular, if $d_1 + 2d_\perp < 4$, the distribution f belongs to $L^q(B(0,1))$ for every

$$1 \leq q < \frac{2N-1}{2N - 5 + d_1 + 2d_\perp}. \tag{21}$$

Likewise, if $(d_1, d_\perp) = (2,1)$ or $(d_1, d_\perp) = (4,0)$, the distributions $x_j f$ belong to $L^q(B(0,1))$ for every

$$1 \leq q < \frac{2N-1}{2N - 6 + d_1 + 2d_\perp}. \tag{22}$$

Remark. When $(d_1, d_\perp) = (2,1)$ or $(d_1, d_\perp) = (4,0)$, the distribution f is not a function in $L^1_{loc}(B(0,1))$. The singularities of f near the origin can present some principal values at the origin or some Dirac masses (see Lemma 3 for more details). However, the distributions $x_j f$ are in $L^1_{loc}(B(0,1))$, so, we can study their L^q-integrability.

The first step of the proof of Propositions 1 and 2 is to describe by some inductive argument the derivatives of $\widehat{f} = R$, in particular their singularities near the origin and their integrability at infinity. By standard integral expressions, we will then deduce the behaviour near the origin and at infinity of the distribution f. More precisely, we first state the form of the derivatives of R.

Proposition 3. *Let $1 \leq j \leq N$ and $p \in \mathbb{N}$. Consider a rational fraction R on $\mathbb{R}^N$ which satisfies formulae* (18) *and* (19)*, and denote $d_\perp = \sum_{j=2}^{N} d_j$. Then, the partial derivative $\partial_j^p R$ writes*

$$\forall \xi \in \mathbb{R}^N, \partial_j^p R(\xi) = \frac{P_{j,p}(\xi)}{(|\xi|^2 + \xi_1^4)^{p+1}}, \tag{23}$$

where $P_{j,p}$ is a polynomial function on $\mathbb{R}^N$. Moreover, there exist some positive real numbers A_p such that the function $P_{1,p}$ satisfies

$$\forall \xi \in B(0,1)^c, |P_{1,p}(\xi)| \leq A_p \sum_{0 \leq k \leq \frac{d_1+3p}{4}} \max\{|\xi_1|, 1\}^{d_1+3p-4k} |\xi_\perp|^{2k+d_\perp}, \tag{24}$$

$$\forall \xi \in B(0,1), |P_{1,p}(\xi)| \leq A_p |\xi|^{p+d_1+d_\perp}. \tag{25}$$

Likewise, if $j \geq 2$, the function $P_{j,p}$ verifies

$$\forall \xi \in B(0,1)^c, |P_{j,p}(\xi)| \leq A_p \sum_{0 \leq k \leq \frac{d_\perp+p}{2}} \max\{|\xi_\perp|, 1\}^{d_\perp+p-2k} \max\{|\xi_1|, 1\}^{d_1+4k}, \tag{26}$$

$$\forall \xi \in B(0,1), |P_{1,p}(\xi)| \leq A_p |\xi|^{p+d_1+d_\perp}. \tag{27}$$

Remark. By linearity, similar estimates hold for any rational fractions of the form (18) (where P is any polynomial function on $\mathbb{R}^N$). However, in the following, all the considered rational fractions will satisfy (18) and (19), so, in order to simplify some computations, we will not investigate this point any further.

As mentioned above, Proposition 3 provides a description of the derivatives of the rational fraction R which is sufficient to describe its singularities near the origin and its integrability at infinity.

Proposition 4. *Let $1 \leq j \leq N$, $p \in \mathbb{N}$ and $q \in [1, +\infty]$. Consider a rational fraction R on $\mathbb{R}^N$ which writes under form* (18)*-*(19)*, and denote*

$$d = \sum_{j=1}^{N} d_j = d_1 + d_\perp.$$

Then, the partial derivative $\partial_j^p R$ belongs to the space $M_{p+2-d}^\infty(B(0,1))$. Moreover, if

$$p > d_1 + 2d_\perp - 4, \tag{28}$$

the partial derivative $\partial_1^p R$ belongs to $L^q(B(0,1)^c)$ for

$$q > \frac{2N-1}{p+4-d_1-2d_\perp}, \tag{29}$$

while it belongs to $L^\infty(B(0,1)^c)$ for

$$p \geq d_1 + 2d_\perp - 4. \tag{30}$$

Likewise, if $j \geq 2$ and

$$p > \frac{d_1}{2} + d_\perp - 2, \tag{31}$$

the partial derivative $\partial_j^p R$ belongs to $L^q(B(0,1)^c)$ for

$$q > \frac{2N-1}{2p+4-d_1-2d_\perp}, \tag{32}$$

while it belongs to $L^\infty(B(0,1)^c)$ *for*

$$p \geq \frac{d_1}{2} + d_\perp - 2. \tag{33}$$

Remarks. 1. By linearity, we can find similar estimates for any rational fractions of form (18) where P is any polynomial function on $\mathbb{R}^N$.

2. We conjecture that Proposition 4 describes sharply the singularities near the origin of the functions $\partial_j^p R$. We believe that they do not belong to $M_\alpha^\infty(B(0,1))$ for any $\alpha > p + 2 - d_1 - d_\perp$.

3. Likewise, the inequalities above may not characterise all the L^q-spaces to which belong the derivatives $\partial_j^p R$. However, Proposition 4 is probably sharp in the sense that the functions $\partial_1^p R$ and $\partial_j^p R$ do not belong to $L^{\frac{2N-1}{p+4-d_1-2d_\perp}}(B(0,1)^c)$ and $L^{\frac{2N-1}{2p+4-d_1-2d_\perp}}(B(0,1)^c)$ (except in the case those spaces are equal to $L^\infty(B(0,1)^c)$).

Our argument then links the behaviour of the tempered distribution f to the behaviour of the derivatives of its Fourier transforms R given by Propositions 3 and 4. This relies on some explicit integral expressions which already appeared in [24] and [26], and which are constructed from the next lemma.

Lemma 1. *Let f be a tempered distribution on $\mathbb{R}^N$ such that its Fourier transform belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$. Assume moreover that there are some integers $1 \leq j \leq N$ and $p \in \mathbb{N}^*$ such that*

(i) $\partial_j^p \widehat{f} \in L^1(B(0,1)^c)$,

(ii) $\partial_j^{p-1} \widehat{f} \in L^1(B(0,1))$,

(iii) $|.|\partial_j^p \widehat{f} \in L^1(B(0,1))$.

The function $x \mapsto x_j^p f(x)$ is then continuous on $\mathbb{R}^N$ and satisfies for every positive real number λ,

$$\begin{aligned} \forall x \in \mathbb{R}^N, x_j^p f(x) = \frac{i^p}{(2\pi)^N} \Bigg(\int_{B(0,\lambda)^c} \partial_j^p \widehat{f}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p-1} \widehat{f}(\xi) d\xi \\ + \int_{B(0,\lambda)} \partial_j^p \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi \Bigg). \end{aligned} \tag{34}$$

Lemma 1 links the algebraic decay at infinity of a distribution f, or its algebraic explosion near the origin, to the integrability of some derivatives of its Fourier transform $\widehat{f}$. Indeed, under the assumptions of Lemma 1, it is sufficient to prove that the right member of equation (34) is uniformly bounded on $B(0,1)$ (or $B(0,1)^c$) to infer that f belongs to $M_p^\infty(B(0,1))$ (or $M_p^\infty(B(0,1)^c)$). In particular, Lemma 1 seems very fruitful to study the algebraic decay and the singularity near the origin of the tempered distributions f whose Fourier transforms $\widehat{f}$ satisfy (18)-(19). Indeed, Proposition 4 yields a lot of information on their integrability. Thus, we can presumably apply formula (34) to compute their algebraic decay or their singularity near the origin.

However, some derivatives of the rational fraction R are not sufficiently integrable at infinity to satisfy the assumptions of Lemma 1. Therefore, we need to fit those assumptions to the integrability properties given by Proposition 4.

Lemma 2. *Let $1 \leq j \leq N$, $\lambda > 0$ and f, a tempered distribution on $\mathbb{R}^N$ such that its Fourier transform belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$. Assume moreover that there exist some integers $1 \leq p \leq m$ and some positive real number A such that*

(i) $\forall \xi \in \mathbb{R}^N, |\widehat{f}(\xi)| \leq A(|\xi|^{-r} + |\xi|^s)$,

(ii) $\forall (k, \xi) \in \{0, \ldots, p\} \times B(0,1), |\xi|^{N-p+k} |\partial_j^k \widehat{f}(\xi)| \leq A$,

(iii) $\partial_j^m \widehat{f} \in L^1(B(0,1)^c)$,

(iv) $\forall k \in \{0, \ldots, m-1\}, \partial_j^k \widehat{f} \in L^{q_{m-k}}(B(0,1)^c)$,

where $r < N$, $s \geq 0$, $1 < q_k < \frac{N}{N-k}$ if $1 \leq k \leq N-1$, and $1 < q_k \leq +\infty$ if $k > N$. Then, the function $x \mapsto x_j^p f(x)$ is continuous on the set $\Omega_j = \{x \in \mathbb{R}^N, x_j \neq 0\}$ and satisfies for every $x \in \Omega_j$,

$$x_j^p f(x) = \frac{i^p}{(2\pi)^N} \left((-ix_j)^{p-m} \int_{B(0,\lambda)^c} \partial_j^m \widehat{f}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \sum_{k=p}^{m-1} (-ix_j)^{p-k-1} \int_{S(0,\lambda)} \xi_j \partial_j^k \widehat{f}(\xi) \right.$$
$$\left. e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p-1} \widehat{f}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^p \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi \right). \tag{35}$$

Remark. Assumptions of Lemma 2 are tailored for the tempered distributions whose Fourier transforms are rational fractions of the form (18)-(19). However, the proof of this lemma, which relies on Lemma 1 and some integrations by parts, can be adapted to many other distributions which do not necessarily satisfy all the hypothesis of Lemma 2.

Indeed, by Proposition 4, Lemma 2 applies to some tempered distributions f whose Fourier transforms are rational fractions R of form (18)-(19).

Proposition 5. *Let $j \in \{1, \ldots, N\}$, $\lambda > 0$, $\Omega_j = \{x \in \mathbb{R}^N, x_j \neq 0\}$, and f, a tempered distribution on $\mathbb{R}^N$ whose Fourier transform $\widehat{f} = R$ writes on form* (18)-(19)*. Denote*

$$d = \sum_{j=1}^N d_j = d_1 + d_\perp,$$

and assume moreover that $d \neq 0$ if $N = 2$, and $d_1 + 2d_\perp \leq 4$. Then, the function $x \mapsto x_j^{p_j} f(x)$ is continuous on Ω_j and is given for every $x \in \Omega_j$, by

$$x_j^{p_j} f(x) = \frac{i^{p_j}}{(2\pi)^N} \left((-ix_j)^{p_j - m_j} \int_{B(0,\lambda)^c} \partial_j^{m_j} R(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \sum_{k=p_j}^{m_j-1} (-ix_j)^{p_j-k-1} \int_{S(0,\lambda)} \xi_j \right.$$
$$\left. \partial_j^k R(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p_j-1} R(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^{p_j} R(\xi)(e^{ix.\xi} - 1) d\xi \right), \tag{36}$$

where $p_j = N - 2 + d$, $m_1 = 2N - 4 + d_1 + 2d_\perp$, and if $j \geq 2$, $m_j = N - 2 + d$.

Remark. We make two additional assumptions $d \neq 0$ if $N = 2$, and $d_1 + 2d_\perp \leq 4$. Indeed, if $d = 0$ and $N = 2$, all the derivatives of the Fourier transform $\widehat{f}$ are not integrable near the origin. In particular, we cannot expect to prove some formula like (36) for this tempered distribution. On the other hand, the second assumption is more technical. Lemma 2 requires some integrability at infinity for the derivatives of the Fourier transform $\widehat{f}$. This is not possible for the derivatives of low order in the case $d_1 + 2d_\perp > 4$. However, we could probably improve Lemma 2 to compute a formula like (36) even in the case $d_1 + 2d_\perp > 4$. Since it is not useful in our context, we will not investigate this point any further.

Formula (36) links the algebraic decay at infinity of the tempered distribution f (or its algebraic explosion near the origin) to the integrability properties of its Fourier transform R. Indeed, it suffices to choose

$$\lambda = \frac{1}{|x|},$$

and to bound the second member of formula (36) by Proposition 4 to obtain the algebraic decay of f stated in Proposition 1. Likewise, we choose

$$\lambda = \frac{1}{|x_j|^{1+\delta_{j,1}}},$$

and bound the second member of formula (36) by Proposition 4 to establish the algebraic explosion of f near the origin. In particular, one interest of formula (36) is the possibility to choose the value of the parameter λ in the most fruitful way. We can fit it to the non-isotropy of our problem to obtain non-isotropic estimates (20) of Proposition 2.

Finally, Theorems 3 and 4, and Corollary 1 are direct consequences of Propositions 1 and 2. They correspond to the cases $(d_1, d_\perp) = (1, 0)$ for the kernel H_0, $(d_1, d_\perp) = (2, 0)$ for the kernel K_0 and $(d_1, d_\perp) = (2 + \delta_{k,1}, 1 - \delta_{k,1})$ for the kernels K_k.

Pointwise limit of the kernel K_0 at infinity.

The integral expressions of Lemmas 1 and 2 have another important application: the computation of the pointwise limit at infinity of the kernel K_0, that is the limit when R tends to $+\infty$ of the function

$$R \mapsto R^N K_0(R\sigma - y),$$

for every $\sigma \in \mathbb{S}^{N-1}$ and every $y \in \mathbb{R}^N$.

Theorem 5. *Let $\sigma \in \mathbb{S}^{N-1}$ and $y \in \mathbb{R}^N$. Then, the following convergence holds*

$$R^N K_0(R\sigma - y) \underset{R \to +\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(1 - N\sigma_1^2). \tag{37}$$

The proof of Theorem 5 relies on formula (36). Indeed, by Proposition 5, the kernel K_0 satisfies for every $j \in \{1, \ldots, N\}$ and $x \in \Omega_j = \{x \in \mathbb{R}^N, x_j \neq 0\}$,

$$\begin{aligned} x_j^N K_0(x) = & \frac{i^N}{(2\pi)^N} \Bigg((-ix_j)^{N-m_j} \int_{B(0,\lambda)^c} \partial_j^{m_j} \widehat{K_0}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \sum_{k=N}^{m_j-1} (-ix_j)^{N-k-1} \int_{S(0,\lambda)} \xi_j \\ & \partial_j^k \widehat{K_0}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{N-1} \widehat{K_0}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{K_0}(\xi)(e^{ix.\xi} - 1) d\xi \Bigg), \end{aligned} \tag{38}$$

where $m_1 = 2N-2$ and $m_j = N$ if $j \geq 2$. By applying the dominated convergence theorem to this formula, we obtain for any $j \in \{1, \dots, N\}$ such that $\sigma_j \neq 0$,

$$R^N K_0(R\sigma - y) \underset{R\to+\infty}{\to} \frac{i^N}{(2\pi\sigma_j)^N}\Bigg(\frac{i}{\sigma_j}\int_{B(0,1)^c} \partial_j^{N+1}\widehat{R_{1,1}}(\xi)e^{i\sigma.\xi}d\xi + \frac{i}{\sigma_j}\int_{\mathbb{S}^{N-1}} \xi_j\partial_j^N\widehat{R_{1,1}}(\xi) e^{i\sigma.\xi}d\xi + \int_{\mathbb{S}^{N-1}} \xi_j\partial_j^{N-1}\widehat{R_{1,1}}(\xi)d\xi + \int_{B(0,1)} \partial_j^N\widehat{R_{1,1}}(\xi)(e^{i\sigma.\xi}-1)d\xi\Bigg). \tag{39}$$

Here, the distributions $R_{k,l}$ are the so-called composed Riesz kernels given by,

$$\forall (k,l) \in \{1, \dots, N\}^2, \widehat{R_{k,l}}(\xi) = \frac{\xi_k\xi_l}{|\xi|^2}. \tag{40}$$

To complete the proof of Theorem 5, it now remains to prove that the right members of equations (37) and (39) are equal.

Theorem 6. *Let $1 \leq j,k,l \leq N$, $y \in \mathbb{R}^N$ and $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_j \neq 0$. Then, the following equality holds*

$$\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{k,l} - N\sigma_k\sigma_l) = \frac{i^N}{(2\pi\sigma_j)^N}\Bigg(\frac{i}{\sigma_j}\int_{B(0,1)^c} \partial_j^{N+1}\widehat{R_{k,l}}(\xi)e^{i\sigma.\xi}d\xi + \frac{i}{\sigma_j}\int_{\mathbb{S}^{N-1}} \xi_j\partial_j^N\widehat{R_{k,l}}(\xi) e^{i\sigma.\xi}d\xi + \int_{\mathbb{S}^{N-1}} \xi_j\partial_j^{N-1}\widehat{R_{k,l}}(\xi)d\xi + \int_{B(0,1)} \partial_j^N\widehat{R_{k,l}}(\xi)(e^{i\sigma.\xi}-1)d\xi\Bigg). \tag{41}$$

Remark. Theorem 6 yields a proof of Conjecture 1 of [26]. We can now compute the first order asymptotic development of any travelling waves for the Gross-Pitaevskii equation in any dimension $N \geq 3$ (and not only the axisymmetric ones as in [26]). We refer to the appendix for more details on this subject.

Theorem 6 follows once more from Lemmas 1 and 2, and from an explicit formula for the composed Riesz kernels. Indeed, by standard Riesz operator theory, the composed Riesz kernels $R_{k,l}$ are equal to

$$R_{k,l}(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Bigg(PV\bigg(\frac{\delta_{k,l}|x|^2 - Nx_kx_l}{|x|^{N+2}}1_{B(0,1)}\bigg) + \frac{\delta_{k,l}|x|^2 - Nx_kx_l}{|x|^{N+2}}1_{B(0,1)^c}\Bigg), \tag{42}$$

where $PV\Big(\frac{\delta_{k,l}|x|^2 - Nx_kx_l}{|x|^{N+2}}1_{B(0,1)}\Big)$ is the principal value at the origin of the function $x \mapsto \frac{\delta_{k,l}|x|^2 - Nx_kx_l}{|x|^{N+2}}1_{B(0,1)}(x)$, given by

$$\Big\langle PV\Big(\frac{\delta_{k,l}|x|^2 - Nx_kx_l}{|x|^{N+2}}1_{B(0,1)}\Big), \phi\Big\rangle = \int_{B(0,1)} \frac{\delta_{k,l}|x|^2 - Nx_kx_l}{|x|^{N+2}}(\phi(x) - \phi(0))dx, \tag{43}$$

for every function $\phi \in C_0^\infty(\mathbb{R}^N)$. In particular, formula (42) leads to equation (41), which completes the analysis of the pointwise convergence of the kernel K_0 described in Theorem 5.

The main interest of Theorem 5 is the computation of the first order term v_∞ of the asymptotic expansion of v at infinity. Indeed, in subsection 2.5, we will use the computation of the pointwise limit of the kernel K_0 to obtain the expression of the limit v_∞ of the function $R \mapsto R^N v(R\sigma)$ when R tends to $+\infty$. However, formulae (10) and (37) already show the patent link between the radial pointwise limit of the kernel K_0 and the function v_∞.

Rigorous formulation of equations (13) **and** (14)**.**

We conclude the study of the kernels H_0, K_0 and K_k by which is one of its justifications: to give a rigorous sense to convolution equations (13) and (14).

Indeed, consider equation (13). By Theorem 3 and Corollary 1, the kernel H_0 belongs to all the spaces $L^q(\mathbb{R}^N)$ for every $\frac{N}{N-1} < q < \frac{2N-1}{2N-4}$. However, we will prove in Theorem 7 that the functions v and ∇v belong to all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q < +\infty$. Therefore, the function $v^p \partial_1 v$ belongs to $L^1(\mathbb{R}^N)$. Thus, by Young's inequalities, equation (13) makes sense in all the spaces $L^q(\mathbb{R}^N)$ for every $\frac{N}{N-1} < q < \frac{2N-1}{2N-4}$. In particular, it makes sense almost everywhere, which will be sufficient in the following.

Likewise, by Theorem 3 and Corollary 1, the kernel K_0 belongs to all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q < \frac{2N-1}{2N-3}$, and by Theorem 7, the function v^{p+1} belongs to $L^1(\mathbb{R}^N)$. Thus, equation (14) makes sense in all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q < \frac{2N-1}{2N-3}$, and therefore, almost everywhere.

However, we will also study the gradient of the function v. Thus, we will consider the gradient of equation (14). This arises another difficulty: by Theorem 4, the first order derivatives of the kernel K_0, given up to a multiplicative coefficient by the kernels K_k, present non-integrable singularities near the origin. We are not allowed to differentiate convolution equation (14) without additional care. In particular, we cannot write

$$\partial_k v = \frac{1}{p+1} \partial_k K_0 * v^{p+1}.$$

The method to overcome this difficulty is reminiscent of some classical arguments in distribution theory, using integral formulae. Indeed, by Theorem 3 and Corollary 1, the kernel K_0 has first order partial derivatives in the sense of distributions, which are equal to

$$\partial_k K_0 = i\ K_k 1_{B(0,1)^c} + i\ PV(K_k 1_{B(0,1)}) + \left(\int_{\mathbb{S}^{N-1}} K_0(y) y_k dy \right) \delta_0,$$

where $PV(K_k 1_{B(0,1)})$ denotes (as above for the composed Riesz kernels) the principal value at the origin of the function K_k,

$$\forall \phi \in C_0^\infty(B(0,1)), < PV(K_k 1_{B(0,1)}), \phi >= \int_{B(0,1)} K_k(x)(\phi(x) - \phi(0)) dx.$$

Provided we know some sufficient smoothness for the function v^{p+1}, we will be able to differentiate equation (14) and to obtain an explicit integral expression for its derivative.

Lemma 3. *Consider a function $f \in C^0(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{N(p+1)}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N$,

and denote $g = K_0 * f$. Then, g is of class C^1 on $\mathbb{R}^N$. Moreover, its partial derivative $\partial_k g$ is given by

$$\forall x \in \mathbb{R}^N, \partial_k g(x) = i \int_{B(0,1)^c} K_k(y) f(x-y) dy + i \int_{B(0,1)} K_k(y)(f(x-y) - f(x)) dy + \left(\int_{\mathbb{S}^{N-1}} K_0(y) y_k dy \right) f(x). \tag{44}$$

In our context, the function v^{p+1} will satisfy the assumptions of Lemma 3, so, we will be able to give a sense to the gradient of equation (14) and to complete our analysis of the asymptotic behaviour of v.

2.3 Integral properties of the solitary waves.

In this section, we specify the L^q-integrability of a solitary wave v and of some of its derivatives. In particular, we extend to every dimension $N \geq 2$ some regularity properties stated by A. de Bouard and J.C. Saut in their articles [13] and [14].

Theorem 7. *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that $0 < p < \frac{4}{2N-3}$. Then, the function v is bounded and continuous on $\mathbb{R}^N$. Moreover, the function v, its gradient ∇v and its double partial derivative $\partial_1^2 v$ belong to $L^q(\mathbb{R}^N)$ for every $q \in]1, +\infty[$.*

Remark. A. de Bouard and J.C. Saut already proved Theorem 7 in dimensions $N = 2$ and $N = 3$ (see the proof of Theorem 4.1 in their first article [13] and the proof of Theorem 1.1 in their second article [14]).

Theorem 7 is an important preliminary to understand the asymptotic behaviour of a solitary wave v. Indeed, it has at least two interests. The first one is to give a rigorous sense to equations (13), (14) and (44) (which was already mentioned in subsection 2.2). The second one is related to our method to establish the decay properties of the solitary waves. Indeed, this method requires some knowledge of the integrability of the solitary waves to estimate the integrals which appear in equations (13), (14) and (44) by standard Holder's inequalities.

The proof of Theorem 7 is reminiscent of the articles of A. de Bouard and J.C. Saut [13, 14]. It relies on the fact that the kernels K_0 and K_k are L^q-multipliers for every $1 < q < +\infty$. Indeed, they satisfy the assumptions of the next theorem due to P.I. Lizorkin [35].

Theorem ([35]). *Let $\widehat{K}$ be a bounded function in $C^\infty(\mathbb{R}^N \setminus \{0\})$ such that*

$$\xi \mapsto \prod_{j=1}^{N} (\xi_j^{k_j}) \partial_1^{k_1} \dots \partial_N^{k_N} \widehat{K}(\xi) \in L^\infty(\mathbb{R}^N)$$

as soon as $(k_1, \dots, k_N) \in \{0, 1\}^N$ satisfies

$$0 \leq \sum_{j=1}^{N} k_j \leq N.$$

Then, $\widehat{K}$ is a L^q-multiplier for every $1 < q < +\infty$.

Theorem 7 then follows from a standard bootstrap argument which uses the superlinearity of the non-linearity v^{p+1}.

2.4 Decay properties of the solitary waves.

In this section, we establish the algebraic decay at infinity of a solitary wave v and of its gradient. In particular, we extend to every dimension $N \geq 2$ the decay properties stated by A. de Bouard and J.C. Saut in their paper [14].

Theorem 8. *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that $0 < p < \frac{4}{2N-3}$. Then, the function v belongs to $M_N^\infty(\mathbb{R}^N)$, while its gradient belongs to $M_{\min\{(p+1)N,N+1\}}^\infty(\mathbb{R}^N)$.*

The proof of Theorem 8 relies on a standard argument which is reminiscent of a paper of J.L. Bona and Yi A. Li, and also appeared in the articles of A. de Bouard and J.C. Saut [14] and M. Maris [40, 41] (see also [24, 26]). This argument links the algebraic decay of the solitary waves to the algebraic decay of the kernels of the convolution equations they satisfy. Those algebraic decays are identical, essentially because the nonlinearity v^{p+1} is superlinear.

To get a feeling for this claim, let us consider the simplified model

$$f = K * f^q,$$

where we assume that $q > 1$, f and K are smooth functions such that f belongs to all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q \leq +\infty$, and the kernel K belongs to $L^1(\mathbb{R}^N)$ and to some space $M_{\alpha_K}^\infty(\mathbb{R}^N)$ for some positive real number α_K. In order to study the algebraic decay of the function f, we then write for every $x \in \mathbb{R}^N$ and for every real number $\alpha > 0$,

$$\begin{aligned} |x|^\alpha |f(x)| \leq & A\bigg(\int_{\mathbb{R}^N} |x-y|^\alpha |K(x-y)||f(y)|^q dy + \int_{\mathbb{R}^N} |K(x-y)||y|^\alpha |f(y)|^q dy \bigg) \\ \leq & A\big(\|K\|_{M_\alpha^\infty(\mathbb{R}^N)} \|f\|_{L^q(\mathbb{R}^N)}^q + \|K\|_{L^1(\mathbb{R}^N)} \|f\|_{M_{\frac{\alpha}{q}}^\infty(\mathbb{R}^N)}^q \big). \end{aligned} \tag{45}$$

Now, the function f belongs to $L^q(\mathbb{R}^N)$, while the kernel K belongs to $L^1(\mathbb{R}^N)$ and to every space $M_\alpha^\infty(\mathbb{R}^N)$, provided that $0 \leq \alpha \leq \alpha_K$. Therefore, if $0 \leq \alpha \leq \alpha_K$, equation (45) reduces to

$$\|f\|_{M_\alpha^\infty(\mathbb{R}^N)} \leq A + A\|f\|_{M_{\frac{\alpha}{q}}^\infty(\mathbb{R}^N)}^q. \tag{46}$$

Thus, equation (46) links the algebraic decay with exponent α of the function f to its algebraic decay with exponent $\frac{\alpha}{q}$. In particular, if we know some algebraic decay with some small exponent $\alpha_0 > 0$, a bootstrap argument yields that the function f belongs to $M_\alpha^\infty(\mathbb{R}^N)$ for $\alpha = q\alpha_0$, $\alpha = q^2\alpha_0$, ..., and at last, for every $\alpha \in [0, \alpha_K]$. This provides a striking optimal decay property for superlinear equations. Indeed, assuming f possesses some algebraic decay, then, if f is solution of such a convolution equation, it decays as fast as the kernel. However, some decay of f must be established first in order to initiate the inductive argument.

The situation for the function v and for the kernels H_0, K_0 and K_k is rather involved, but, this simple model shows that the decay of the solitary wave is determined by the decay of the kernels. Actually, the main difficulties come from the singularities near the origin of the kernels which do not appear in the previous example. In particular, in the case of the kernels K_k, we must adapt our argument to equation (44), which is not anymore a convolution equation.

However, in order to apply the above argument to equations (13), (14) and (44), we first determine some initial decay for the function v. Indeed, we already know the integrability of the solitary wave v by Theorem 7, and the integrability and the algebraic decay of the kernels H_0, K_0 and K_k by Theorems 3 and 4, and Corollary 1. Therefore, it only remains to establish some algebraic decay for the function v. We deduce it from the next proposition due to A. de Bouard and J.C. Saut [14].

Proposition 6 ([14]). *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that $0 < p < \frac{4}{2N-3}$. Then,*

$$\int_{\mathbb{R}^N} |x|^2 (|\nabla v(x)|^2 + |\partial_1^2 v(x)|^2) dx < +\infty. \tag{47}$$

Remark. Actually, in [14], A. de Bouard and J.C. Saut proved Proposition 6 in dimensions two and three. However, their proof is still relevant in dimensions $N \geq 4$. Indeed, it follows from the multiplication of equation (8) by the multiplier $x \mapsto |x|^2 v(x)$ (after some standard process of truncation). Thus, we will omit the proof of Proposition 6 here, and refer to [14] for more details.

Proposition 6 gives some very weak decay for the gradient of v and its double partial derivative $\partial_1^2 v$. In particular, it does not provide some local algebraic decay at infinity but some integral algebraic decay. However, it is sufficient to apply the argument above and to prove Theorem 8.

2.5 Asymptotic expansion of the solitary waves

In Theorem 8, we established the decay properties of a solitary wave v and of its gradient. In order to complete the proof of Theorem 1, we now state the existence of a first order asymptotic expansion of a solitary wave v, i.e. we compute the limit when $|x|$ tends to $+\infty$ of the function $x \mapsto |x|^N v(x)$. Our argument is reminiscent of [26] and relies once more on convolution equation (14).

Indeed, by equation (14), this reduces to compute the limit when R tends to $+\infty$ of the functions v_R defined by

$$\forall \sigma \in \mathbb{S}^{N-1}, v_R(\sigma) = R^N v(R\sigma) = \frac{R^N}{p+1} \int_{\mathbb{R}^N} K_0(R\sigma - y) v^{p+1}(y) dy, \tag{48}$$

and to prove that this convergence is uniform on the sphere $\mathbb{S}^{N-1}$, provided that $p \geq \frac{1}{N}$. Thus, our argument splits in two steps. We invoke the dominated convergence theorem to compute the pointwise limit of the functions v_R above, and then, Ascoli-Arzela's theorem to prove the uniformity of the convergence.

Step 1. *Pointwise convergence of the functions v_R.*

Proposition 7. *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that $0 < p < \frac{4}{2N-3}$. Then, the following convergence holds*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_R(\sigma) \underset{R \to +\infty}{\to} v_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(p+1)} (1 - N\sigma_1^2) \int_{\mathbb{R}^N} v^{p+1}(y) dy. \tag{49}$$

Proposition 7 follows from the dominated convergence theorem applied to formula (48). Indeed, by Theorem 5, the integrand of formula (48) satisfies for every $y \in \mathbb{R}^N$ and $\sigma \in \mathbb{S}^{N-1}$,

$$R^N K_0(R\sigma - y) v^{p+1}(y) \underset{R \to +\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(p+1)} (1 - N\sigma_1^2) v^{p+1}(y).$$

It then remains to dominate this integrand by invoking Theorems 3, 4 and 7 to obtain Proposition 7.

Step 2. *Uniform convergence of the functions v_R.*

Proposition 8. *Let $v \in Y$ be a solution of speed 1 of equation (6). Assume that $\frac{1}{N} \leq p < \frac{4}{2N-3}$. Then, the following convergence holds*

$$\|v_R - v_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \underset{R \to +\infty}{\to} 0. \tag{50}$$

Proposition 8 is a consequence of Ascoli-Arzela's theorem. Indeed, we already know the existence of a pointwise limit at infinity, so, this theorem will give the uniformity of the convergence. However, Ascoli-Arzela's theorem requires some compactness: we deduce it from the algebraic decay of the gradient of the function v. Provided that $p \geq \frac{1}{N}$, this gradient belongs to $M^\infty_{N+1}(\mathbb{R}^N)$ by Theorem 8. Therefore, the gradients on the sphere $\mathbb{S}^{N-1}$ of the functions v_R are uniformly bounded on $\mathbb{S}^{N-1}$. This yields the compactness which is necessary to invoke Ascoli-Arzela's theorem.

Remark. Our argument based on Ascoli-Arzela's theorem fails to prove the uniformity of the convergence when $0 < p < \frac{1}{N}$. However, the nonlinearity v^{p+1} is less and less smooth at the origin when p tends to 0. Thus, it may be possible that the convergence is not anymore uniform when p is too small.

Finally, Theorem 1 follows from Theorem 8 (which states the decay properties of the solitary waves) and from Propositions 7 and 8 (which describe their asymptotic expansion).

3 Plan of the paper.

The paper is divided into two parts. The first part is devoted to the analysis of the kernels H_0, K_0 and K_k. In the first section, we establish some properties of the rational fractions of form (18)-(19) stated in Propositions 3 and 4. In the second one, we prove Lemmas 1 and 2 to obtain integral formula (36): it is the starting point of the proofs of Propositions 1 and 2. The third section deals with the algebraic decay at infinity of the kernels H_0, K_0 and K_k, and their explosion near the origin stated in Theorems 3 and 4, and Corollary 1. The fourth section is concerned with the pointwise limit of the kernel K_0 obtained by Theorems 5 and 6, while in the last section, we prove Lemma 3 to give a rigorous sense to the derivative of convolution equation (14).

The proof of Theorem 1 forms the core of the second part. The first ingredient is the integral properties of the solitary waves stated in Theorem 7: the first section is devoted to its proof which follows from Lizorkin's theorem [35]. In the second one, we show Theorem 8, which gives the optimal algebraic decay of a solitary wave and of its gradient. In the third section, we end the proof of Theorem 1 by computing the asymptotic expansion of a solitary wave of Propositions 7 and 8. Finally, in the last section, we focus on the standard Kadomtsev-Petviashvili equation. We link the asymptotic expansion of a solitary wave to its energy and its action by Theorem 2. As mentioned above, Theorem 2 follows from the standard Pohozaev identities which were derived by A. de Bouard and J.C. Saut in [13] [2].

[2] Actually, in [13], A. de Bouard and J.C. Saut proved Lemma 4 in dimensions two and three. However, their proof is still relevant in dimensions $N \geq 4$. Indeed, it follows from the multiplication of equation (6) by the multipliers v and $x \mapsto x_k v(x)$ for $k \in \{1, \dots, N\}$ (after some standard process of truncation). Thus, we will omit the proof of Lemma 4 here, and refer to [13] for more details.

Lemma 4 ([13]). *Consider some positive real number p and a solution $v \in Y$ of speed 1 of equation (6). Then, the following identities hold for every $k \in \{2, \ldots, N\}$,*

$$\int_{\mathbb{R}^N} \Big(-v(x)^2 + \frac{2}{p+2} v(x)^{p+2} - 3\partial_1 v(x)^2 + \sum_{j=2}^{N} v_j(x)^2 \Big) dx = 0, \tag{51}$$

$$\int_{\mathbb{R}^N} \Big(v(x)^2 - \frac{2}{(p+1)(p+2)} v(x)^{p+2} + \partial_1 v(x)^2 - 2v_k(x)^2 + \sum_{j=2}^{N} v_j(x)^2 \Big) dx = 0, \tag{52}$$

$$\int_{\mathbb{R}^N} \Big(v(x)^2 - \frac{1}{p+1} v(x)^{p+2} + \partial_1 v(x)^2 + \sum_{j=2}^{N} v_j(x)^2 \Big) dx = 0. \tag{53}$$

Moreover, we will mention another straightforward consequence of Lemma 4: the non-existence of non-trivial solutions of equation (6) in Y when $p \geq \frac{4}{2N-3}$ in dimension $N \geq 4$.[3]

Corollary 2. *Consider a solution $v \in Y$ of speed 1 of equation (6) and, assume that $N \geq 4$ and*

$$p \geq \frac{4}{2N-3}.$$

Then, v is constant.

Finally, we complete the paper by an appendix where we mention the proof of Conjecture 1 of [26]. This proof, which is a straightforward consequence of Theorem 6, yields a complete description of the first order asymptotic development of any travelling wave for the Gross-Pitaevskii equation in any dimension $N \geq 2$.

1 Main properties of the kernels H_0, K_0 and K_k.

In this part, we state some properties of the kernels H_0, K_0 and K_k. We first study their algebraic decay at infinity and their explosion near the origin. It follows from a careful analysis of the integrability properties of their Fourier transforms and from some integral expressions. Then, we compute the pointwise limit at infinity of the kernel K_0, and we derive rigorously convolution equations (13), (14) and (44).

1.1 Properties of the Fourier transforms of the kernels H_0, K_0 and K_k.

This first section is devoted to the analysis of the Fourier transforms of the kernels H_0, K_0 and K_k. This analysis relies on the form of the functions $\widehat{H_0}$, $\widehat{K_0}$ and $\widehat{K_k}$. By formulae (15), (16) and (17), they are rational fractions of form (18)-(19). In Proposition 3, we describe precisely the derivatives of all the rational fractions of this form. Here, the main difficulty comes from their non-isotropy in direction ξ_1. We must differentiate the terms which involve the variable ξ_1 and the other terms. In particular, we must differentiate the derivatives in this direction and in the other directions. More precisely, Proposition 3 relies on the next inductive argument.

[3] Their non-existence was already proved in dimensions two and three by A. de Bouard and J.C. Saut in their article [13].

Proof of Proposition 3. By a straightforward inductive argument, the derivatives $\partial_j^p R$ satisfy equation (23). Here, $P_{j,p}$ are polynomial functions on $\mathbb{R}^N$ given by

$$\begin{cases} P_{j,0}(\xi) = P(\xi) = \prod_{j=1}^{N} \xi_j^{d_j}, \\ P_{j,p+1}(\xi) = (|\xi|^2 + \xi_1^4)\partial_j P_{j,p}(\xi) - 2(p+1)(\xi_j + 2\delta_{j,1}\xi_1^3)P_{j,p}(\xi). \end{cases} \tag{54}$$

In particular, the inductive definitions of $P_{j,p}$ are different according $j = 1$ or $j \geq 2$. Thus, we split our analysis in two cases depending on the value of j.

Case 1. $j = 1$.

The polynomial function $P_{1,p}$ then writes

$$P_{1,p}(\xi) = \sum_{k=0}^{+\infty} a_{k,p}(\xi_1)|\xi_\perp|^{2k} \prod_{j=2}^{N} \xi_j^{d_j}, \tag{55}$$

where the functions $a_{k,p}$ are polynomial functions on $\mathbb{R}$. Indeed, by formulae (54), if the function $P_{1,p}$ writes under the above form, then, the function $P_{1,p+1}$ writes

$$P_{1,p+1}(\xi) = \sum_{k=0}^{+\infty} \left(a'_{k,p}(\xi_1)\Big(|\xi_\perp|^2 + \xi_1^2 + \xi_1^4\Big) - 2(p+1)\Big(\xi_1 + 2\xi_1^3\Big)a_{k,p}(\xi_1)\right)|\xi_\perp|^{2k} \prod_{j=2}^{N} \xi_j^{d_j}.$$

Therefore, with the usual convention $a_{-1,p} = 0$, the functions $a_{k,p}$ are given by the inductive definition

$$\forall k \in \mathbb{N}, \begin{cases} a_{k,0}(\xi_1) = \delta_{k,0}\xi_1^{d_1}, \\ a_{k,p+1}(\xi_1) = a'_{k-1,p}(\xi_1) + (\xi_1^2 + \xi_1^4)a'_{k,p}(\xi_1) - 2(p+1)\xi_1(1 + 2\xi_1^2)a_{k,p}(\xi_1). \end{cases}$$

Thus, they are polynomial functions on $\mathbb{R}$. Moreover, denoting their degree $d_{k,p} = \deg(a_{k,p}) \in \mathbb{N} \cup \{-\infty\}$ (with the usual convention $\deg(0) = -\infty$), we obtain

$$d_{k,p+1} \leq \max\{d_{k-1,p} - 1, d_{k,p} + 3\}. \tag{56}$$

It follows that

$$d_{k,p} \leq d_1 + 3p - 4k. \tag{57}$$

Indeed, by assumption (19), the degrees $d_{k,0}$ are equal to

$$\begin{cases} d_{0,0} = d_1, \\ \forall k \geq 1, d_{k,0} = -\infty, \end{cases}$$

so, formula (57) holds for $p = 0$ and every $k \in \mathbb{N}$. Now, assume that it holds for every $0 \leq p \leq p_0$ and every $k \in \mathbb{N}$. In particular, it gives

$$d_{k-1,p_0} - 1 \leq d_1 + 3p_0 - 4k + 3,$$

and

$$d_{k,p_0} + 3 \leq d_1 + 3p_0 - 4k + 3.$$

Therefore, by inequality (56), we deduce

$$d_{k,p_0+1} \leq d_1 + 3(p_0 + 1) - 4k.$$

Hence, by induction, inequality (57) holds for every $(k,p) \in \mathbb{N}^2$. Thus, by equation (55) and inequality (57), we notice that

$$\forall \xi \in B(0,1)^c, |P_{1,p}(\xi)| \leq A_p \sum_{0 \leq k \leq \frac{d_1+3p}{4}} \max\{|\xi_1|, 1\}^{d_1+3p-4k} |\xi_\perp|^{2k+d_\perp},$$

which yields inequality (24).

On the other hand, by equations (54) and a straightforward inductive argument, either the function $P_{1,p}$ is identically equal to 0, either its terms of lowest degree are of degree

$$v_{1,p} \geq d_1 + d_\perp + p.$$

It follows that

$$\forall \xi \in B(0,1), |P_{1,p}(\xi)| \leq A_p |\xi|^{v_{1,p}} \leq A_p |\xi|^{d_1+d_\perp+p},$$

which is exactly inequality (25).

Case 2. $j \geq 2$.

The polynomial function $P_{j,p}$ then writes

$$P_{j,p}(\xi) = \sum_{k=0}^{+\infty} b_{k,p}(\xi_\perp) \xi_1^{2k+d_1}, \tag{58}$$

where the functions $b_{k,p}$ are polynomial functions on $\mathbb{R}^{N-1}$. Indeed, by formulae (54), if the function $P_{j,p}$ writes under the above form, then, the function $P_{j,p+1}$ writes

$$P_{j,p+1}(\xi) = \sum_{k=0}^{+\infty} \Big(\partial_j b_{k,p}(\xi_\perp) \Big(\xi_1^4 + \xi_1^2 + |\xi_\perp|^2 \Big) - 2(p+1) \xi_j b_{k,p}(\xi_\perp) \Big) \xi_1^{2k+d_1},$$

so, with the usual convention $b_{-1,p} = 0$, the functions $b_{k,p}$ are given for every $k \in \mathbb{N}$ by the inductive definition

$$\begin{cases} b_{k,0}(\xi_\perp) = \delta_{k,0} \prod_{j=2}^{N} \xi_j^{d_j}, \\ b_{k,p+1}(\xi_\perp) = \partial_j b_{k-2,p}(\xi_\perp) + \partial_j b_{k-1,p}(\xi_\perp) + |\xi_\perp|^2 \partial_j b_{k,p}(\xi_\perp) - 2(p+1) \xi_j b_{k,p}(\xi_\perp). \end{cases}$$

Thus, they are polynomial functions on $\mathbb{R}^{N-1}$. Moreover, denoting their degree $d'_{k,p} = \deg(b_{k,p}) \in \mathbb{N} \cup \{-\infty\}$, we obtain

$$d'_{k,p+1} \leq \max\{d'_{k-2,p} - 1, d'_{k-1,p} - 1, d'_{k,p} + 1\}. \tag{59}$$

It follows that

$$d'_{k,p} \leq d_\perp + p - k - \nu(k), \tag{60}$$

where $\nu(k) = 0$ if k is even, and $\nu(k) = 1$ if k is odd. Indeed, by assumption (19), the degrees $d'_{k,0}$ are equal to

$$\begin{cases} d'_{0,0} = d_\perp, \\ \forall k \geq 1, d'_{k,0} = -\infty, \end{cases}$$

so, formula (60) holds for $p = 0$ and every $k \in \mathbb{N}$. Likewise, by assumption (19) and equation (54), $P_{j,1}$ writes

$$\forall \xi \in \mathbb{R}^N, P_{j,1}(\xi) = \big(d_j \xi_1^4 \xi_j^{d_j-1} + d_j \xi_1^2 \xi_j^{d_j-1} + d_j |\xi_\perp|^2 \xi_j^{d_j-1} - 2\xi_j^{d_j+1} \big) \prod_{k=1, k\neq j}^{N} \xi_k^{d_k}.$$

Therefore,

$$\begin{cases} d'_{0,1} = d_\perp + 1, \\ d'_{1,1} \leq d_\perp - 1 \\ d'_{2,1} \leq d_\perp - 1, \\ \forall k \geq 3, d'_{k,1} = -\infty, \end{cases}$$

and formula (60) holds for $p = 1$ and every $k \in \mathbb{N}$. Now, assume that it holds for every $0 \leq p \leq p_0$ and every $k \in \mathbb{N}$. In particular, it gives for every $k \in \mathbb{N}$,

$$d'_{k-2,p_0} - 1 \leq d_\perp + p_0 + 1 - k - \nu(k),$$

$$d'_{k-1,p_0} - 1 \leq d_\perp + p_0 - k - \nu(k-1),$$

and

$$d'_{k,p_0} + 1 \leq d_\perp + p_0 + 1 - k.$$

Therefore, by inequality (59), since $\nu(k-1) \geq \nu(k) - 1$, we deduce

$$d'_{k,p_0+1} \leq d_\perp + p_0 + 1 - k.$$

Hence, by induction, inequality (60) holds for every $(k,p) \in \mathbb{N}^2$. Thus, by equation (58), we find that for every $\xi \in B(0,1)^c$,

$$\begin{aligned} |P_{j,p}(\xi)| \leq & A_p \sum_{d'_{k,p} \geq 0} \max\{|\xi_\perp|, 1\}^{d'_{k,p}} |\xi_1|^{2k+d_1} \\ \leq & A_p \Bigg(\sum_{0 \leq k \leq \frac{d_\perp + p}{2}} \max\{|\xi_\perp|, 1\}^{d_\perp - p - 2k} |\xi_1|^{4k+d_1} + \sum_{0 \leq k \leq \frac{d_\perp + p}{2} - 1} |\xi_1|^{4k+2+d_1} \\ & \max\{|\xi_\perp|, 1\}^{d_\perp - p - 2k - 2} \Bigg) \\ \leq & A_p \sum_{0 \leq k \leq \frac{d_\perp + p}{2}} \max\{|\xi_\perp|, 1\}^{d_\perp - p - 2k} \max\{|\xi_1|, 1\}^{4k+d_1}, \end{aligned}$$

which is inequality (26).

On the other hand, by equations (54) and a straightforward inductive argument, either the function $P_{j,p}$ is identically equal to 0, either its terms of lowest degree are of degree

$$v_{j,p} \geq d_1 + d_\perp + p.$$

It follows that

$$\forall \xi \in B(0,1), |P_{j,p}(\xi)| \leq A_p |\xi|^{v_{j,p}} \leq A_p |\xi|^{d_1 + d_\perp + p},$$

which completes the proof of Proposition 3. □

Proposition 3 yields rather keen estimates of the derivatives of the rational fractions R of form (18)-(19). In particular, they are sufficiently keen to prove Proposition 4 which describes the singularities of the derivatives of R near the origin and their integrability at infinity.

Proof of Proposition 4. Proposition 4 is a consequence of Proposition 3. Indeed, let us first consider the behaviour of the derivative $\partial_j^p R$ near the origin. By equation (23), and estimates (25) and (27) of Proposition 3, we compute for every $\xi \in B(0,1)$,

$$|\partial_j^p R(\xi)| \leq A_p \frac{|\xi|^{p+d}}{(|\xi|^2 + \xi_1^4)^{p+1}} \leq \frac{A_p}{|\xi|^{p+2-d}}.$$

Therefore, the partial derivative $\partial_j^p R$ belongs to $M_{p+2-d}^{\infty}(B(0,1))$.
Then, consider the integrability properties of the derivative $\partial_j^p R$ at infinity. By equation (23), and inequalities (24) and (26) of Proposition 3, we can estimate the L^q-norm of the function $\partial_j^p R$ for every $q \in [1,+\infty]$. However, estimates (24) and (26) differ because of the non-isotropy of the rational fraction R in direction ξ_1. Thus, we split our study in two cases depending on the value of j.

Case 1. $j = 1$.

We first assume $q < +\infty$ and compute from Proposition 3,

$$\begin{aligned}\int_{B(0,1)^c} |\partial_1^p R(\xi)|^q d\xi \leq & \int_{B(0,1)^c} \frac{|P_{1,p}(\xi)|^q}{(|\xi|^2+\xi_1^4)^{q(p+1)}} d\xi \\ \leq & A \sum_{0\leq k\leq \frac{d_1+3p}{4}} \int_{B(0,1)^c} \frac{\max\{|\xi_1|,1\}^{q(d_1+3p-4k)} \max\{1,|\xi_\perp|\}^{q(2k+d_\perp)}}{(|\xi|^2+\xi_1^4)^{q(p+1)}} d\xi,\end{aligned}$$

so, denoting

$$\forall \lambda \in [1,+\infty[, \forall (\alpha,\beta,\gamma) \in (\mathbb{R}_+^*)^3, J_{\alpha,\beta,\gamma}(\lambda) = \int_{S(0,\lambda)} \frac{\max\{|\xi_1|,1\}^{\alpha} \max\{1,|\xi_\perp|\}^{\beta}}{(|\xi|^2+\xi_1^4)^{\gamma}} d\xi, \quad (61)$$

we have

$$\int_{B(0,1)^c} |\partial_1^p R(\xi)|^q d\xi \leq A \sum_{0\leq k\leq \frac{d_1+3p}{4}} \int_1^{+\infty} J_{q(d_1+3p-4k),q(2k+d_\perp),q(p+1)}(r) dr. \quad (62)$$

However, by using the spherical coordinates

$$x = (r\cos(\theta_1), r\sin(\theta_1)\cos(\theta_2), \dots, r\sin(\theta_1)\dots\sin(\theta_{N-1})),$$

and the successive changes of variables $u = \tan(\theta_1)$ and $v = \frac{u}{\sqrt{\lambda}}$, we obtain

$$\begin{aligned}J_{\alpha,\beta,\gamma}(\lambda) \leq & A \int_0^{\frac{\pi}{2}} \frac{\max\{\cos(\theta_1),\frac{1}{\lambda}\}^{\alpha}}{(1+\lambda^2\cos(\theta_1)^4)^{\gamma}} d\theta_1 \lambda^{N-1+\alpha+\beta-2\gamma} \\ \leq & A\Big(\lambda^{N-1+\beta-2\gamma} \int_{\cos(\theta_1)\leq\frac{1}{\lambda}} d\theta_1 + \lambda^{N-1+\alpha+\beta-2\gamma} \int_0^{+\infty} \frac{(1+u^2)^{2\gamma-\frac{\alpha}{2}-1}}{((1+u^2)^2+\lambda^2)^{\gamma}} du\Big) \\ \leq & A\Big(\lambda^{N-2+\beta-2\gamma} + \lambda^{N-1+\alpha+\beta-2\gamma}\Big(\int_0^1 \frac{du}{(1+\lambda^2)^{\gamma}} + \int_1^{+\infty} \frac{u^{4\gamma-\alpha-2}}{(u^4+\lambda^2)^{\gamma}} du\Big)\Big) \\ \leq & A\Big(\lambda^{N-2+\beta-2\gamma} + \lambda^{N-1+\alpha+\beta-4\gamma} + \lambda^{N-\frac{3}{2}+\frac{\alpha}{2}+\beta-2\gamma}\Big(\int_0^{+\infty} \frac{v^{4\gamma-\alpha-2}}{(v^4+1)^{\gamma}} dv\Big)\Big),\end{aligned}$$

so,

$$\forall \lambda > 1, \forall (\alpha,\beta,\gamma) \in (\mathbb{R}_+^*)^3, J_{\alpha,\beta,\gamma}(\lambda) \leq A\Big(\lambda^{N-2+\beta-2\gamma} + \lambda^{N-1+\alpha+\beta-4\gamma} + \lambda^{N-\frac{3}{2}+\frac{\alpha}{2}+\beta-2\gamma}\Big). \quad (63)$$

Thus, by equations (62) and (63),

$$\int_{B(0,1)^c} |\partial_1^p R(\xi)|^q d\xi \leq A \sum_{0\leq k\leq \frac{d_1+3p}{4}} \int_1^{+\infty} r^{N-2}\Big(r^{q(d_\perp+2k-2p-2)} + r^{1+q(d_1+d_\perp-p-2k-4)}$$
$$+ r^{\frac{1}{2}+q(\frac{d_1}{2}+d_\perp-\frac{p}{2}-2)}\Big) dr$$
$$\leq A \int_1^{+\infty} \Big(r^{N-1+q(d_1+d^\perp-p-4)} + r^{N-\frac{3}{2}+q(\frac{d_1}{2}+d^\perp-\frac{p}{2}-2)}\Big) dr.$$

In particular, if

$$\begin{cases} N-1+q(d_1+d^\perp-p-4) < -1, \\ N-\frac{3}{2}+q(\frac{d_1}{2}+d^\perp-\frac{p}{2}-2) < -1, \end{cases} \tag{64}$$

the derivative $\partial_1^p R$ belongs to $L^q(B(0,1)^c)$. Moreover, by assumption (28), the system of inequalities (64) reduces to

$$q > \frac{2N-1}{p+4-d_1-2d_\perp},$$

which completes the proof of assertion (29).

Now, consider the case $q = +\infty$. It follows from Proposition 3 that for every $\xi \in \mathbb{R}^N$,

$$|\partial_1^p R(\xi)| \leq \frac{|P_{1,p}(\xi)|}{(|\xi|^2+\xi_1^4)^{p+1}} \leq A \sum_{0\leq k\leq \frac{d_1+3p}{4}} \frac{\max\{|\xi_1|,1\}^{d_1+3p-4k}|\xi_\perp|^{2k+d_\perp}}{(|\xi|^2+\xi_1^4)^{p+1}}$$
$$\leq A \sum_{0\leq k\leq \frac{d_1+3p}{4}} \Big(\frac{\max\{\xi_1^2,|\xi|\}^{\frac{d_1}{2}+\frac{3p}{2}+d_\perp}}{(|\xi|^2+\xi_1^4)^{p+1}} + |\xi|^{2k+d_\perp-2p-2}\Big)$$
$$\leq A\Big(\max\{\xi_1^2,|\xi|\}^{\frac{d_1}{2}-\frac{p}{2}+d_\perp-2} + |\xi|^{\frac{d_1}{2}-\frac{p}{2}+d_\perp-2}\Big).$$

Hence, the function $\partial_1^p R$ belongs to $L^\infty(B(0,1)^c)$ if assumption (30) holds.

Case 2. $j \geq 2$.

We first assume $q < +\infty$, and compute likewise from Proposition 3 and definition (61),

$$\int_{B(0,1)^c} |\partial_j^p R(\xi)|^q d\xi \leq \int_{B(0,1)^c} \frac{|P_{j,p}(\xi)|^q}{(|\xi|^2+\xi_1^4)^{q(p+1)}} d\xi$$
$$\leq A \sum_{0\leq k\leq \frac{d_\perp+p}{2}} \int_{B(0,1)^c} \frac{\max\{|\xi_\perp|,1\}^{q(d_\perp+p-2k)} \max\{|\xi_1|,1\}^{q(4k+d_1)}}{(|\xi|^2+\xi_1^4)^{q(p+1)}} d\xi$$
$$\leq A \sum_{0\leq k\leq \frac{d_\perp+p}{2}} \int_1^{+\infty} J_{q(4k+d_1),q(d_\perp+p-2k),q(p+1)}(r) dr,$$

so, by formula (63),

$$\int_{B(0,1)^c} |\partial_j^p R(\xi)|^q d\xi \leq A \sum_{0\leq k\leq \frac{d_\perp+p}{2}} \int_1^{+\infty} r^{N-2}\Big(r^{q(d_\perp-p-2k-2)} + r^{1+q(d_1+d_\perp-3p+2k-4)}$$
$$+ r^{\frac{1}{2}+q(\frac{d_1}{2}+d_\perp-p-2)}\Big) dr$$
$$\leq A \int_1^{+\infty} \Big(r^{N-1+q(d_1+2d^\perp-2p-4)} + r^{N-\frac{3}{2}+q(\frac{d_1}{2}+d^\perp-p-2)}\Big) dr.$$

In particular, if assumption (31) holds, the derivative $\partial_j^p R$ belongs to $L^q(B(0,1)^c)$ as soon as

$$q > \frac{2N-1}{2p+4-d_1-2d_\perp},$$

which completes the proof of assertion (32).

Finally, consider the case $q=+\infty$. We deduce from Proposition 3 that for every $\xi \in \mathbb{R}^N$,

$$\begin{aligned}
|\partial_j^p R(\xi)| \leq \frac{|P_{j,p}(\xi)|}{(|\xi|^2+\xi_1^4)^{p+1}} \leq & A \sum_{0\leq k\leq \frac{d_\perp+p}{2}} \frac{\max\{|\xi_\perp|,1\}^{d_\perp+p-2k}\max\{|\xi_1|,1\}^{4k+d_1}}{(|\xi|^2+\xi_1^4)^{p+1}} \\
\leq & A \sum_{0\leq k\leq \frac{d_\perp+p}{2}} \Big(\max\{\xi_1^2,|\xi|\}^{\frac{d_1}{2}+d_\perp-p-2} + \max\{|\xi_\perp|,1\}^{d_\perp-p-2k-2} \\
& + \max\{|\xi_1|,1\}^{d_1+4k-4p-4} + 1\Big) \\
\leq & A\Big(1+\max\{\xi_1^2,|\xi|\}^{\frac{d_1}{2}+d_\perp-p-2} + \max\{|\xi_\perp|,1\}^{d_\perp-p-2} \\
& + \max\{|\xi_1|,1\}^{d_1+2d_\perp-2p-4}\Big).
\end{aligned}$$

Hence, the function $\partial_j^p R$ belongs to $L^\infty(B(0,1)^c)$ if assumption (33) holds. This ends the proof of Proposition 4. □

1.2 Integral representations of some classes of tempered distributions.

In this section, we derive the integral expressions stated in Lemmas 1 and 2, and Proposition 5. They link the properties of a tempered distribution f with the properties of the derivatives of its Fourier transform (in particular, when this Fourier transform is a rational fraction R of form (18)-(19)). For sake of completeness, we first show Lemma 1 (which is probably well-known to the experts and which already appeared in a slightly different form in [24] and [26]). Indeed, though it cannot be applied to our kernels, Lemma 1 is the key ingredient of the proof of Lemma 2.

Proof of Lemma 1. Consider some positive real number λ and a smooth function $\psi \in C_c^\infty(\mathbb{R}^N,[0,1])$ such that

$$\begin{cases} \operatorname{supp}(\psi) \subset B(0,2\lambda), \\ \forall x \in B(0,\lambda), \psi(x)=1. \end{cases} \tag{65}$$

By standard duality, we state for every function $g \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned}
< x_j^p f, \widehat{g} > =& < x_j^p f, \widehat{\psi g} > + < x_j^p f, \widehat{(1-\psi) g} > \\
=& < x_j^{p-1} f, x_j \widehat{\psi g} > + < \widehat{x_j^p f}, (1-\psi) g > \\
=& -i < x_j^{p-1} f, \widehat{\partial_j(\psi g)} > + i^p < \partial_j^p \widehat{f}, (1-\psi) g > \\
=& \, i^p(- < \partial_j^{p-1}\widehat{f}, \partial_j(\psi g) > + < \partial_j^p \widehat{f}, (1-\psi) g >).
\end{aligned}$$

However, by assumptions (i) and (ii), and the smoothness of $\widehat{f}$, the tempered distributions $\partial_j^p \widehat{f}$ and $\partial_j^{p-1}\widehat{f}$ belong to $L^1(B(0,\lambda)^c)$ and $L^1(B(0,2\lambda))$. Hence, by assumption (65),

$$< x_j^p f, \widehat{g} >= i^p\Big(-\int_{B(0,2\lambda)} \partial_j^{p-1}\widehat{f}(\xi)\partial_j(\psi g)(\xi) d\xi + \int_{B(0,\lambda)^c} \partial_j^p \widehat{f}(\xi)(1-\psi(\xi))g(\xi)d\xi.\Big) \tag{66}$$

However, by assumptions (iii) and (65), and some integration by parts,

$$\int_{B(0,2\lambda)} \partial_j^{p-1}\widehat{f}(\xi)\partial_j(\psi g)(\xi)d\xi = -\int_{B(0,\lambda)} \partial_j^p\widehat{f}(\xi)(g(\xi)-g(0))d\xi - \frac{1}{\lambda}\int_{S(0,\lambda)} \partial_j^{p-1}\widehat{f}(\xi)\xi_j d\xi \; g(0) - \int_{\lambda<|\xi|<2\lambda} \partial_j^p\widehat{f}(\xi)\psi(\xi)g(\xi)d\xi,$$

while by assumptions (65),

$$\int_{B(0,\lambda)^c} \partial_j^p\widehat{f}(\xi)(1-\psi(\xi))g(\xi)d\xi = \int_{\lambda<|\xi|<2\lambda} \partial_j^p\widehat{f}(\xi)(1-\psi(\xi))g(\xi)d\xi + \int_{B(0,2\lambda)^c} \partial_j^p\widehat{f}(\xi)g(\xi)d\xi.$$

Hence, equation (66) becomes

$$< x_j^p f, \widehat{g} >= i^p\Big(\int_{B(0,\lambda)} \partial_j^p\widehat{f}(\xi)(g(\xi)-g(0))d\xi + \frac{1}{\lambda}\int_{S(0,\lambda)} \partial_j^{p-1}\widehat{f}(\xi)\xi_j d\xi \; g(0) + \int_{B(0,\lambda)^c} \partial_j^p\widehat{f}(\xi)g(\xi)d\xi\Big).$$

However, g is in $S(\mathbb{R}^N)$, so,

$$\forall \xi \in \mathbb{R}^N, g(\xi) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{g}(x)e^{ix.\xi}dx,$$

which yields

$$< x_j^p f, \widehat{g} >= \frac{i^p}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{g}(x)\Bigg(\int_{B(0,\lambda)} \partial_j^p\widehat{f}(\xi)(e^{ix.\xi}-1)d\xi + \frac{1}{\lambda}\int_{S(0,\lambda)} \partial_j^{p-1}\widehat{f}(\xi)\xi_j d\xi + \int_{B(0,\lambda)^c} \partial_j^p\widehat{f}(\xi)e^{ix.\xi}d\xi\Bigg)dx.$$

Therefore, by standard duality, the tempered distribution $x_j^p f$ is equal to the tempered distribution Φ given for every $x \in \mathbb{R}^N$ by

$$\Phi(x) = \frac{i^p}{(2\pi)^N}\Bigg(\int_{B(0,\lambda)} \partial_j^p\widehat{f}(\xi)(e^{ix.\xi}-1)d\xi + \frac{1}{\lambda}\int_{S(0,\lambda)} \partial_j^{p-1}\widehat{f}(\xi)\xi_j d\xi + \int_{B(0,\lambda)^c} \partial_j^p\widehat{f}(\xi)e^{ix.\xi}d\xi\Bigg). \tag{67}$$

Indeed, by assumptions (i) and (iii), Φ belongs to $L^1_{loc}(\mathbb{R}^N)$ and satisfies

$$\forall x \in \mathbb{R}^N, |\Phi(x)| \leq A(1+|x|).$$

Therefore, Φ is well-defined on $\mathbb{R}^N$ and is a tempered distribution. Moreover, it is continuous on $\mathbb{R}^N$ by assumptions (i) and (iii) once more, and a standard application of the dominated convergence theorem. Thus, the function $x \mapsto x_j^p f(x)$ is continuous on $\mathbb{R}^N$ and verifies formula (34) by equation (67). □

Lemma 2 then follows from Lemma 1.

Proof of Lemma 2. Consider some positive real number ε and denote f_ε, the tempered distribution defined by

$$\forall \xi \in \mathbb{R}^N, \widehat{f_\varepsilon}(\xi) = \widehat{f}(\xi) e^{-\varepsilon|\xi|^2}. \tag{68}$$

The functions $\widehat{f_\varepsilon}$ decay much faster at infinity than the function $\widehat{f}$. In particular, we can apply them Lemma 1, which is not always possible for the function $\widehat{f}$. Thus, Lemma 2 follows from applying Lemma 1 to the distribution f_ε, and after several integrations by parts, taking the limit $\varepsilon \to 0$. In order to do so, we first establish some properties of the distributions f_ε.

Step 1. *Let $(j, p) \in \{1, \dots, N\} \times \mathbb{N}$. The function $\widehat{f_\varepsilon}$ belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$ and satisfies*

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \partial_j^p \widehat{f_\varepsilon}(\xi) = \sum_{k=0}^{p} C_p^k \varepsilon^{\frac{k}{2}} \partial_j^{p-k} \widehat{f}(\xi) S_k(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2}. \tag{69}$$

Here, the functions S_k are polynomial functions on $\mathbb{R}$ of degree less or equal to k. In particular, S_0 is identically equal to the constant function 1.

Indeed, consider the function Ψ given by

$$\forall x \in \mathbb{R}, \Psi(x) = e^{-x^2}.$$

By a straightforward inductive argument, there exist some polynomial functions $(S_k)_{k\in\mathbb{N}}$ on $\mathbb{R}$ such that

$$\forall (x, k) \in \mathbb{R} \times \mathbb{N}, \Psi^{(k)}(x) = S_k(x) e^{-x^2}.$$

Moreover, the degree of the polynomial function S_k is less or equal to k, and S_0 is identically equal to the constant function 1.

However, the function $\widehat{f_\varepsilon}$ writes by formula (68),

$$\forall \xi \in \mathbb{R}^N, \widehat{f_\varepsilon}(\xi) = \widehat{f}(\xi) \prod_{k=1}^{N} \Psi(\sqrt{\varepsilon}\xi_k).$$

Therefore, since f is in $C^\infty(\mathbb{R}^N \setminus \{0\})$, $\widehat{f_\varepsilon}$ is also in $C^\infty(\mathbb{R}^N \setminus \{0\})$. Moreover, by Leibnitz' formula, we compute for every $j \in \{1, \dots, N\}$ and $p \in \mathbb{N}$,

$$\begin{aligned}\forall \xi \in \mathbb{R}^N \setminus \{0\}, \partial_j^p \widehat{f_\varepsilon}(\xi) &= \sum_{k=0}^{p} C_p^k \partial_j^{p-k} \widehat{f}(\xi) \partial_j^k \Big(\Psi(\sqrt{\varepsilon}\xi_j)\Big) \prod_{k=1, k\neq j}^{N} \Psi(\sqrt{\varepsilon}\xi_k) \\ &= \sum_{k=0}^{p} C_p^k \varepsilon^{\frac{k}{2}} \partial_j^{p-k} \widehat{f}(\xi) S_k(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2},\end{aligned}$$

which completes the proof of Step 1.

As claimed above, we then apply Lemma 1 to the tempered distribution f_ε.

Step 2. *Let $j \in \{1, \dots, N\}$ and $\lambda \in \mathbb{R}_+^*$. Then, the function $x \mapsto x_j^p f_\varepsilon(x)$ is continuous on $\mathbb{R}^N$ and writes*

$$\begin{aligned}\forall x \in \mathbb{R}^N, x_j^p f_\varepsilon(x) = \frac{i^p}{(2\pi)^N} \Bigg(&\int_{B(0,\lambda)^c} \partial_j^p \widehat{f_\varepsilon}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p-1} \widehat{f_\varepsilon}(\xi) d\xi \\ &+ \int_{B(0,\lambda)} \partial_j^p \widehat{f_\varepsilon}(\xi)(e^{ix.\xi} - 1) d\xi \Bigg).\end{aligned} \tag{70}$$

Indeed, by formula (68), f_ε is a tempered distribution on $\mathbb{R}^N$ whose Fourier transform is in $C^\infty(\mathbb{R}^N \setminus \{0\})$. Moreover, it follows from formula (69) that

$$\forall \xi \in B(0,1)^c, |\partial_j^p \widehat{f_\varepsilon}(\xi)| \leq A_\varepsilon \sum_{k=0}^{p} |\xi|^k |\partial_j^{p-k} \widehat{f}(\xi)| e^{-\varepsilon|\xi|^2},$$

so, since the functions $\partial_j^k \widehat{f}$ belongs to $L^{q_{m-k}}(B(0,1)^c)$ for every $k \in \{0, \dots, p\}$ by assumption (iv), the derivative $\partial_j^p \widehat{f_\varepsilon}$ belongs to $L^1(B(0,1)^c)$. On the other hand, by formula (69),

$$\forall \xi \in B(0,1), |\partial_j^{p-1} \widehat{f_\varepsilon}(\xi)| \leq A_\varepsilon \sum_{k=0}^{p-1} |\partial_j^{p-1-k} \widehat{f}(\xi)|,$$

so, by assumption (ii),

$$\forall \xi \in B(0,1), |\partial_j^{p-1} \widehat{f_\varepsilon}(\xi)| \leq A_\varepsilon \sum_{k=0}^{p-1} |\xi|^{1-N+k} \leq \frac{A_\varepsilon}{|\xi|^{N-1}}.$$

It follows that the function $\partial_j^{p-1} \widehat{f_\varepsilon}$ is in $L^1(B(0,1))$. Likewise, by formula (69) and assumption (ii),

$$\forall \xi \in B(0,1), |\xi| |\partial_j^p \widehat{f_\varepsilon}(\xi)| \leq \frac{A_\varepsilon}{|\xi|^{N-1}},$$

so, the function $|.|\partial_j^p \widehat{f_\varepsilon}$ also belongs to $L^1(B(0,1))$. Finally, the tempered distribution f_ε satisfies all the assumptions of Lemma 1. Thus, the function $x \mapsto x_j^p f_\varepsilon(x)$ is continuous on $\mathbb{R}^N$ and satisfies equation (70).

Now, we integrate by parts equation (70) in order to compute formula (35) for the tempered distribution f_ε.

Step 3. *Let $\Omega_j = \{x \in \mathbb{R}^N, x_j \neq 0\}$. Then, the following equation holds for every $x \in \Omega_j$,*

$$x_j^p f_\varepsilon(x) = \frac{i^p}{(2\pi)^N} \Bigg((-ix_j)^{p-m} \int_{B(0,\lambda)^c} \partial_j^m \widehat{f_\varepsilon}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \sum_{k=p}^{m-1} (-ix_j)^{p-k-1} \int_{S(0,\lambda)} \xi_j \partial_j^k \widehat{f_\varepsilon}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p-1} \widehat{f_\varepsilon}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^p \widehat{f_\varepsilon}(\xi) (e^{ix.\xi} - 1) d\xi \Bigg). \tag{71}$$

Indeed, consider the integrals defined for every $x \in \Omega_j$ and $k \in \{p, \dots, m\}$ by

$$I_k(x) = (-ix_j)^{-k} \int_{B(0,\lambda)^c} \partial_j^k \widehat{f_\varepsilon}(\xi) e^{ix.\xi} d\xi. \tag{72}$$

By formula (69), we compute

$$\forall \xi \in B(0,1)^c, |\partial_j^k \widehat{f_\varepsilon}(\xi)| \leq A_\varepsilon \sum_{l=0}^{k} |\xi|^l |\partial_j^{k-l} \widehat{f}(\xi)| e^{-\varepsilon|\xi|^2}.$$

Since the functions $\partial_j^l \widehat{f}$ belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$ and $L^{q_{m-l}}(B(0,1)^c)$ for every $l \in \{0, \dots, m\}$ (with $q_m = 1$) by assumptions (iii) and (iv), we deduce that the derivative $\partial_j^k \widehat{f_\varepsilon}$ belongs to $L^1(B(0,1)^c)$. Thus, the integral $I_k(x)$ is well-defined for every $x \in \Omega_j$.

On the other hand, an integration by parts of $I_k(x)$ yields for every $k \in \{p, \ldots, m-1\}$,

$$I_k(x) = -(-ix_j)^{-k-1} \int_{B(0,\lambda)^c} \partial_j^k \widehat{f}_\varepsilon(\xi) \partial_j e^{ix.\xi} d\xi$$
$$= I_{k+1}(x) + \frac{(-ix_j)^{-k-1}}{\lambda} \int_{S(0,\lambda)} \partial_j^k \widehat{f}_\varepsilon(\xi) \xi_j e^{ix.\xi} d\xi.$$

By adding this equation for any k between p and $m-1$, we obtain

$$I_p(x) = I_m(x) + \frac{1}{\lambda} \sum_{k=p}^{m-1} (-ix_j)^{-k-1} \int_{S(0,\lambda)} \partial_j^k \widehat{f}_\varepsilon(\xi) \xi_j e^{ix.\xi} d\xi,$$

which gives by formula (72),

$$\int_{B(0,\lambda)^c} \partial_j^p \widehat{f}_\varepsilon(\xi) e^{ix.\xi} d\xi = (-ix_j)^{p-m} \int_{B(0,\lambda)^c} \partial_j^m \widehat{f}_\varepsilon(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \sum_{k=p}^{m-1} (-ix_j)^{p-k-1}$$
$$\int_{S(0,\lambda)} \partial_j^k \widehat{f}_\varepsilon(\xi) \xi_j e^{ix.\xi} d\xi.$$

Equation (71) then follows from replacing the first term of the right member of equation (70) by the expression just above.

In order to derive formula (35), it only remains to study the convergences when ε tends to 0 of both members of equation (71). We begin by the convergence of its left member in the sense of distributions.

Step 4. *Let $j \in \{1, \ldots, N\}$ and denote $\mathcal{D}'(\mathbb{R}^N)$, the standard space of distributions. Then, the following convergence holds in $\mathcal{D}'(\mathbb{R}^N)$:*

$$x_j^p f_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} x_j^p f. \tag{73}$$

Indeed, by standard duality, we have

$$\forall \phi \in C_c^\infty(\mathbb{R}^N), < x_j^p f_\varepsilon, \phi >=< f_\varepsilon, x_j^p \phi >=< \widehat{f}_\varepsilon, \widehat{x_j^p \phi} >= i^p < \widehat{f}_\varepsilon, \partial_j^p \widehat{\phi} > .$$

Moreover, by assumption (i) and formula (68), $\widehat{f}_\varepsilon$ is in $L^1(\mathbb{R}^N)$, so,

$$< x_j^p f_\varepsilon, \phi >= i^p \int_{\mathbb{R}^N} \widehat{f}_\varepsilon(\xi) \partial_j^p \widehat{\phi}(\xi) d\xi = i^p \int_{\mathbb{R}^N} \widehat{f}(\xi) \partial_j^p \widehat{\phi}(\xi) e^{-\varepsilon|\xi|^2} d\xi.$$

On the other hand, it follows from assumption (i) that

$$\forall \xi \in \mathbb{R}^N, |\widehat{f}(\xi) \partial_j^p \widehat{\phi}(\xi) e^{-\varepsilon|\xi|^2}| \leq A|\xi|^{-r} |\partial_j^p \phi(\xi)|.$$

Therefore, by the dominated convergence theorem,

$$\int_{\mathbb{R}^N} \widehat{f}(\xi) \partial_j^p \widehat{\phi}(\xi) e^{-\varepsilon|\xi|^2} d\xi \underset{\varepsilon \to 0}{\longrightarrow} \int_{\mathbb{R}^N} \widehat{f}(\xi) \partial_j^p \widehat{\phi}(\xi) d\xi.$$

However, by assumption (i), the function $\widehat{f}$ belongs to $L^1_{loc}(\mathbb{R}^N)$, so, by standard duality,

$$\int_{\mathbb{R}^N} \widehat{f}(\xi) \partial_j^p \widehat{\phi}(\xi) d\xi =< \widehat{f}, \partial_j^p \widehat{\phi} >= (-i)^p < x_j^p f, \phi > .$$

Finally,

$$< x_j^p f_\varepsilon, \phi > \underset{\varepsilon \to 0}{\longrightarrow} < x_j^p f, \phi >,$$

and the distribution $x_j^p f_\varepsilon$ converges towards $x_j^p f$ in $\mathcal{D}'(\mathbb{R}^N)$.

Then, we consider the convergence of the right member of equation (71) in $L^\infty_{loc}(\Omega_j)$.

Step 5. *Let $g : \Omega_j \mapsto \mathbb{R}$, the function defined by*

$$\forall x \in \Omega_j, g(x) = \frac{i^p}{(2\pi)^N}\left((-ix_j)^{p-m}\int_{B(0,\lambda)^c}\partial_j^m \widehat{f}(\xi)e^{ix.\xi}d\xi + \frac{1}{\lambda}\sum_{k=p}^{m-1}(-ix_j)^{p-k-1}\int_{S(0,\lambda)}\xi_j \right.$$
$$\left. \partial_j^k\widehat{f}(\xi)e^{ix.\xi}d\xi + \frac{1}{\lambda}\int_{S(0,\lambda)}\xi_j\partial_j^{p-1}\widehat{f}(\xi)d\xi + \int_{B(0,\lambda)}\partial_j^p\widehat{f}(\xi)(e^{ix.\xi}-1)d\xi\right). \tag{74}$$

Then, g is continuous on Ω_j and satisfies

$$x_j^p f_\varepsilon \underset{\varepsilon\to 0}{\to} g \text{ in } L^\infty_{loc}(\Omega_j). \tag{75}$$

Indeed, by assumptions (ii) and (iii), the functions $\partial_j^m\widehat{f}$ and $|.|\partial_j^p\widehat{f}$ respectively belongs to $L^1(B(0,\lambda)^c)$ and $L^1(B(0,\lambda))$. Moreover, the function $\widehat{f}$ is in $C^\infty(\mathbb{R}^N\setminus\{0\})$. Therefore, by equation (74) and a standard application of the dominated convergence theorem, the function g is well-defined and continuous on $\mathbb{R}^N$.
On the other hand, it follows from equations (69), (71) and (74) that for every $x \in \Omega_j$,

$$x_j^p f_\varepsilon(x) - g(x) = \frac{i^p}{(2\pi)^N}\left((-ix_j)^{p-m}\int_{B(0,\lambda)^c}\sum_{k=1}^m C_m^k\varepsilon^{\frac{k}{2}}\partial_j^{m-k}\widehat{f}(\xi)S_k(\sqrt{\varepsilon}\xi_j)e^{-\varepsilon|\xi|^2+ix.\xi}d\xi\right.$$
$$+\frac{1}{\lambda}\sum_{k=p}^{m-1}(-ix_j)^{p-k-1}\int_{S(0,\lambda)}\xi_j\sum_{l=1}^{k}C_k^l\varepsilon^{\frac{l}{2}}\partial_j^{k-l}\widehat{f}(\xi)S_l(\sqrt{\varepsilon}\xi_j)e^{-\varepsilon|\xi|^2+ix.\xi}d\xi$$
$$+\frac{1}{\lambda}\int_{S(0,\lambda)}\xi_j\sum_{k=1}^{p-1}C_{p-1}^k\varepsilon^{\frac{k}{2}}\partial_j^{p-k-1}\widehat{f}(\xi)S_k(\sqrt{\varepsilon}\xi_j)e^{-\varepsilon|\xi|^2}d\xi$$
$$\left.+\int_{B(0,\lambda)}\sum_{k=1}^{p}C_p^k\varepsilon^{\frac{k}{2}}\partial_j^{p-k}\widehat{f}(\xi)S_k(\sqrt{\varepsilon}\xi_j)e^{-\varepsilon|\xi|^2}(e^{ix.\xi}-1)d\xi\right). \tag{76}$$

Assumption (iv) and Step 1 then yield for the first term of the right member,

$$\left|\sum_{k=1}^m\int_{B(0,\lambda)^c}C_m^k\varepsilon^{\frac{k}{2}}\partial_j^{m-k}\widehat{f}(\xi)S_k(\sqrt{\varepsilon}\xi_j)e^{-\varepsilon|\xi|^2+ix.\xi}d\xi\right|$$
$$\leq A\sum_{k=1}^m\int_{B(0,\lambda)^c}\varepsilon^{\frac{k}{2}}|\partial_j^{m-k}\widehat{f}(\xi)||S_k(\sqrt{\varepsilon}\xi_j)|e^{-\varepsilon|\xi|^2}d\xi$$
$$\leq A\sum_{k=1}^m\varepsilon^{\frac{k}{2}}\|\partial_j^{m-k}\widehat{f}\|_{L^{q_k}(B(0,\lambda)^c)}\left(\int_{B(0,\lambda)^c}|S_k(\sqrt{\varepsilon}\xi_j)|^{q'_k}e^{-\varepsilon q'_k|\xi|^2}d\xi\right)^{\frac{1}{q'_k}}$$
$$\leq A\sum_{k=1}^m\varepsilon^{\frac{k}{2}-\frac{N}{2q'_k}}\left(\int_{\mathbb{R}^N}|S_k(\xi_j)|^{q'_k}e^{-q'_k|\xi|^2}d\xi\right)^{\frac{1}{q'_k}}$$

where $q'_k = \frac{q_k}{q_k-1}$. Moreover, we know that

$$\begin{cases} 1 < q_k < \frac{N}{N-k} \text{ if } 1 \leq k \leq N-1, \\ 1 < q_k \leq +\infty \text{ otherwise,}\end{cases}$$

so, there is some positive real number δ such that for every $k \in \{1, \dots, m\}$,

$$\frac{k}{2} - \frac{N}{2q'_k} \geq \delta.$$

Thus, we obtain

$$\left| \sum_{k=1}^{m} \int_{B(0,\lambda)^c} C_m^k \varepsilon^{\frac{k}{2}} \partial_j^{m-k} \widehat{f}(\xi) S_k(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2 + ix.\xi} d\xi \right| \leq A\varepsilon^{\delta}. \tag{77}$$

By Step 1 and the smoothness of $\widehat{f}$ on $S(0,\lambda)$, we then estimate the second term of the right member of equation (76),

$$\begin{aligned} &\left| \frac{1}{\lambda} \sum_{k=p}^{m-1} (-ix_j)^{p-k-1} \int_{S(0,\lambda)} \xi_j \sum_{l=1}^{k} C_k^l \varepsilon^{\frac{l}{2}} \partial_j^{k-l} \widehat{f}(\xi) S_l(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2 + ix.\xi} d\xi \right| \\ \leq & A\varepsilon^{\frac{1}{2}} \sum_{k=p}^{m-1} |x_j|^{p-k-1} \int_{S(0,\lambda)} \sum_{l=1}^{k} (1 + \varepsilon^l \lambda^l) d\xi, \end{aligned}$$

which reduces to

$$\left| \sum_{k=p}^{m-1} \frac{(-ix_j)^{p-k-1}}{\lambda} \int_{S(0,\lambda)} \xi_j \sum_{l=1}^{k} C_k^l \varepsilon^{\frac{l}{2}} \partial_j^{k-l} \widehat{f}(\xi) S_l(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2 + ix.\xi} d\xi \right| \leq A\varepsilon^{\frac{1}{2}} \sum_{k=p}^{m-1} |x_j|^{p-k-1}. \tag{78}$$

Likewise, the third term of the right member of equation (76) satisfies by Step 1 and the smoothness of $\widehat{f}$ on $S(0,\lambda)$,

$$\left| \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \sum_{k=1}^{p-1} C_{p-1}^k \varepsilon^{\frac{k}{2}} \partial_j^{p-k-1} \widehat{f}(\xi) S_k(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2} d\xi \right| \leq A\varepsilon^{\frac{1}{2}}. \tag{79}$$

Finally, the last term of the right member of equation (76) verifies by Step 1 and assumption (ii),

$$\begin{aligned} &\left| \int_{B(0,\lambda)} \sum_{k=1}^{p} C_p^k \varepsilon^{\frac{k}{2}} \partial_j^{p-k} \widehat{f}(\xi) S_k(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2} (e^{ix.\xi} - 1) d\xi \right| \\ \leq & A|x| \int_{B(0,\lambda)} \sum_{k=1}^{p} \varepsilon^{\frac{k}{2}} |\partial_j^{p-k} \widehat{f}(\xi)| |\xi| |S_k(\sqrt{\varepsilon}\xi_j)| d\xi \\ \leq & A|x| \varepsilon^{\frac{1}{2}} \int_{B(0,\lambda)} \sum_{k=1}^{p} |\xi|^{k-N+1} d\xi, \end{aligned}$$

which gives

$$\left| \int_{B(0,\lambda)} \sum_{k=1}^{p} C_p^k \varepsilon^{\frac{k}{2}} \partial_j^{p-k} \widehat{f}(\xi) S_k(\sqrt{\varepsilon}\xi_j) e^{-\varepsilon|\xi|^2} (e^{ix.\xi} - 1) d\xi \right| \leq A\varepsilon^{\frac{1}{2}} |x|. \tag{80}$$

Thus, equations (76), (77), (78), (79) and (80) yield

$$\forall x \in \Omega_j, |x_j^p f_\varepsilon(x) - g(x)| \leq A\Big(\varepsilon^{\delta} + \varepsilon^{\frac{1}{2}} \Big(1 + |x| + \sum_{k=p}^{m-1} |x_j|^{p-k-1}\Big)\Big).$$

Therefore, the functions $x \mapsto x_j^p f_\varepsilon$ converge towards the function g in $L^\infty_{loc}(\Omega_j)$ when ε tends to 0.

Finally, by Step 4, the function $x \mapsto x_j^p f_\varepsilon$ converges towards the function $x \mapsto x_j^p f$ in $\mathcal{D}'(\mathbb{R}^N)$ when ε tends to 0, while by Step 5, it also converges towards the function g in $L^\infty_{loc}(\Omega_j)$. Therefore, the function $x \mapsto x_j^p f$ is identically equal to g on Ω_j. In particular, by Step 5, it is continuous on Ω_j and satisfies equation (35), which ends the proof of Lemma 2. □

Lemma 2 then yields the integral expression (36) of Proposition 5 for some tempered distributions f whose Fourier transform is a rational fraction R of form (18)-(19).

Proof of Proposition 5. Let us denote $p_j = N - 2 + d$, $m_1 = 2N - 4 + d_1 + 2d_\perp$ and $m_j = N - 2 + d$ if $j \geq 2$. In order to prove formula (36), we apply Lemma 2 to the distribution f with $p = p_j$ and $m = m_j$.

Indeed, by Proposition 4, the tempered distribution f satisfies all the assumptions of Lemma 2. By formulae (18)-(19), its Fourier transform $\widehat{f} = R$ belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$ and satisfies

$$\forall \xi \in \mathbb{R}^N, |\widehat{f}(\xi)| \leq A\bigg(|\xi|^d + |\xi|^{d-2}\bigg).$$

Thus, since $d \neq 0$ if $N = 2$, assumption (i) is satisfied by f.

Moreover, by Proposition 4, there is some positive real number A such that

$$\forall (k, \xi) \in \{0, \ldots, N-1\} \times B(0,1), |\xi|^{k+2-d} |\partial_j^k \widehat{f}(\xi)| \leq A.$$

Therefore, f verifies assumption (ii) with $p_j = N - 2 + d$.

On the other hand, by Proposition 4, the function $\partial_j^{m_j} \widehat{f}$ belongs to $L^1(B(0,1))^c$. Moreover, since $d_1 + 2d_\perp \leq 4$, for every $k \in \{0, \ldots, m_j - 1\}$, the function $\partial_j^k \widehat{f}$ belongs to some space $L^q(B(0,1))$. More precisely, they belong to all the spaces $L^q(B(0,1)^c)$ for every $q > q^j_{m_j - k} = \frac{2N-1}{(1+\delta_{j,1})k + 4 - d_1 - 2d_\perp}$. In particular, we remark that

$$\forall k \in \{1, \ldots, N-1\}, q_k^1 = \frac{2N-1}{2N-k} < \frac{N}{N-k},$$

while for every $j \in \{2, \ldots, N\}$,

$$\forall k \in \{1, \ldots, N-1\}, q_k^j = \frac{2N-1}{2N-2k+d_1} < \frac{N}{N-k}.$$

Therefore, the distribution f also verifies assumptions (iii) and (iv) of Lemma 2, and we can apply it with $p = p_j$ and $m = m_j$. Thus, the function $x \mapsto x_j^{p_j} f(x)$ is continuous on Ω_j, and by equation (35), formula (36) holds. □

1.3 Algebraic decay and explosion near the origin of the kernels H_0, K_0 and K_k.

This section focuses on the proofs of Propositions 1 and 2 (which immediately yield Theorems 3 and 4, and Corollary 1). Their proofs follow from expression (36) of Proposition 5. We estimate it for any value of the parameter λ before choosing it appropriately (either to compute the algebraic decay at infinity or the explosion near the origin of the considered distribution). For instance, let us begin by the algebraic decay properties stated in Proposition 1 and Theorem 3.

Proof of Proposition 1. By Proposition 5, the functions $x \mapsto x_j^{p_j} f(x)$ are continuous on Ω_j for every $j \in \{1, \ldots, N\}$. In particular, it follows that the restriction of the tempered distribution f to $\mathbb{R}^N \setminus \{0\}$ is a continuous function on $\mathbb{R}^N \setminus \{0\}$. Now, consider $x \in B(0,1)^c$ and $j \in \{1, \ldots, N\}$ such that

$$|x_j| \geq \frac{|x|}{\sqrt{N}}, \tag{81}$$

and choose

$$\lambda = \frac{1}{|x|}.$$

By formula (36) of Proposition 5, and since the restriction to $B(0,1)^c$ of the tempered distribution f is a continuous function, we compute

$$\begin{aligned} |x|^{N-2+d}|f(x)| \leq & A\frac{|x|^{N-2+d}}{|x_j|^{N-2+d}}\Bigg(|x_j|^{N-2+d-m_j}\int_{B(0,\frac{1}{|x|})^c}|\partial_j^{m_j}\widehat{f}(\xi)|d\xi + \sum_{k=N-2+d}^{m_j-1}|x_j|^{N-3+d-k} \\ & \int_{S(0,\frac{1}{|x|})}|\partial_j^{k}\widehat{f}(\xi)|d\xi + \int_{S(0,\frac{1}{|x|})}|\partial_j^{N-3+d}\widehat{f}(\xi)|d\xi + |x|\int_{B(0,\frac{1}{|x|})}|\partial_j^{N-2+d}\widehat{f}(\xi)| \\ & |\xi|d\xi\Bigg). \end{aligned}$$

Therefore, by assumption (81), Proposition 4, and since $m_j \geq N-2+d$,

$$\begin{aligned} |x|^{N-2+d}|f(x)| \leq & A_N\Bigg(|x|^{N-2+d-m_j}\bigg(\int_{B(0,1)^c}|\partial_j^{m_j}\widehat{f}(\xi)|d\xi + \int_{\frac{1}{|x|}\leq|\xi|\leq 1}\frac{d\xi}{|\xi|^{m_j+2-d}}\bigg) \\ & + \sum_{k=N-2+d}^{m_j-1}|x|^{N-3+d-k}\int_{S(0,\frac{1}{|x|})}\frac{d\xi}{|\xi|^{k+2-d}} + \int_{S(0,\frac{1}{|x|})}\frac{d\xi}{|\xi|^{N-1}} + |x| \\ & \int_{B(0,\frac{1}{|x|})}\frac{d\xi}{|\xi|^{N-1}}\Bigg) \\ \leq & A(|x|^{N-2+d-m_j}+1) \leq A. \end{aligned}$$

Thus, the distribution f belongs to $M^{\infty}_{N-2+d}(B(0,1)^c)$. □

Then, we deduce the proof of Theorem 3.

Proof of Theorem 3. Theorem 3 is a direct application of Proposition 1 with $(d_1, d_\perp) = (1,0)$ for the kernel H_0, $(d_1, d_\perp) = (2,0)$ for the kernel K_0 and $(d_1, d_\perp) = (2+\delta_{k,1}, 1-\delta_{k,1})$ for the kernels K_k. □

Now, we turn to the study of the singularities of the distribution f near the origin.

Proof of Proposition 2. We first derive local estimates (20) for the distribution f. Then, we describe its singularities in terms of L^q-spaces.

The proof relies on the use of expression (36), which we estimate by the same arguments as in the proof of Proposition 4. Indeed, the distribution f satisfies the assumptions of Proposition 5. Therefore, by the argument mentioned above in the proof of Proposition 1, its restriction to the open set $B(0,1)\setminus\{0\}$ is a continuous function, which satisfies moreover equation (36). Thus, it remains to bound the second term of equation (36) independently of x to obtain local estimates (20). However, there are at least two difficulties. The first one follows from the form of equation (36). Indeed, this equation is suitable to give the optimal algebraic decay of the distribution f, but not its optimal algebraic explosion near the origin. More precisely, a simple way to estimate the last term of equation (36) is to write by Proposition 4,

$$\left|\int_{B(0,\lambda)} \partial_j^{p_j} \widehat{f}(\xi)(e^{ix.\xi}-1)d\xi\right| \leq A|x| \int_{B(0,1)} |\partial_j^{p_j} \widehat{f}(\xi)||\xi|d\xi + A \int_{1\leq|\xi|\leq\lambda} |\partial_j^{p_j} \widehat{f}(\xi)|d\xi$$
$$\leq A|x| + A_\lambda.$$

By this argument, we cannot expect to prove (for any choice of λ) that this term is bounded by $A|x_j|^d$ for some positive real numbers A and d (which is actually the goal of Proposition 2). Thus, in order to describe precisely the singularities of the distribution f near the origin, we integrate by parts the last term of equation (36).

Indeed, consider $j \in \{1,\dots,N\}$, $\lambda > 1$ and $x \in B(0,1) \cap \Omega_j$. By Proposition 4, the functions $\partial_j^k \widehat{f}$ belong to the space $L^1(B(0,\lambda))$ for every $0 \leq k < p_j$. Therefore, by integrating by parts the integral above, we deduce for any integer $1 \leq q_j \leq p_j$,

$$\int_{B(0,\lambda)} \partial_j^{p_j} \widehat{f}(\xi)(e^{ix.\xi}-1)d\xi = -\frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j \partial_j^{p_j-1}\widehat{f}(\xi)d\xi + \sum_{k=1}^{q_j} \frac{(-ix_j)^{k-1}}{\lambda}\int_{S(0,\lambda)} \xi_j e^{ix.\xi}$$
$$\partial_j^{p_j-k}\widehat{f}(\xi)d\xi + (-ix_j)^{q_j}\int_{B(0,\lambda)} \partial_j^{p_j-q_j}\widehat{f}(\xi)e^{ix.\xi}d\xi.$$

Hence, equation (36) becomes

$$x_j^{p_j} f(x) = \frac{i^{p_j}}{(2\pi)^N}\Bigg((-ix_j)^{p_j-m_j}\int_{B(0,\lambda)^c} \partial_j^{m_j}\widehat{f}(\xi)e^{ix.\xi}d\xi + \frac{1}{\lambda}\sum_{k=p_j}^{m_j-1}(-ix_j)^{p_j-k-1}\int_{S(0,\lambda)} \xi_j$$
$$\partial_j^k \widehat{f}(\xi)e^{ix.\xi}d\xi + \sum_{k=1}^{q_j}\frac{(-ix_j)^{k-1}}{\lambda}\int_{S(0,\lambda)} \xi_j\partial_j^{p_j-k}\widehat{f}(\xi)e^{ix.\xi}d\xi + (-ix_j)^{q_j}\int_{B(0,\lambda)} e^{ix.\xi}$$
$$\partial_j^{p_j-q_j}\widehat{f}(\xi)d\xi\Bigg). \tag{82}$$

In particular, the simple argument used above to bound the integral on $B(0,\lambda)$ now gives by Proposition 4,

$$\left|(-ix_j)^{q_j}\int_{B(0,\lambda)} e^{ix.\xi}\partial_j^{p_j-q_j}\widehat{f}(\xi)d\xi\right|$$
$$\leq A|x_j|^{q_j}\left(\int_{B(0,1)} |\partial_j^{p_j-q_j}\widehat{f}(\xi)|d\xi + \int_{1\leq|\xi|\leq\lambda} |\partial_j^{p_j-q_j}\widehat{f}(\xi)|d\xi\right)$$
$$\leq A|x_j|^{q_j} + A_\lambda|x_j|^{q_j}.$$

Therefore, by choosing q_j and λ appropriately, we can now expect to describe sharply the singularities of the distribution f near the origin.

On the other hand, a second difficulty arises from the non-isotropy of the considered distributions. Their explosion near the origin is not the same in all the directions, so, we will split our analysis in two steps: the first one for the direction x_1, the second one for the directions x_j with $j \geq 2$. In particular, we will choose different values of q_j and λ according to the value of j.

Step 1. *Estimates of the distribution f in direction x_1.*

Let us first study the singularity of f near the origin in direction x_1. In order to do so, consider $x \in B(0,1) \cap \Omega_1$ and $\lambda \geq 1$. By Proposition 5, the restriction to $B(0,1) \cap \Omega_1$ of the distribution f is continuous and satisfies by equation (82),

$$\begin{aligned}|x_1|^{m_1}|f(x)| \leq & A\Bigg(\int_{B(0,\lambda)^c}|\partial_1^{m_1}\widehat{f}(\xi)|d\xi + \frac{1}{\lambda}\sum_{k=p_1-q_1}^{m_1-1}|x_1|^{m_1-1-k}\int_{S(0,\lambda)}|\partial_1^k\widehat{f}(\xi)||\xi_1|d\xi \\ & + |x_1|^{m_1-p_1+q_1}\int_{B(0,\lambda)}|\partial_1^{p_1-q_1}\widehat{f}(\xi)|d\xi\Bigg),\end{aligned} \tag{83}$$

where we denote as above $p_1 = N - 2 + d$ and $m_1 = 2N - 4 + d_1 + 2d_\perp$, and choose $q_1 = \min\{p_1, 2\}$. Indeed, since $d \neq 0$ if $N = 2$, q_1 satisfies in any case

$$1 \leq q_1 \leq p_1.$$

The first term of the right member of equation (83) then becomes by Proposition 3,

$$\begin{aligned}\int_{B(0,\lambda)^c}|\partial_1^{m_1}\widehat{f}(\xi)|d\xi = & \int_{B(0,\lambda)^c}\frac{|P_{1,m_1}(\xi)|}{(|\xi|^2+\xi_1^4)^{m_1+1}}d\xi \\ \leq & A\sum_{0\leq k\leq\frac{d_1+3m_1}{4}}\int_{B(0,\lambda)^c}\frac{\max\{|\xi_1|,1\}^{d_1+3m_1-4k}|\xi_\perp|^{2k+d_\perp}}{(|\xi|^2+\xi_1^4)^{m_1+1}}d\xi,\end{aligned}$$

so, by equations (61) and (63),

$$\begin{aligned}\int_{B(0,\lambda)^c}|\partial_1^{m_1}\widehat{f}(\xi)|d\xi \leq & A\sum_{0\leq k\leq\frac{d_1+3m_1}{4}}\int_\lambda^{+\infty}J_{d_1+3m_1-4k,2k+d_\perp,m_1+1}(r)dr \\ \leq & A\sum_{0\leq k\leq\frac{d_1+3m_1}{4}}\int_\lambda^{+\infty}(r^{2k-3N+4-2d_1-3d_\perp}+r^{-2k-N-1-d_\perp}+r^{-\frac{3}{2}})dr \\ \leq & A\int_\lambda^{+\infty}r^{-\frac{3}{2}}dr,\end{aligned}$$

and finally,

$$\int_{B(0,\lambda)^c}|\partial_1^{m_1}\widehat{f}(\xi)|d\xi \leq \frac{A}{\sqrt{\lambda}}. \tag{84}$$

Likewise, Proposition 3, and equations (61) and (63) yield for every $k \in \{p_1 - q_1, \ldots, m_1 - 1\}$,

$$\begin{aligned}\int_{S(0,\lambda)}|\xi_1||\partial_1^k\widehat{f}(\xi)|d\xi = & \int_{S(0,\lambda)}\frac{|\xi_1||P_{1,k}(\xi)|}{(|\xi|^2+\xi_1^4)^{k+1}}d\xi \\ \leq & A\sum_{0\leq l\leq\frac{d_1+3k}{4}}\int_{S(0,\lambda)}\frac{\max\{|\xi_1|,1\}^{1+d_1+3k-4l}|\xi_\perp|^{2l+d_\perp}}{(|\xi|^2+\xi_1^4)^{k+1}}d\xi \\ \leq & A\sum_{0\leq l\leq\frac{d_1+3k}{4}}J_{1+d_1+3k-4l,2l+d_\perp,k+1}(\lambda) \\ \leq & A\sum_{0\leq l\leq\frac{d_1+3k}{4}}\lambda^{N-4}\Big(\lambda^{2l+d_\perp-2k}+\lambda^{d_1+d_\perp-k-2l}+\lambda^{1+\frac{d_1}{2}+d_\perp-\frac{k}{2}}\Big),\end{aligned}$$

which gives

$$\int_{S(0,\lambda)} |\xi_1||\partial_1^k \widehat{f}(\xi)|d\xi \leq A\lambda^{N-3+\frac{d_1}{2}+d_\perp-\frac{k}{2}}. \tag{85}$$

Finally, the last term of the second member of equation (83) satisfies by Propositions 3 and 4, and equations (61) and (63),

$$\begin{aligned}
\int_{B(0,\lambda)} |\partial_1^{p_1-q_1}\widehat{f}(\xi)|d\xi \leq &\int_{B(0,1)} |\partial_1^{p_1-q_1}\widehat{f}(\xi)|d\xi + \int_{1\leq|\xi|\leq\lambda} |\partial_1^{p_1-q_1}\widehat{f}(\xi)|d\xi \\
\leq &A\left(1+\int_{1\leq|\xi|\leq\lambda} \frac{|P_{1,p_1-q_1}(\xi)|}{(|\xi|^2+\xi_1^4)^{p_1-q_1+1}}d\xi\right) \\
\leq &A\left(1+\sum_{0\leq k\leq\frac{d_1+3p_1-3q_1}{4}} \int_1^\lambda J_{d_1+3p_1-3q_1-4k,2k+d_\perp,p_1-q_1+1}(r)dr\right) \\
\leq &A\left(1+\int_1^\lambda r^{\frac{N-5+d_\perp+q_1}{2}}dr\right),
\end{aligned}$$

which yields

$$\int_{B(0,\lambda)} |\partial_1^{p_1-q_1}\widehat{f}(\xi)|d\xi \leq A\Big(1+(1-\delta)\lambda^{\frac{N-3+d_\perp+q_1}{2}}+\delta\ln(\lambda)\Big), \tag{86}$$

where $\delta=\delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0}$. It then follows from equations (83), (84), (85) and (86) that

$$\begin{aligned}
|x_1|^{m_1}|f(x)| \leq &A\Bigg(\lambda^{-\frac{1}{2}}+\sum_{k=p_1-q_1}^{m_1-1}\Big(|x_1|^{2N-5+d_1+2d_\perp-k}\lambda^{N-4+d_\perp+\frac{d_1-k}{2}}\Big)+|x_1|^{N-2+d_\perp+q_1}\Big(1 \\
&+(1-\delta)\lambda^{\frac{N-3+d_\perp+q_1}{2}}+\delta\ln(\lambda)\Big)\Bigg).
\end{aligned}$$

However, since $|x|<1$, we can set

$$\lambda=\frac{1}{x_1^2}>1,$$

to obtain

$$|x_1|^{m_1}|f(x)| \leq A\Big(|x_1|+|x_1|^3+|x_1|^{N-2+d_\perp+q_1}\Big(1+\delta|\ln(|x_1|)|\Big)\Big).$$

Finally, we deduce the estimate of the kernel f near the origin in direction x_1,

$$|x_1|^{2N-5+d_1+2d_\perp}|f(x)| \leq A\Big(1+\delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0}|\ln(|x_1|)|\Big). \tag{87}$$

Step 2. *Estimates of the distribution f in directions x_j, $j\neq 1$.*

Now, we consider the singularity of f near the origin in every direction x_j with $j\in\{2,\ldots,N\}$. In order to do so, let $x\in B(0,1)\cap\Omega_j$ and $\lambda\geq 1$. Equation (82) then yields

$$\begin{aligned}
|x_j|^{p_j}|f(x)| \leq &A\Bigg(\int_{B(0,\lambda)^c}|\partial_j^{p_j}\widehat{f}(\xi)|d\xi+\sum_{k=1}^{q_j}\frac{|x_j|^{k-1}}{\lambda}\int_{S(0,\lambda)}|\xi_j||\partial_j^{p_j-k}\widehat{f}(\xi)|d\xi \\
&+|x_j|^{q_j}\int_{B(0,\lambda)}|\partial_j^{p_j-q_j}\widehat{f}(\xi)|d\xi\Bigg),
\end{aligned} \tag{88}$$

where we denote $p_j = N-2+d$ and choose $q_j = \min\{k \in \mathbb{N}, k > \frac{d_1+1}{2}\} - \delta$ with $\delta = \delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0}$ as above. Indeed, since $d \neq 0$ if $N=2$, q_j satisfies in any case

$$1 \leq q_j \leq p_j.$$

On one hand, by Proposition 3, and equations (61) and (63),

$$\begin{aligned}\int_{B(0,\lambda)^c} |\partial_j^{p_j} \widehat{f}(\xi)| d\xi = & \int_{B(0,\lambda)^c} \frac{|P_{j,p_j}(\xi)|}{(|\xi|^2+\xi_1^4)^{p_j+1}} d\xi \\ \leq & A \sum_{0 \leq k \leq \frac{d_\perp + p_j}{2}} \int_{B(0,\lambda)^c} \frac{\max\{|\xi_1|,1\}^{4k+d_1} \max\{|\xi_\perp|,1\}^{d_\perp+p_j-2k}}{(|\xi|^2+\xi_1^4)^{p_j+1}} d\xi \\ \leq & A \sum_{0 \leq k \leq \frac{d_\perp + p_j}{2}} \int_\lambda^{+\infty} J_{4k+d_1, d_\perp+p_j-2k, p_j+1}(r) dr \\ \leq & A \int_\lambda^{+\infty} \frac{dr}{r^{\frac{3+d_1}{2}}},\end{aligned}$$

so,

$$\int_{B(0,\lambda)^c} |\partial_j^{p_j} \widehat{f}(\xi)| d\xi \leq \frac{A}{\lambda^{\frac{1+d_1}{2}}}. \tag{89}$$

On the other hand, Proposition 3 and equation (61) yield for every $k \in \{1, \ldots, q_j\}$,

$$\begin{aligned}\int_{S(0,\lambda)} |\xi_j| |\partial_j^{p_j-k} \widehat{f}(\xi)| d\xi = & \int_{S(0,\lambda)} \frac{|P_{j,p_j-k}(\xi)| |\xi_j|}{(|\xi|^2+\xi_1^4)^{p_j-k+1}} d\xi \\ \leq & A \sum_{0 \leq l \leq \frac{p_j-k+d_\perp}{2}} J_{d_1+4l, d_\perp+1+p_j-k-2l, p_j-k+1}(\lambda),\end{aligned}$$

which gives by equation (63),

$$\int_{S(0,\lambda)} |\xi_j| |\partial_j^k \widehat{f}(\xi)| d\xi \leq A \lambda^{-\frac{d_1+1}{2}+k}. \tag{90}$$

Finally, we deduce from Proposition 3 and equation (61),

$$\begin{aligned}\int_{B(0,\lambda)} |\partial_j^{p_j-q_j} \widehat{f}(\xi)| d\xi \leq & \int_{B(0,1)} |\partial_j^{p_j-q_j} \widehat{f}(\xi)| d\xi + \int_{1 \leq |\xi| \leq \lambda} |\partial_j^{p_j-q_j} \widehat{f}(\xi)| d\xi \\ \leq & A\Bigg(1 + \int_{1 \leq |\xi| \leq \lambda} \frac{|P_{j,p_j-q_j}(\xi)|}{(|\xi|^2+\xi_1^4)^{p_j-q_j+1}} d\xi\Bigg) \\ \leq & A + A \sum_{0 \leq k \leq \frac{p_j-q_j+d_\perp}{2}} \int_1^\lambda J_{4k+d_1, d_\perp+p_j-q_j-2k, p_j-q_j+1}(r) dr,\end{aligned}$$

which yields by equations (63),

$$\int_{B(0,\lambda)} |\partial_j^{p_j-q_j} \widehat{f}(\xi)| d\xi \leq A\Big(1 + (1-\delta)\lambda^{\frac{-1-d_1}{2}+q_j} + \delta \ln(\lambda)\Big), \tag{91}$$

where $\delta = \delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0}$ as above. It then follows from equations (88), (89), (90) and (91) that

$$|x_j|^{p_j} |f(x)| \leq A\Bigg(\lambda^{-\frac{1+d_1}{2}} + \sum_{k=1}^{q_j} \Big(|x_j|^{k-1} \lambda^{-\frac{d_1+3}{2}+k}\Big) + |x_j|^{q_j} \Big(1 + (1-\delta)\lambda^{\frac{-1-d_1}{2}+q_j} + \delta \ln(\lambda)\Big)\Bigg).$$

However, since $|x| < 1$, we can set

$$\lambda = \frac{1}{|x_j|} > 1,$$

to obtain the estimate of the kernel f near the origin in direction x_j,

$$|x_j|^{N-\frac{5}{2}+\frac{d_1}{2}+d_\perp}|f(x)| \leq A\Big(1+\delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0}|\ln(|x_j|)|\Big). \tag{92}$$

Step 3. *Proof of local estimate* (20).

Estimate (20) follows from Steps 1 and 2. Indeed, assume first that $N = 2$, $d_1 = 1$ and $d_\perp = 0$. Since the restriction of the distribution f to $B(0,1) \setminus \{0\}$ is continuous, equations (87) and (92) yield

$$\forall x \in B(0,1) \setminus \{0\}, |f(x)| \leq A \min\{|\ln(|x_1|)|, |\ln(|x_2|)|\},$$

so,

$$\forall x \in B(0,1) \setminus \{0\}, |f(x)| \leq A|\ln(|x|)|,$$

which is exactly estimate (20). Likewise, if $N \neq 2$, $d_1 \neq 1$ or $d_\perp \neq 0$, the restriction of the distribution f to $B(0,1) \setminus \{0\}$ is also continuous, and equations (87) and (92) give for every $x \in B(0,1) \setminus \{0\}$,

$$\Big(x_1^2 + |x_\perp|\Big)^{N-\frac{5}{2}+\frac{d_1}{2}+d_\perp}|f(x)| \leq A\Big(|x_1|^{2N-5+d_1+2d_\perp} + \sum_{j=2}^N |x_j|^{N-\frac{5}{2}+\frac{d_1}{2}+d_\perp}\Big)|f(x)| \leq A,$$

which concludes the proof of estimate (20).

Step 4. *L^q-integrability of the distribution f near the origin.*

Now, we turn to integral estimates of the distribution f near the origin. They will follow from local estimates (20) by a standard argument of distribution theory. Indeed, let $j \in \{1, \ldots, N\}$ and denote f_j and g_j, the tempered distributions defined on $\mathbb{R}^N$ and $\mathbb{R}^N \setminus \{0\}$ by

$$\begin{aligned} f_j &= x_j^{\delta'} f, \\ g_j &= x_j^{\delta'} f_{|\mathbb{R}^N\setminus\{0\}}. \end{aligned} \tag{93}$$

Here, δ' is equal to 1 if $(d_1, d_\perp) = (2,1)$ or $(d_1, d_\perp) = (4,0)$, and 0, otherwise, while $f_{|\mathbb{R}^N\setminus\{0\}}$ denotes the restriction of the distribution f to $\mathbb{R}^N \setminus \{0\}$. As mentioned above, $f_{|\mathbb{R}^N\setminus\{0\}}$ is actually a continuous function on $\mathbb{R}^N \setminus \{0\}$, so, g_j is also continuous on $\mathbb{R}^N \setminus \{0\}$. Moreover, equation (20) yields for every $x \in \mathbb{R}^N \setminus \{0\}$,

$$|g_j(x)| \leq A\frac{(1+\delta|\ln(|x|)|)|x|^{\delta'}}{(|x_1|^2 + |x_\perp|)^s}, \tag{94}$$

where we denote $\delta = \delta_{N,2}\delta_{d_1,1}\delta_{d_\perp,0}$ and $s = N - \frac{5}{2} + \frac{d_1}{2} + d_\perp$. Thus, if $N = 2$, $d_1 = 1$ and $d_\perp = 0$, we compute for every $q \geq 1$,

$$\int_{B(0,1)} |g_j(x)|^q dx \leq A \int_{B(0,1)} |\ln(|x|)|^q dx < +\infty,$$

so, the distribution g_j belongs to $L^q(B(0,1))$ for every $q \geq 1$, i.e.

$$1 \leq q < \frac{2N-1}{2N-5+d_1+2d_\perp}.$$

On the other hand, if $N \neq 2$, $d_1 \neq 1$ or $d_\perp \neq 0$, equation (94) yields for every $q \geq 1$,

$$\int_{B(0,1)} |g_j(x)|^q dx \leq A \int_{B(0,1)} \frac{|x|^{q\delta'}}{(|x_1|^2+|x_\perp|)^{qs}} dx,$$

which gives by using the spherical coordinates and the successive changes of variables $u = \tan(\theta_1)$ and $v = \frac{u}{r}$,

$$\begin{aligned}\int_{B(0,1)} |g_j(x)|^q dx \leq & A \int_0^1 r^{N-1+q(\delta'-s)} \Big(\int_0^{\frac{\pi}{2}} \frac{\sin(\theta_1)^{N-2} d\theta_1}{(r\cos^2(\theta_1)+\sin(\theta_1))^{qs}} \Big) dr \\ \leq & A \int_0^1 r^{N-1+q(\delta'-s)} \Big(\int_0^{+\infty} \frac{u^{N-2}(1+u^2)^{qs-\frac{N}{2}} du}{(r+u\sqrt{1+u^2})^{qs}} \Big) dr \\ \leq & A \int_0^1 r^{2N-2+q(\delta'-2s)} \Big(\int_0^{+\infty} \frac{v^{N-2}(1+r^2v^2)^{qs-\frac{N}{2}} dv}{(1+v\sqrt{1+r^2v^2})^{qs}} \Big) dr.\end{aligned}$$

However, we compute for every $r \in]0,1]$,

$$\begin{aligned}\int_0^{+\infty} \frac{v^{N-2}(1+r^2v^2)^{qs-\frac{N}{2}} dv}{(1+v\sqrt{1+r^2v^2})^{qs}} \leq & A \Big(\int_0^{\frac{1}{r}} \frac{v^{N-2} dv}{(1+v)^{qs}} + \int_{\frac{1}{r}}^{+\infty} \frac{r^{2qs-N} v^{2qs-2} dv}{(1+rv^2)^{qs}} \Big) \\ \leq & A \Big(1 + \int_1^{\frac{1}{r}} \frac{dv}{v^{qs-N+2}} + r^{qs-N} \int_{\frac{1}{r}}^{+\infty} \frac{dv}{v^2} \Big),\end{aligned}$$

so,

$$\int_0^{+\infty} \frac{v^{\frac{N}{2}}(1+r^2v^2)^{qs-\frac{N}{2}} dv}{(1+v\sqrt{1+r^2v^2})^{qs}} \leq \begin{cases} A & \text{if} \quad qs > N-1, \\ A|\ln(r)| & \text{if} \quad qs = N-1, \\ Ar^{qs-N+1} & \text{if} \quad qs < N-1. \end{cases}$$

Thus, we deduce

$$\int_{B(0,1)} |g_j(x)|^q dx \leq A \begin{cases} \int_0^1 r^{2N-2+q\delta'-2qs} dr & \text{if} \quad qs > N-1, \\ \int_0^1 r^{q\delta'} |\ln(r)| dr & \text{if} \quad qs = N-1, \\ \int_0^1 r^{N+q\delta'-qs-1} dr & \text{if} \quad qs < N-1. \end{cases}$$

Hence, the distribution g_j belongs to $L^q(B(0,1))$ for

$$1 \leq q < \frac{2N-1}{2s-\delta'},$$

which is exactly condition (21) if $\delta' = 0$, and condition (22) if $\delta' = 1$.

Now, we deduce the same result for the distributions f_j. Indeed, by the conclusion just above, g_j is continuous on $\mathbb{R}^N \setminus \{0\}$, and belongs to $L^1(B(0,1))$. Therefore, it defines a distribution on the whole space $\mathbb{R}^N$ (and not only on the subset $\mathbb{R}^N \setminus \{0\}$). It then follows from definition (93) that the support of the distribution $f_j - g_j$ is included in the singleton $\{0\}$. Thus, by Schwartz' theorem, there exist some integer M and some real numbers λ_α such that

$$f_j - g_j = \sum_{|\alpha| \leq M} \lambda_\alpha \partial_\alpha \delta_\alpha. \tag{95}$$

However, by Proposition 1, definition (93) and the proof just above, the distribution g_j belongs to all the spaces $L^q(B(0,1))$ for every $1 \leq q < \frac{2N-1}{2N-5+d_1+2d_\perp-\delta'}$, and to all the spaces $L^q(B(0,1)^c)$ for every $q > \frac{N}{N-2+d-\delta'}$. Therefore, it belongs to all the spaces $L^q(\mathbb{R}^N)$ for

$$\frac{N}{N-2+d-\delta'} < q < \frac{2N-1}{2N-5+d_1+2d_\perp-\delta'}.$$

In particular, if $N > 4-2d$, g_j belongs to some space $L^q(\mathbb{R}^N)$ with $1 \leq q \leq 2$. Therefore, its Fourier transform $\widehat{g_j}$ belongs to $L^{q'}(\mathbb{R}^N)$ (where $q' = \frac{q}{q-1}$). Since equation (95) then writes for almost every $\xi \in \mathbb{R}^N$,

$$i\partial_j^{\delta'}\Big(\frac{\prod_{l=1}^N \xi_l^{d_l}}{|\xi|^2+\xi_1^4}\Big) = \widehat{g_j}(\xi) + \sum_{|\alpha|\leq M} \lambda_\alpha i^{|\alpha|}\xi^\alpha,$$

the distribution g_j belongs to $L^{q'}(\mathbb{R}^N)$ only if all the real numbers λ_α vanish. Finally, if $N > 4-2d$, we deduce from equation (95) that the distribution f_j is equal to the function g_j. Thus, f_j belongs to $L^q(B(0,1))$ when condition (21) (if $\delta' = 0$) or condition (22) (if $\delta' = 1$) holds.

Now, assume that $N \leq 4-2d$. In this case, the coefficient δ' is necessarily equal to 0, and we can simplify our notations by denoting g, the distributions g_j above. Moreover, we will denote f', the tempered distribution whose Fourier transform is given by

$$\widehat{f'}(\xi) = \frac{\prod_{j=1}^N \xi_j^{d'_j}}{|\xi|^2+\xi_1^4}, \tag{96}$$

where $d'_j = d_j + 2\delta_{j,1}$ for every $j \in \{1,\dots,N\}$, and g', the restriction of f' to the set $\mathbb{R}^N \setminus \{0\}$. On one hand, since $d'_1 \geq 2$ and $N \leq 4-2d$, we compute that $d' \neq 0$ if $N = 2$, $d'_1 + 2d'_\perp \leq 4$ and $N > 4-2d'$ (with the usual notation $d' = \sum_{j=1}^N d'_j = d'_1 + d'_\perp$). Hence, by the argument just above, the distribution f' is equal to the distribution g'. On the other hand, by equation (96), the distribution f' is equal to the distribution $-\partial_1^2 f$. Likewise, by definition (93), g' is equal to the distribution $-\partial_1^2 g$. Thus, equation (95) yields

$$0 = f' - g' = \sum_{|\alpha|\leq M} \lambda_\alpha \partial_1^2 \partial^\alpha \delta_\alpha.$$

It follows that all the real numbers λ_α vanish. Therefore, the distribution f is equal to g: it belongs to $L^q(B(0,1))$ when condition (21) holds, which completes the analysis of the L^q-integrability of the distribution f near the origin and the proof of Proposition 2. □

Finally, Theorem 4 and Corollary 1 follow from Proposition 2.

Proof of Theorem 4. Theorem 4 is a direct consequence of equation (20) of Proposition 2 with $(d_1, d_\perp) = (1,0)$ for the kernel H_0, $(d_1, d_\perp) = (2,0)$ for the kernel K_0 and $(d_1, d_\perp) = (2+\delta_{k,1}, 1-\delta_{k,1})$ for the kernels K_k. □

Proof of Corollary 1. Likewise, Corollary 1 follows from conditions (21) and (22) of Proposition 2 with $(d_1, d_\perp) = (1,0)$ for the kernel H_0, $(d_1, d_\perp) = (2,0)$ for the kernel K_0 and $(d_1, d_\perp) = (2+\delta_{k,1}, 1-\delta_{k,1})$ for the kernels K_k. □

1.4 Pointwise limit at infinity of the kernel K_0.

This section is devoted to the proofs of Theorems 5 and 6. Since the proof of Theorem 5 involves formula (41) of Theorem 6, we will first show Theorem 6. However, their proofs follow from the same argument: the use of explicit integral expressions. Indeed, we will apply Proposition 5 to the kernel K_0 to compute formula (38), while we will apply Lemma 2 to the composed Riesz kernels $R_{k,l}$ to get formula (35). Then, we will invoke the dominated convergence theorem to compute the limits at infinity of such expressions, and to prove that they are equal. Finally, we will obtain an explicit expression of this limit by formulae (42) and (43).

Proof of Theorem 6. As mentioned above, our argument relies on expression (35). Indeed, for any $1 \leq k, l \leq N$, the composed Riesz kernel $R_{k,l}$ satisfies all the assumptions of Lemma 2. By formula (40), its Fourier transform is in $C^\infty(\mathbb{R}^N \setminus \{0\})$ and verifies

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, |\widehat{R_{k,l}}(\xi)| \leq 1.$$

Therefore, assumption (i) holds with $r = s = 0$. Moreover, by formula (40) once more, we compute for every $j \in \{1, \dots, N\}$ and $n \in \mathbb{N}$,

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, |\partial_j^n \widehat{R_{k,l}}(\xi)| \leq \frac{A}{|\xi|^n}.$$

Hence, assumption (ii) and (iii) hold with $p = N$ and $m = N + 1$. Moreover, the partial derivative $\partial_j^n \widehat{R_{k,l}}$ belongs to $L^q(B(0,1)^c)$ for any $q > q_{N+1-n} = \frac{N}{n}$. In particular, we notice that

$$\forall n \in \{1, \dots, N-1\}, q_n = \frac{N}{N+1-n} < \frac{N}{N-n}.$$

Thus, the kernel $R_{k,l}$ also verifies assumption (iv) and we can write by Lemma 2 for every positive real number λ and every $x \in \Omega_j = \{x \in \mathbb{R}^N, x_j \neq 0\}$,

$$\begin{aligned} x_j^N R_{k,l}(x) = & \frac{i^N}{(2\pi)^N} \Bigg(\frac{i}{x_j} \int_{B(0,\lambda)^c} \partial_j^{N+1} \widehat{R_{k,l}}(\xi) e^{ix.\xi} d\xi + \frac{i}{\lambda x_j} \int_{S(0,\lambda)} \xi_j \partial_j^N \widehat{R_{k,l}}(\xi) e^{ix.\xi} d\xi \\ & + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{N-1} \widehat{R_{k,l}}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{R_{k,l}}(\xi) (e^{ix.\xi} - 1) d\xi \Bigg). \end{aligned} \tag{97}$$

On the other hand, by formulae (42) and (43), the restriction of $R_{k,l}$ to the set $\mathbb{R}^N \setminus \{0\}$ is a smooth function which writes

$$\forall x \in \mathbb{R}^N \setminus \{0\}, R_{k,l}(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\delta_{k,l}|x|^2 - N x_k x_l}{|x|^{N+2}}.$$

Thus, equation (97) becomes for every $x \in \Omega_j$,

$$\begin{aligned} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\delta_{k,l}|x|^2 - N x_k x_l}{|x|^{N+2}} = & \frac{i^N}{(2\pi x_j)^N} \Bigg(\frac{i}{x_j} \int_{B(0,\lambda)^c} \partial_j^{N+1} \widehat{R_{k,l}}(\xi) e^{ix.\xi} d\xi + \frac{i}{\lambda x_j} \int_{S(0,\lambda)} \xi_j e^{ix.\xi} \\ & \partial_j^N \widehat{R_{k,l}}(\xi) d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{N-1} \widehat{R_{k,l}}(\xi) d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{R_{k,l}}(\xi) \\ & (e^{ix.\xi} - 1) d\xi \Bigg). \end{aligned}$$

In particular, if we consider $y \in \mathbb{R}^N$, $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_j \neq 0$, and some positive real number R such that $R|\sigma_j| > 2|y_j|$, we compute for $x = R\sigma - y$ and $\lambda = \frac{1}{R}$ after the change of variables $u = R\xi$,

$$\begin{aligned}&\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\frac{\delta_{k,l}|\sigma-\frac{y}{R}|^2 - N(\sigma_k - \frac{y_k}{R})(\sigma_l - \frac{y_l}{R})}{|\sigma - \frac{y}{R}|^{N+2}}\\ =&\frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N}\Bigg(\frac{i}{\sigma_j - \frac{y_j}{R}}\int_{B(0,1)^c} R^{-N-1}\partial_j^{N+1}\widehat{R_{k,l}}\Big(\frac{u}{R}\Big)e^{i(\sigma-\frac{y}{R}).u}du + \frac{i}{\sigma_j - \frac{y_j}{R}}\\ &\int_{\mathbb{S}^{N-1}} u_j R^{-N}\partial_j^N\widehat{R_{k,l}}\Big(\frac{u}{R}\Big)e^{i(\sigma-\frac{y}{R}).u}du + \int_{\mathbb{S}^{N-1}} u_j R^{1-N}\partial_j^{N-1}\widehat{R_{k,l}}\Big(\frac{u}{R}\Big)du\\ &+\int_{B(0,1)} R^{-N}\partial_j^N\widehat{R_{k,l}}\Big(\frac{u}{R}\Big)(e^{i(\sigma-\frac{y}{R}).u}-1)du\Bigg).\end{aligned} \tag{98}$$

In order to get formula (41), it now remains to compute the limit when R tends to $+\infty$ of all the terms of equation (98). Here, our argument relies on the homogeneity of the Fourier transform of the distribution $R_{k,l}$. Indeed, by formula (40), the Fourier transform $\widehat{R_{k,l}}$ is a homogeneous rational fraction of degree 0. Therefore, its partial derivative of order $k \in \mathbb{N}$, $\partial_j^k\widehat{R_{k,l}}$ is a homogeneous rational fraction of degree $-k$. Thus, equation (98) becomes

$$\begin{aligned}&\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\frac{\delta_{k,l}|\sigma-\frac{y}{R}|^2 - N(\sigma_k - \frac{y_k}{R})(\sigma_l - \frac{y_l}{R})}{|\sigma - \frac{y}{R}|^{N+2}}\\ =&\frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N}\Bigg(\frac{i}{\sigma_j - \frac{y_j}{R}}\Bigg(\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i(\sigma-\frac{y}{R}).u}du + \int_{\mathbb{S}^{N-1}} u_j\partial_j^N\widehat{R_{k,l}}(u)\\ &e^{i(\sigma-\frac{y}{R}).u}du\Bigg) + \int_{\mathbb{S}^{N-1}} u_j\partial_j^{N-1}\widehat{R_{k,l}}(u)du + \int_{B(0,1)}\partial_j^N\widehat{R_{k,l}}(u)(e^{i(\sigma-\frac{y}{R}).u}-1)du\Bigg).\end{aligned} \tag{99}$$

Now, we invoke the dominated convergence theorem to compute the limit of the right member of equation (99). Indeed, by homogeneity of the partial derivatives of the Fourier transform $\widehat{R_{k,l}}$, we compute for the first term of the second member

$$\forall u \in B(0,1)^c, |\partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i(\sigma-\frac{y}{R}).u}| \leq \frac{A}{|u|^{N+1}}.$$

Likewise, the second term of the second member satisfies

$$\forall u \in \mathbb{S}^{N-1}, |u_j\partial_j^N\widehat{R_{k,l}}(u)e^{i(\sigma-\frac{y}{R}).u}| \leq A,$$

while the fourth term verifies

$$\forall u \in B(0,1), |\partial_j^N\widehat{R_{k,l}}(u)(e^{i(\sigma-\frac{y}{R}).u}-1)| \leq \frac{A}{|u|^{N-1}}\Big|\sigma - \frac{y}{R}\Big| \leq \frac{A}{|u|^{N-1}},$$

provided that $R \geq 2|y|$. Therefore, by taking the limit $R \to +\infty$ in equation (99), the dominated convergence theorem yields

$$\begin{aligned}\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{k,l} - N\sigma_k\sigma_l) = \frac{i^N}{(2\pi\sigma_j)^N}&\Bigg(\frac{i}{\sigma_j}\Bigg(\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i\sigma.u}du + \int_{\mathbb{S}^{N-1}}\partial_j^N\widehat{R_{k,l}}(u)e^{i\sigma.u}\\ &u_jdu\Bigg) + \int_{\mathbb{S}^{N-1}} u_j\partial_j^{N-1}\widehat{R_{k,l}}(u)du + \int_{B(0,1)}\partial_j^N\widehat{R_{k,l}}(u)(e^{i\sigma.u}-1)du\Bigg),\end{aligned}$$

which is exactly formula (41). □

We then deduce the pointwise limit at infinity of the kernel K_0 given by Theorem 5.

Proof of Theorem 5. Let $\sigma \in \mathbb{S}^{N-1}$ and $y \in \mathbb{R}^N$, and consider some integer $j \in \{1, \dots, N\}$ such that $\sigma_j \neq 0$ and some positive real number $R > \max\{2|y|, \frac{2|y_j|}{|\sigma_j|}\}$. The kernel K_0 fulfils all the assumptions of Proposition 5 with $d_1 = 2$ and $d_\perp = 0$. Therefore, by equation (36), formula (38) holds with $m_1 = 2N-2$ and $m_j = N$ if $j \geq 2$. In particular, after the change of variables $u = R\xi$, this formula becomes for $x = R\sigma - y$ and $\lambda = \frac{1}{R}$,

$$\begin{aligned} R^N K_0(R\sigma - y) =& \frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N} \Bigg(\Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-m_j} \int_{B(0,1)^c} R^{-m_j} \partial_j^{m_j} \widehat{K_0}\Big(\frac{u}{R}\Big) \\ & e^{i(\sigma - \frac{y}{R}).u} du + \sum_{k=N}^{m_j-1} \Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j R^{-k} \partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) \\ & e^{i(\sigma - \frac{y}{R}).u} du + \int_{\mathbb{S}^{N-1}} u_j R^{1-N} \partial_j^{N-1} \widehat{K_0}\Big(\frac{u}{R}\Big) du + \int_{B(0,1)} R^{-N} \partial_j^N \widehat{K_0}\Big(\frac{u}{R}\Big) \\ & (e^{i(\sigma - \frac{y}{R}).u} - 1) du \Bigg). \end{aligned} \tag{100}$$

Then, denoting

$$N_j = \max\{N+1, m_j\}, \tag{101}$$

we claim that

$$\begin{aligned} R^N K_0(R\sigma - y) =& \frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N} \Bigg(\Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-N_j} \int_{B(0,1)^c} R^{-N_j} \partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) \\ & e^{i(\sigma - \frac{y}{R}).u} du + \sum_{k=N}^{N_j-1} \Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j R^{-k} \partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) \\ & e^{i(\sigma - \frac{y}{R}).u} du + \int_{\mathbb{S}^{N-1}} u_j R^{1-N} \partial_j^{N-1} \widehat{K_0}\Big(\frac{u}{R}\Big) du + \int_{B(0,1)} R^{-N} \partial_j^N \widehat{K_0}\Big(\frac{u}{R}\Big) \\ & (e^{i(\sigma - \frac{y}{R}).u} - 1) du \Bigg). \end{aligned} \tag{102}$$

Indeed, by Proposition 4 (with $d_1 = 2$ and $d_\perp = 0$), the function $\partial_j^k \widehat{K_0}$ belongs to $L^1(B(0,1)^c)$ for every $k \geq m_j$. In particular, we compute by integrating by parts,

$$\begin{aligned} & \Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-m_j} \int_{B(0,1)^c} R^{-m_j} \partial_j^{m_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du \\ =& \Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-m_j-1} \int_{B(0,1)^c} R^{-m_j-1} \partial_j^{m_j+1} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du \\ &+ \Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-m_j-1} \int_{\mathbb{S}^{N-1}} u_j R^{-m_j} \partial_j^{m_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du. \end{aligned}$$

Therefore, after several integrations by parts, we obtain

$$\begin{aligned}
&\Big(- i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-m_j} \int_{B(0,1)^c} R^{-m_j} \partial_j^{m_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du \\
=&\Big(- i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-N_j} \int_{B(0,1)^c} R^{-N_j} \partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du \\
+& \sum_{k=m_j}^{N_j-1} \Big(- i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j R^{-k} \partial_j^{k} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du,
\end{aligned}$$

which yields equation (102) by equation (100).

In order to compute formula (37), it now remains to compute the limit when R tends to $+\infty$ of equation (102). In particular, we must compute the limit when $R \to +\infty$ of the functions

$$u \mapsto R^{-k} \partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big)$$

for any $k \in \{N-1, \ldots, N_j\}$. In order to do so, we must describe a little more precisely the partial derivative $\partial_j^k \widehat{K_0}$. Indeed, the Fourier transform $\widehat{K_0}$ satisfies all the assumptions of Proposition 3 with $d_1 = 2$ and $d_\perp = 0$. By equations (23), (25) and (27), the partial derivative $\partial_j^k \widehat{K_0}$ then writes for every $k \in \mathbb{N}$,

$$\forall \xi \in \mathbb{R}^N, \partial_j^k \widehat{K_0}(\xi) = \frac{P_{j,k}(\xi)}{(|\xi|^2 + \xi_1^4)^{k+1}}, \tag{103}$$

where $P_{j,k}$ is a polynomial function on $\mathbb{R}^N$ which satisfies,

$$\forall \xi \in B(0,1), |P_{j,k}(\xi)| \leq A_k |\xi|^{k+2}. \tag{104}$$

Moreover, denoting

$$\forall \xi \in \mathbb{R}^N, P_{j,k}(\xi) = \sum_{i=0}^{d_k} Q_i(\xi), \tag{105}$$

where Q_i are homogeneous polynomial functions either equal to 0 or of degree i, we claim that the non-vanishing function Q_{i_0} of lowest degree is equal to

$$\forall \xi \in \mathbb{R}^N, Q_{i_0}(\xi) = |\xi|^{2k+2} \partial_j^k \widehat{R_{1,1}}(\xi). \tag{106}$$

Indeed, by a straightforward inductive argument, the partial derivative $\partial_j^k \widehat{R_{1,1}}(\xi)$ writes

$$\forall \xi \in \mathbb{R}^N, \partial_j^k \widehat{R_{1,1}}(\xi) = \frac{S_k(\xi)}{|\xi|^{2k+2}},$$

where the functions S_k are polynomial functions given by

$$\begin{cases} S_0(\xi) = \xi_1^2, \\ S_{k+1}(\xi) = |\xi|^2 \partial_j S_k(\xi) - 2(k+1) \xi_j S_k(\xi). \end{cases}$$

In particular, the polynomial function S_k is of degree $k+2$. Likewise, by equations (54), the functions $P_{j,k}$ are given by

$$\begin{cases} P_{j,0}(\xi) = \xi_1^2, \\ P_{j,k+1}(\xi) = (|\xi|^2 + \xi_1^4) \partial_j P_{j,k}(\xi) - 2(k+1)(\xi_j + 2\delta_{j,1}\xi_1^3) P_{j,k}(\xi). \end{cases}$$

Therefore, the homogeneous term of lowest degree Q_{i_0} of $P_{j,k}$ satisfies exactly the same equation as the function S_k: the function Q_{i_0} is equal to S_k which leads to formula (106). In particular, we deduce that the degree i_0 of Q_{i_0} is equal to $k+2$.

Now, we compute the limit when R tends to $+\infty$ of the function

$$u \mapsto R^{-k}\partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big).$$

More precisely, we claim that for every $k \in \mathbb{N}$,

$$\forall u \in \mathbb{R}^N \setminus \{0\}, R^{-k}\partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) \underset{R\to+\infty}{\to} \partial_j^k \widehat{R_{1,1}}(u). \tag{107}$$

Indeed, by equations (103) and (105), we compute for every $u \in \mathbb{R}^N \setminus \{0\}$,

$$R^{-k}\partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) = \frac{\sum\limits_{i=k+2}^{d_k} R^{k+2-i} Q_i(u)}{(|u|^2 + R^{-2}u_1^4)^{k+1}}.$$

This gives

$$R^{-k}\partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) \underset{R\to+\infty}{\to} \frac{Q_{k+2}(u)}{|u|^{2k+2}},$$

and yields assertion (107) by equation (106).

Finally, we invoke the dominated convergence theorem to compute the limit of the right member of equation (102). Indeed, the first term of the second member of equation (102) writes by the change of variables $u = R\xi$,

$$\int_{B(0,1)^c} R^{-N_j}\partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du = \int_{1<|u|<R} R^{-N_j}\partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du$$
$$+ \int_{B(0,1)^c} R^{N-N_j}\partial_j^{N_j} \widehat{K_0}(\xi) e^{i(R\sigma - y).\xi} d\xi.$$

On one hand, by Proposition 5, the function $\partial_j^{N_j}\widehat{K_0}$ belongs to $L^1(B(0,1)^c)$, so,

$$\int_{B(0,1)^c} R^{N-N_j}\partial_j^{N_j} \widehat{K_0}(\xi) e^{i(R\sigma - \xi).u} du \leq R^{N-N_j} \int_{B(0,1)^c} |\partial_j^{N_j} \widehat{K_0}(\xi)| d\xi \underset{R\to+\infty}{\to} 0.$$

On the other hand, by equations (103) and (104),

$$\forall 1 < |u| < R, |R^{-N_j}\partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u}| \leq \frac{A}{|u|^{N_j}},$$

so, since $N_j \geq N+1$ by definition (101), by the dominated convergence theorem and assertion (107),

$$\int_{1<|u|<R} R^{-N_j}\partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du \underset{R\to+\infty}{\to} \int_{B(0,1)^c} \partial_j^{N_j} \widehat{R_{1,1}}(u) e^{i\sigma.u} du.$$

Thus, we deduce

$$\int_{B(0,1)^c} R^{-N_j}\partial_j^{N_j} \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du \underset{R\to+\infty}{\to} \int_{B(0,1)^c} \partial_j^{N_j} \widehat{R_{1,1}}(u) e^{i\sigma.u} du. \tag{108}$$

Equations (103) and (104) then yield for every $k \in \{N, \ldots, N_j - 1\}$,

$$\forall u \in \mathbb{S}^{N-1}, |u_j R^{-k} \partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u}| \leq A,$$

and

$$\forall u \in \mathbb{S}^{N-1}, \Big|u_j R^{1-N} \partial_j^{N-1} \widehat{K_0}\Big(\frac{u}{R}\Big)\Big| \leq A,$$

while the last term of the right member of equation (102) verifies likewise

$$\forall u \in B(0,1), |R^{-N} \partial_j^N \widehat{K_0}\Big(\frac{u}{R}\Big)(e^{i(\sigma - \frac{y}{R}).u} - 1)| \leq \frac{A}{|u|^{N-1}}\Big|\sigma - \frac{y}{R}\Big| \leq \frac{A}{|u|^{N-1}},$$

provided that $R \geq 2|y|$. Therefore, by assertion (107), the dominated convergence theorem yields

$$\begin{aligned}
&\sum_{k=N}^{N_j-1} \Big(-i\Big(\sigma_j - \frac{y_j}{R}\Big)\Big)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j R^{-k} \partial_j^k \widehat{K_0}\Big(\frac{u}{R}\Big) e^{i(\sigma - \frac{y}{R}).u} du + \int_{\mathbb{S}^{N-1}} u_j R^{1-N} \\
&\partial_j^{N-1} \widehat{K_0}\Big(\frac{u}{R}\Big) du + \int_{B(0,1)} R^{-N} \partial_j^N \widehat{K_0}\Big(\frac{u}{R}\Big)(e^{i(\sigma - \frac{y}{R}).u} - 1) du \\
\underset{R \to +\infty}{\to} &\sum_{k=N}^{N_j-1} (-i\sigma_j)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j \partial_j^k \widehat{R_{1,1}}(u) e^{i\sigma.u} du + \int_{\mathbb{S}^{N-1}} u_j \partial_j^{N-1} \widehat{R_{1,1}}(u) du \\
&+ \int_{B(0,1)} \partial_j^N \widehat{R_{1,1}}(u)(e^{i\sigma.u} - 1) du.
\end{aligned}$$

Thus, it follows from equations (102) and (108) that

$$\begin{aligned}
R^N K_0(R\sigma - y) \underset{R \to +\infty}{\to} \frac{i^N}{(2\pi\sigma_j)^N} \Bigg(&(-i\sigma_j)^{N-N_j} \int_{B(0,1)^c} \partial_j^{N_j} \widehat{R_{1,1}}(u) e^{i\sigma.u} du \\
&+ \sum_{k=N}^{N_j-1} (-i\sigma_j)^{N-k-1} \int_{\mathbb{S}^{N-1}} u_j \partial_j^k \widehat{R_{1,1}}(u) e^{i\sigma.u} du + \int_{\mathbb{S}^{N-1}} \partial_j^{N-1} \widehat{R_{1,1}}(u) \\
&u_j du + \int_{B(0,1)} \partial_j^N \widehat{R_{1,1}}(u)(e^{i\sigma.u} - 1) du.
\end{aligned} \tag{109}$$

In order to obtain assertion (37), we then integrate by parts the first term of the right member of equation (109). Indeed, we previously stated that any partial derivative $\partial_j^k \widehat{R_{1,1}}$ is a homogeneous rational fraction which satisfies

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, |\partial_j^k \widehat{R_{1,1}}(\xi)| \leq \frac{A}{|\xi|^k}.$$

Hence, it belongs to $L^1(B(0,1)^c)$ for every $k \geq N+1$. In particular, since $N_j \geq N+1$ by definition (101), it yields by integrating by parts,

$$\begin{aligned}
(-i\sigma_j)^{N-N_j} \int_{B(0,1)^c} \partial_j^{N_j} \widehat{R_{1,1}}(u) e^{i\sigma.u} du = &(-i\sigma_j)^{N-N_j+1} \int_{B(0,1)^c} \partial_j^{N_j-1} \widehat{R_{1,1}}(u) e^{i\sigma.u} du \\
&-(-i\sigma_j)^{N-N_j} \int_{\mathbb{S}^{N-1}} u_j \partial_j^{N_j-1} \widehat{R_{1,1}}(u) e^{i\sigma.u} du.
\end{aligned}$$

Thus, we deduce from several integrations by parts,

$$\begin{aligned}(-i\sigma_j)^{N-N_j}\int_{B(0,1)^c}\partial_j^{N_j}\widehat{R_{1,1}}(u)e^{i\sigma.u}du =&\frac{i}{\sigma_j}\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R_{1,1}}(u)e^{i\sigma.u}du\\ &-\sum_{k=N+1}^{N_j-1}(-i\sigma_j)^{N-k-1}\int_{\mathbb{S}^{N-1}}u_j\partial_j^k\widehat{R_{1,1}}(u)e^{i\sigma.u}du,\end{aligned}$$

so, by equation (109),

$$\begin{aligned}R^N K_0(R\sigma-y)\underset{R\to+\infty}{\to}\frac{i^N}{(2\pi\sigma_j)^N}\Bigg(\frac{i}{\sigma_j}\bigg(\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R_{1,1}}(u)e^{i\sigma.u}du+\int_{\mathbb{S}^{N-1}}u_j\partial_j^N\widehat{R_{1,1}}(u)\\ e^{i\sigma.u}du\bigg)+\int_{\mathbb{S}^{N-1}}u_j\partial_j^{N-1}\widehat{R_{1,1}}(u)du+\int_{B(0,1)}\partial_j^N\widehat{R_{1,1}}(u)(e^{i\sigma.u}-1)du\Bigg).\end{aligned}$$

Finally, assertion (37) holds by formula (41) in the case $k=l=1$. This completes the proof of Theorem 5. □

1.5 Rigorous derivation of the convolution equations.

The aim of this last section is to give a rigorous sense to convolution equations (13), (14) and (44). Indeed, as mentioned in the introduction, our analysis of the asymptotic behaviour of the generalised Kadomtsev-Petviashvili solitary waves relies on the use of those equations.

In the introduction, we already proved that equations (13) and (14) hold almost everywhere. Indeed, by Theorem 3 and Corollary 1, the kernels H_0 and K_0 belong to all the spaces $L^q(\mathbb{R}^N)$ for every $\frac{N}{N-1}<q<\frac{2N-1}{2N-4}$, respectively $1<q<\frac{2N-1}{2N-3}$. Moreover, by Theorem 7, the functions v and ∇v belong to all the spaces $L^q(\mathbb{R}^N)$ for every $1<q<+\infty$. Therefore, by Young's inequalities, equation (13) makes sense in all the spaces $L^q(\mathbb{R}^N)$ for every $\frac{N}{N-1}<q<\frac{2N-1}{2N-4}$, while equation (14) makes sense in all the spaces $L^q(\mathbb{R}^N)$ for every $1<q<\frac{2N-1}{2N-3}$. In particular, they both make sense almost everywhere.

However, as mentioned in the introduction, we will also study the gradient of equation (14), whose derivation is rather more difficult. In order to give it a rigorous sense, we establish equation (44) of Lemma 3, which holds for smooth functions f with sufficient decay at infinity. In particular, the function v^{p+1} which appears in equation (14) will satisfy such assumptions by Theorems 7 and 8. Therefore, we will be able to derive rigorously the gradient of equation (14).

Now, let us establish Lemma 3.

Proof of Lemma 3. Let $k\in\{1,\ldots,N\}$ and consider the function h_k given by

$$\begin{aligned}\forall x\in\mathbb{R}^N, h_k(x)=&i\int_{B(0,1)^c}K_k(y)f(x-y)dy+i\int_{B(0,1)}K_k(y)(f(x-y)-f(x))dy\\ &+\bigg(\int_{\mathbb{S}^{N-1}}K_0(y)y_kdy\bigg)f(x).\end{aligned}\tag{110}$$

Our proof splits in two steps. In the first one, we will state the continuity of the functions g, h_1, ..., h_N on $\mathbb{R}^N$. In the second one, we will establish that the partial derivative $\partial_k g$ of g in the sense of distributions is equal to the function h_k. Then, we will conclude that the function g is of class C^1 on $\mathbb{R}^N$ and that its first order partial derivatives are given by formula (44).

Step 1. *Continuity of the functions g, h_1, ..., h_N.*

The function f is continuous on $\mathbb{R}^N$, so, by assumption (i), it belongs to all the spaces $L^q(\mathbb{R}^N)$ for every $q \geq 1$. It then follows from Young's inequalities that the function g is well-defined in all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q < \frac{2N-1}{2N-3}$. In particular, it is given for almost every $x \in \mathbb{R}^N$, by

$$g(x) = \int_{\mathbb{R}^N} K_0(y) f(x-y) dy. \tag{111}$$

Moreover, by Corollary 1, the kernel K_0 belongs to $L^1(B(0,1))$. Therefore, by continuity of f, assumption (i) and a standard application of the dominated convergence theorem, the function

$$g_1 : x \mapsto \int_{B(0,1)} K_0(y) f(x-y) dy$$

is continuous on $\mathbb{R}^N$. On the other hand, the function

$$g_2 : x \mapsto \int_{B(0,1)^c} K_0(y) f(x-y) dy$$

is also continuous on $\mathbb{R}^N$. Indeed, consider $x_0 \in \mathbb{R}^N$ and compute by assumption (i) and Theorem 3,

$$\forall x \in B(x_0,1), \forall y \in B(0,1)^c, |K_0(y) f(x-y)| \leq \frac{A}{|y|^N (1 + |x-y|^{N(p+1)})}.$$

In particular, it yields for every $x \in B(x_0,1)$,

$$\forall 1 < |y| < |x_0| + 1, |K_0(y) f(x-y)| \leq A_{x_0},$$

and

$$\forall y \in B(0, |x_0|+1)^c, |K_0(y) f(x-y)| \leq \frac{A}{|y|^N (1 + (|y| - |x_0| - 1)^{N(p+1)})}.$$

Thus, by a standard application of the dominated convergence theorem, the function g_2 is continuous at the point x_0. Hence, it is continuous on $\mathbb{R}^N$. Finally, by equation (111), the function g is equal to $g_1 + g_2$, so, it is also continuous on $\mathbb{R}^N$.

On the other hand, by Theorem 3, the kernel K_k belongs to $L^1(B(0,1)^c)$, so, by continuity of f, assumption (i) and a standard application of the dominated convergence theorem, the function

$$h_k^1 : x \mapsto i \int_{B(0,1)^c} K_k(y) f(x-y) dy$$

is continuous on $\mathbb{R}^N$. Likewise, by Proposition 5, the kernel K_0 belongs to $C^0(\mathbb{R}^N \setminus \{0\})$, so, by continuity of f, the function

$$h_k^2 : x \mapsto \Big(\int_{\mathbb{S}^{N-1}} K_0(y) y_k dy \Big) f(x)$$

is continuous on $\mathbb{R}^N$. Finally, assumption (ii) yields

$$\forall y \in B(0,1), |K_k(y)(f(x-y) - f(x))| \leq \|\nabla f\|_{L^\infty(\mathbb{R}^N)} \sum_{j=1}^N |y_j| |K_k(y)|.$$

Since the functions $y \mapsto y_j K_k(y)$ belong to $L^1(B(0,1)$ by Corollary 1, it follows that the function

$$h_k^3 : x \mapsto i \int_{B(0,1)} K_k(y)(f(x-y) - f(x))dy$$

is also continuous on $\mathbb{R}^N$. However, by equation (110), the function h_k is equal to $h_k^1 + h_k^2 + h_k^3$, so, it is continuous on $\mathbb{R}^N$.

Step 2. *First order partial derivatives of g in the sense of distributions.*

Now, consider some test function $\phi \in C_c^\infty(\mathbb{R}^N)$. By definition (111), we compute

$$< \partial_k g, \phi >= - \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} f(y) K_0(x-y) dy \Big) \partial_k \phi(x) dx. \tag{112}$$

Let us then denote

$$\forall y \in \mathbb{R}^N, \Lambda_\phi(y) = \int_{\mathbb{R}^N} K_0(x-y) \partial_k \phi(x) dx, \tag{113}$$

and fix some point $y \in \mathbb{R}^N$. Since ϕ belongs to $C_c^\infty(\mathbb{R}^N)$, there is some positive real number $R > |y| + 2$ such that

$$\begin{aligned} \Lambda_\phi(y) &= \int_{B(0,R)} K_0(x-y) \partial_k \phi(x) dx \\ &= \int_{B(y,1)} K_0(x-y) \partial_k (\phi(x) - \phi(y)) dx + \int_{B(0,R) \setminus B(y,1)} K_0(x-y) \partial_k \phi(x) dx. \end{aligned}$$

However, by Proposition 5, the kernels K_0 and K_k belong to $C^0(\mathbb{R}^N \setminus \{0\})$. Therefore, since the kernel K_k is equal to $-i\partial_k K_0$, the kernel K_0 is of class C^1 on $\mathbb{R}^N \setminus \{0\}$ such that

$$\forall z \in \mathbb{R}^N \setminus \{0\}, \partial_k K_0(z) = i K_k(z).$$

Hence, by integrating by parts, we infer

$$\begin{aligned} \Lambda_\phi(y) = &\int_{B(y,1)} K_0(x-y) \partial_k (\phi(x) - \phi(y)) dx - i \int_{B(0,R) \setminus B(y,1)} K_k(x-y) \phi(x) dx \\ &- \int_{S(y,1)} K_0(x-y)(x_k - y_k) \phi(x) dx. \end{aligned} \tag{114}$$

Moreover, by Corollary 1, the kernel K_0 belongs to $L^1(B(0,1))$, so,

$$\int_{B(y,1)} K_0(x-y) \partial_k (\phi(x) - \phi(y)) dx = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x-y| < 1} K_0(x-y) \partial_k (\phi(x) - \phi(y)) dx. \tag{115}$$

Since the kernel K_0 is of class C^1 on $\mathbb{R}^N \setminus \{0\}$, it then follows that for every $\varepsilon \in]0,1[$,

$$\begin{aligned} \int_{\varepsilon < |x-y| < 1} K_0(x-y) \partial_k (\phi(x) - \phi(y)) dx = &- i \int_{\varepsilon < |x-y| < 1} K_k(x-y)(\phi(x) - \phi(y)) dx \\ &+ \int_{S(y,1)} K_0(x-y)(x_k - y_k)(\phi(x) - \phi(y)) dx \\ &- \frac{1}{\varepsilon} \int_{S(y,\varepsilon)} K_0(x-y)(x_k - y_k)(\phi(x) - \phi(y)) dx. \end{aligned} \tag{116}$$

On one hand, ϕ belongs to $C_c^\infty(\mathbb{R}^N)$, which gives

$$\forall x \in B(y,1), |K_k(x-y)(\phi(x)-\phi(y))| \leq A \sum_{l=1}^{N} |x_l - y_l||K_k(x-y)|,$$

so, since the functions $x \mapsto x_l K_k(x)$ belong to $L^1(B(0,1))$ by Corollary 1,

$$\int_{\varepsilon<|x-y|<1} K_k(x-y)(\phi(x)-\phi(y))dx \underset{\varepsilon\to 0}{\to} \int_{B(y,1)} K_k(x-y)(\phi(x)-\phi(y))dx. \tag{117}$$

On the other hand, the kernel K_0 belongs to $L^1(B(0,1))$. Therefore, there exists a sequence of positive real numbers δ_n which tends to 0 when $n \to +\infty$, and which satisfies

$$\forall n \in \mathbb{N}, \exists \varepsilon_n \in]0, \delta_n[, \delta_n \int_{S(0,\varepsilon_n)} |K_0(z)|dz \leq \int_{B(0,\delta_n)} |K_0(z)|dz.$$

Thus, since K_0 and ϕ belong respectively to $L^1(B(0,1))$ and $C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\forall n \in \mathbb{N}, & \left|\frac{1}{\varepsilon_n} \int_{S(y,\varepsilon_n)} K_0(x-y)(x_k-y_k)(\phi(x)-\phi(y))dx\right| \\ \leq & \frac{A}{\varepsilon_n} \int_{S(y,\varepsilon_n)} |K_0(x-y)||x-y|^2 dx \\ \leq & A\varepsilon_n \int_{S(0,\varepsilon_n)} |K_0(z)|dz \\ \leq & A \int_{B(0,\delta_n)} |K_0(z)|dz \underset{n\to+\infty}{\to} 0.\end{aligned}$$

Therefore, by equations (115), (116) and (117), we obtain

$$\begin{aligned}\int_{B(y,1)} K_0(x-y)\partial_k(\phi(x)-\phi(y))dx = & -i \int_{B(y,1)} K_k(x-y)(\phi(x)-\phi(y))dx \\ & + \int_{S(y,1)} K_0(x-y)(x_k-y_k)(\phi(x)-\phi(y))dx,\end{aligned}$$

which gives by equation (114),

$$\begin{aligned}\Lambda_\phi(y) = & -i \int_{B(y,1)} K_k(x-y)(\phi(x)-\phi(y))dx - \Big(\int_{S(y,1)} K_0(x-y)(x_k-y_k)dx\Big)\phi(y) \\ & -i \int_{B(0,R)\setminus B(y,1)} K_k(x-y)\phi(x)dx.\end{aligned} \tag{118}$$

On the other hand, by definition (113), the function Λ_ϕ belongs to all the spaces $L^q(\mathbb{R}^N)$ for $1 < q \leq +\infty$. Indeed, if the distance between y and the support of ϕ is more than 1, we compute by Theorem 3,

$$|\Lambda_\phi(y)| \leq A \int_{\mathrm{Supp}(\phi)} |x-y|^{-N}|\phi(x)|dx \leq \frac{A}{\mathrm{d}(y, \mathrm{supp}(\phi))^N} \|\phi\|_{L^1(\mathbb{R}^N)},$$

while if this distance is less than 1, by Corollary 1,

$$|\Lambda_\phi(y)| \leq A\|\phi\|_{L^\infty(\mathbb{R}^N)}\|K_0\|_{L^1(\mathrm{Supp}(\phi))}.$$

Thus, since the support of ϕ is compact, Λ_ϕ does belong to all the spaces $L^q(\mathbb{R}^N)$ for $1 < q \leq +\infty$. By assumption (i), the function f then belongs to all the spaces $L^q(\mathbb{R}^N)$ for $1 \leq q \leq +\infty$. Therefore, it follows from equations (112) and (118), and Fubini's theorem that

$$\begin{aligned}
< \partial_k g, \phi >= & - \int_{\mathbb{R}^N} f(y)\Lambda_\phi(y)dy \\
= & \int_{\mathbb{R}^N} f(y)\bigg(i\int_{B(y,1)} K_k(x-y)(\phi(x)-\phi(y))dx + \bigg(\int_{S(y,1)} K_0(x-y)(x_k-y_k) \\
& dx\bigg)\phi(y) + i\int_{B(y,1)^c} K_k(x-y)\phi(x)dx\bigg)dy.
\end{aligned}$$

It now remains to make the change of variables $z = x - y$ to get

$$\begin{aligned}
< \partial_k g, \phi >= & \int_{\mathbb{R}^N} f(y)\bigg(i\int_{B(0,1)} K_k(z)(\phi(z+y)-\phi(y))dz\bigg)dy + \int_{\mathbb{R}^N}\bigg(\int_{S(0,1)} K_0(z)z_k dz\bigg) \\
& f(y)\phi(y)dy + i\int_{\mathbb{R}^N} f(y)\bigg(\int_{B(0,1)^c} K_k(z)\phi(z+y)dz\bigg)dy \\
= & i\int_{B(0,1)} K_k(z)\bigg(\int_{\mathbb{R}^N} f(y)\phi(z+y)dy - \int_{\mathbb{R}^N} f(y)\phi(y)dy\bigg)dz + \int_{\mathbb{R}^N} f(y) \\
& \phi(y)\bigg(\int_{S(0,1)} K_0(z)z_k dz\bigg)dy + i\int_{B(0,1)^c} K_k(z)\bigg(\int_{\mathbb{R}^N} f(y)\phi(z+y)dy\bigg)dz,
\end{aligned}$$

so, by the change of variables $x = y + z$,

$$\begin{aligned}
< \partial_k g, \phi >= & i\int_{B(0,1)} K_k(z)\bigg(\int_{\mathbb{R}^N} f(x-z)\phi(x)dx - \int_{\mathbb{R}^N} f(x)\phi(x)dx\bigg)dz + \int_{\mathbb{R}^N} f(x) \\
& \phi(x)\bigg(\int_{S(0,1)} K_0(z)z_k dz\bigg)dx + i\int_{B(0,1)^c} K_k(z)\bigg(\int_{\mathbb{R}^N} f(x-z)\phi(x)dx\bigg)dz \\
= & \int_{\mathbb{R}^N} \phi(x)\bigg(i\int_{B(0,1)} K_k(z)(f(x-z)-f(x))dz + \bigg(\int_{S(0,1)} K_0(z)z_k dz\bigg)f(x) \\
& + i\int_{B(0,1)^c} K_k(z)f(x-z)dz\bigg)dx,
\end{aligned}$$

which gives by definition (110),

$$< \partial_k g, \phi >=< h_k, \phi > .$$

Finally, the partial derivative $\partial_k g$ of g in the sense of distributions is equal to h_k. Since g, h_1, ..., h_N are continuous functions by Step 1, we conclude that g is of class C^1 on $\mathbb{R}^N$ with partial derivatives given by formula (44). This completes the proof of Lemma 3. □

2 Asymptotic behaviour of the solitary waves.

The second part is mainly devoted to the proof of Theorem 1. We first establish some integral properties of the solitary waves in Theorem 7: they follow from an argument of A. de Bouard and J.C. Saut which relies on Lizorkin's theorem [35]. We then obtain the optimal algebraic decay of solitary waves stated in Theorem 8 by the standard argument mentioned in the introduction. In the third section, we complete the proof of Theorem 1 by inferring the asymptotic expansion of a solitary wave from Propositions 7 and 8. Finally, the last section is devoted to the proof of Theorem 2 which links the asymptotic expansion of a solitary wave to its energy and its action in the case of the standard Kadomtsev-Petviashvili equation. In particular, the proof of Theorem 2 relies on Lemma 4, which also yields the non-existence of non-trivial solitary waves when $p \geq \frac{4}{2N-3}$ (see Corollary 2 for more details).

2.1 Integrability of the solitary waves.

In this section, we derive the integrability properties of Theorem 7. As mentioned in the introduction, A. de Bouard and J.C. Saut [13, 14] already established them in dimensions $N = 2$ and $N = 3$. Moreover, their argument is still relevant in dimensions $N \geq 4$. Indeed, their proof follows from two ingredients which can be extended in every dimension $N \geq 2$. The first one is the embedding theorem for anisotropic Sobolev spaces, which states that the space Y embeds in $L^q(\mathbb{R}^N)$ for every $2 \leq q \leq \frac{4N+1}{2N-3}$ (see the book of O.V. Besov, V.P. Il'in and S.M. Nikolskii [3]). This ensures that any solitary wave v belongs to some space $L^q(\mathbb{R}^N)$, and enables to use some standard argument of multipliers theory in $L^q(\mathbb{R}^N)$. Indeed, the second ingredient is to use the fact that the kernels K_0, K_k and L_0, given by formulae (16), (17) and

$$L_0(\xi) = \frac{\xi_1^4}{|\xi|^2 + \xi_1^4}, \tag{119}$$

are L^q-multipliers for every $1 < q < +\infty$. Indeed, they satisfy the assumptions of Lizorkin's theorem [35] mentioned in the introduction. Theorem 7 then follows from a standard bootstrap argument which relies on the superlinearity of the non-linearity v^{p+1}: if a solitary wave v belongs to some space $L^q(\mathbb{R}^N)$, then, by equation (14) and since the kernel K_0 is a L^q-multiplier, it belongs to $L^{\frac{q}{p+1}}(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ with $\frac{1}{r} = \frac{p+1}{q} - \frac{2}{2N+1}$. Hence, by induction, it belongs to all the spaces $L^q(\mathbb{R}^N)$ for every $1 < q < +\infty$, which is the desired result.

For sake of completeness, we now extend the argument of A. de Bouard and J.C. Saut [13, 14] to every dimension $N \geq 2$.

Proof of Theorem 7. We split the proof in two steps: we first initialise our inductive argument by showing that the function v belongs to some space $L^q(\mathbb{R}^N)$.

Step 1. *The function v belongs to the space $L^q(\mathbb{R}^N)$ provided that*

$$2 \leq q \leq \frac{4N-2}{2N-3}.$$

Indeed, the function v belongs to the space Y. However, by standard embedding theorem for anisotropic Sobolev spaces [3], the space Y embeds into all the spaces $L^q(\mathbb{R}^N)$ for every

$$2 \leq q \leq \frac{4N-2}{2N-3}.$$

Therefore, the function v belongs to all those spaces.

We then settle the inductive argument mentioned above.

Step 2. *Consider some real number* $q_0 \in]p+1, +\infty[$*, and assume that the function* v *belongs to* $L^{q_0}(\mathbb{R}^N)$*. Then, if* $q_0 \neq q_{\rm crit} = \frac{(2N-1)(p+1)}{2}$*, the function* v *is in* $L^q(\mathbb{R}^N)$ *for*

$$\frac{q_0}{p+1} \leq q \leq r_0,$$

where

$$r_0 = \frac{(2N-1)q_0}{(2N-1)(p+1) - 2q_0},$$

if $q_0 < q_{\rm crit}$*, and* $r_0 = +\infty$ *otherwise. Likewise, if* $q_0 = q_{\rm crit}$*, the function* v *is in* $L^q(\mathbb{R}^N)$ *for* $\frac{q_0}{p+1} \leq q < +\infty$*. Moreover, the functions* ∇v *and* $\partial_1^2 v$ *belong to* $L^{\frac{q_0}{p+1}}(\mathbb{R}^N)$*.*

Indeed, by a straightforward inductive argument, the Fourier transforms $\widehat{K_0}$, $\widehat{K_k}$ and $\widehat{L_0}$, given by formulae (16), (17) and (119), satisfy all the assumptions of Lizorkin's theorem [35]. Therefore, they are multipliers on $L^q(\mathbb{R}^N)$ for every $q \in]1, +\infty[$. On the other hand, the function v is solution of convolution equation (14), which writes

$$v = \frac{1}{p+1} K_0 * v^{p+1}.$$

In particular, it yields for every $k \in \{1, \ldots, N\}$,

$$\begin{cases} \partial_k v = \frac{i}{p+1} K_k * v^{p+1}, \\ \partial_1^2 v = -\frac{1}{p+1} L_0 * v^{p+1}. \end{cases}$$

However, since v belongs to $L^{q_0}(\mathbb{R}^N)$, the function v^{p+1} belongs to $L^{\frac{q_0}{p+1}}(\mathbb{R}^N)$. Thus, since $\frac{q_0}{p+1} > 1$ and since the kernels K_0, K_k and L_0 are L^q-multipliers for $1 < q < +\infty$, the functions v, ∇v and $\partial_1^2 v$ belong to $L^{\frac{q_0}{p+1}}(\mathbb{R}^N)$. We then invoke the embedding theorem for anisotropic Sobolev spaces once more [3] to conclude that the function v also belongs to $L^{r_0}(\mathbb{R}^N)$ with

$$r_0 = \frac{(2N-1)q_0}{(2N-1)(p+1) - 2q_0},$$

if $q_0 < q_{\rm crit}$, to $L^\infty(\mathbb{R}^N)$ if $q_0 > q_{\rm crit}$, and to all the spaces $L^q(\mathbb{R}^N)$ for $q_0 \leq q < +\infty$, if $q_0 = q_{\rm crit}$. Then, Step 2 follows from standard interpolation theory between L^p-spaces.

Finally, Steps 1 and 2 yield the desired conclusion.

Step 3. *The functions* v*,* ∇v *and* $\partial_1^2 v$ *belong to all the spaces* $L^q(\mathbb{R}^N)$ *for every* $q \in]1, +\infty[$*. Moreover, the function* v *is continuous and bounded on* $\mathbb{R}^N$*.*

On one hand, by Step 1, the function v belongs to the space $L^{q_0}(\mathbb{R}^N)$ for $q_0 = \frac{4N-2}{2N-3}$. However, since

$$p < \frac{4}{2N-3},$$

we compute

$$p + 1 < \frac{2N+1}{2N-3} < \frac{4N-2}{2N-3}.$$

Therefore, by applying inductively Step 2, v belongs to all the spaces $L^q(\mathbb{R}^N)$ for $\frac{q_0}{(p+1)^{n+1}} \leq q \leq q_0$, as long as the integer n satisfies the condition $\frac{q_0}{(p+1)^n} > p+1$. However, since $p+1 > 1$, the geometric sequence $(\frac{q_0}{(p+1)^{n+1}})_{n \in \mathbb{N}}$ converges to 0 at infinity, so, the function v belongs to all the spaces $L^q(\mathbb{R}^N)$ for $p+1 < q \leq q_0$. Another application of Step 2 then yields that the function v belongs to all the spaces $L^q(\mathbb{R}^N)$ for $1 < q \leq q_0$.

On the other hand, consider the function f defined by

$$\forall r \in I = \left]\frac{(2N-1)p}{2}, q_{\text{crit}}\right[, f(r) = \frac{(2N-1)r}{(2N-1)(p+1)-2r}.$$

We compute

$$\forall r \in I, f(r) - r = \frac{r(2r-(2N-1)p)}{(2N-1)(p+1)-2r} > 0,$$

so, the function f is increasing on interval I. In particular, if we consider a sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_0 \in I$ and

$$\forall n \in \mathbb{N}, r_{n+1} = \frac{(2N-1)r_n}{(2N-1)(p+1)-2r_n},$$

this sequence is increasing. Moreover, since the function f has no fixed point in $I \cup \{q_{\text{crit}}\}$, there is some integer n_0 such that

$$r_{n_0} \geq q_{\text{crit}}.$$

However, we assumed that

$$p < \frac{4}{2N-3},$$

so,

$$\frac{(2N-1)p}{2} < \frac{4N-2}{2N-3} = q_0.$$

Therefore, either q_0 belongs to the interval I, either $q_0 \geq q_{\text{crit}}$. In the second case, by Step 2, the function v belongs to all the spaces $L^q(\mathbb{R}^N)$ for $q_0 \leq q < +\infty$. However, in the first case, since $q_0 \in I$, we can consider the sequence $(q_n)_{n\in\mathbb{N}}$ given inductively by

$$\forall n \in \mathbb{N}, q_{n+1} = \frac{(2N-1)q_n}{(2N-1)(p+1)-2q_n}.$$

By the argument above, this sequence is well-defined till some index $n_0 \in \mathbb{N}$. Moreover, all the real numbers q_n belong to I when $n < n_0$, and q_{n_0} belongs to the interval $[q_{\text{crit}}, +\infty[$. On the other hand, by Step 2, the function v belongs to $L^q(\mathbb{R}^N)$ for $q_0 \leq q \leq q_{n_0}$, and since $q_{n_0} \geq q_{\text{crit}}$, it also belongs to $L^q(\mathbb{R}^N)$ for $q_{n_0} \leq q < +\infty$. Thus, in all cases, the function v belongs to all the spaces $L^q(\mathbb{R}^N)$ for $q_0 \leq q < +\infty$.

In conclusion, v belongs to all the spaces $L^q(\mathbb{R}^N)$ for $1 < q < +\infty$. In particular, by Step 2, the functions ∇v and $\partial_1^2 v$ also belong to all the spaces $L^q(\mathbb{R}^N)$ for $1 < q < +\infty$. By the embedding theorem for anisotropic Sobolev spaces [3], the function v is then continuous and bounded on $\mathbb{R}^N$, which completes the proofs of Step 3 and of Theorem 7. □

2.2 Algebraic decay of the solitary waves.

We now establish the algebraic decay of the solitary waves stated in Theorem 8. As mentioned in the introduction, the proof of this theorem follows from a standard inductive argument which links the algebraic decay of the solitary waves to the algebraic decay of the associated kernels. We first determine some small algebraic decay for the solitary waves. It follows from Proposition 6 which gives some integral algebraic decay for the functions ∇v and $\partial_1^2 v$. We then improve inductively the algebraic decay of the functions v and ∇v by using the superlinearity of equations (14) and (44). This is possible as long as the rate of decay is less important than the rate of decay of the kernels K_0 and K_k. Thus, the solitary waves decay at least as fast as the kernel K_0, while their gradient decays at least as fast as the kernels K_k. This leads to Theorem 8 whose proof follows below.

Proof of Theorem 8. We split the proof in five steps. In the first one, we infer from Proposition 6 some small algebraic decay for the functions v and ∇v.

Step 1. *There is some positive real number α_0 such that the functions v and ∇v belong to every space $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0, \alpha_0]$.*

Step 1 follows from Proposition 6 and equations (13) and (14). Indeed, as mentioned in the introduction, by Theorems 3 and 7, and Corollary 1, equation (13) holds almost everywhere. In particular, this means that for every real number $\beta \geq 0$, and almost every $x \in \mathbb{R}^N$,

$$\begin{aligned} |x|^\beta |v(x)| =& |x|^\beta \left| \int_{\mathbb{R}^N} H_0(x-y) v^p(y) \partial_1 v(y) dy \right| \\ \leq& A\bigg(\int_{\mathbb{R}^N} |H_0(x-y)||y|^\beta |v(y)|^p |\partial_1 v(y)| dy + \int_{\mathbb{R}^N} |x-y|^\beta |H_0(x-y)||v(y)|^p \\ & |\partial_1 v(y)| dy \bigg). \end{aligned} \tag{120}$$

However, if $0 \leq \beta \leq N-1$, by Theorem 3 and Corollary 1, we compute for every positive real number $1 < q < \frac{2N-1}{2N-4}$,

$$\begin{aligned} & \int_{\mathbb{R}^N} |x-y|^\beta |H_0(x-y)||v(y)|^p |\partial_1 v(y)| dy \\ \leq & A\bigg(\int_{B(x,1)^c} |x-y|^{\beta-N+1} |v(y)|^p |\partial_1 v(y)| dy + \int_{B(x,1)} |H_0(x-y)||v(y)|^p |\partial_1 v(y)| dy \bigg) \\ \leq & A\bigg(\int_{B(x,1)^c} |v(y)|^p |\partial_1 v(y)| dy + \|H_0\|_{L^q(B(0,1))} \|v^p \partial_1 v\|_{L^{q'}(B(x,1))} \bigg). \end{aligned}$$

Therefore, by Theorem 7, there exists some real number A such that for every $0 \leq \beta \leq N-1$,

$$\int_{\mathbb{R}^N} |x-y|^\beta |H_0(x-y)||v(y)|^p |\partial_1 v(y)| dy \leq A. \tag{121}$$

On the other hand, by Theorem 3 and Corollary 1, we compute for every positive real numbers $1 < q < \frac{2N-1}{2N-4}$ and $r > \frac{N}{N-1}$,

$$\begin{aligned} & \int_{\mathbb{R}^N} |H_0(x-y)||y|^\beta |v(y)|^p |\partial_1 v(y)| dy \\ \leq & A\bigg(\int_{B(x,1)^c} |x-y|^{1-N} |y|^\beta |v(y)|^p |\partial_1 v(y)| dy + \int_{B(x,1)} |H_0(x-y)||y|^\beta |v(y)|^p |\partial_1 v(y)| dy \bigg) \\ \leq & A\Bigg(\bigg(\int_{B(0,1)^c} |y|^{(1-N)r} dy \bigg)^{\frac{1}{r}} \bigg(\int_{B(x,1)^c} |y|^{\beta r'} |v(y)|^{pr'} |\partial_1 v(y)|^{r'} dy \bigg)^{\frac{1}{r'}} \\ + & \bigg(\int_{B(0,1)} |H_0(z)|^q dz \bigg)^{\frac{1}{q}} \bigg(\int_{B(x,1)} |y|^{\beta r'} |v(y)|^{pq'} |\partial_1 v(y)|^{q'} dy \bigg)^{\frac{1}{q'}} \Bigg). \end{aligned}$$

However, by Theorem 7, the function v is bounded on $\mathbb{R}^N$, so, by Proposition 6, for every positive real numbers $s > 1$ and $0 < \beta < \min\{1, \frac{2}{s}\}$,

$$\begin{aligned}\int_{\mathbb{R}^N} |y|^{\beta s}|v(y)|^{ps}|\partial_1 v(y)|^s dy \leq & A\int_{\mathbb{R}^N} |y|^{\beta s}|\partial_1 v(y)|^s dy \\ \leq & A\Big(\int_{\mathbb{R}^N} |y|^2\partial_1 v(y)^2 dy\Big)^{\frac{\beta s}{2}}\Big(\int_{\mathbb{R}^N} |\partial_1 v(y)|^{\frac{2(1-\beta)s}{2-\beta s}} dy\Big)^{1-\frac{\beta s}{2}} \\ \leq & A,\end{aligned}$$

so, invoking once more Theorem 7 in case $\beta = 0$, for every $0 \leq \beta < \min\{1, \frac{2}{r'}, \frac{2}{q'}\}$,

$$\int_{\mathbb{R}^N} |H_0(x-y)||y|^\beta |v(y)|^p|\partial_1 v(y)|dy \leq A. \tag{122}$$

Thus, by equations (120), (121) and (122), there is some positive real number α_1 such that the function v belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $0 \leq \beta \leq \alpha_1$.

Likewise, by Theorems 3 and 7, and Corollary 1, equation (14) holds in all the spaces $L^q(\mathbb{R}^N)$ for $1 < q < \frac{2N-1}{2N-3}$. Moreover, by Theorem 3 and Corollary 1, the kernel K_0 belongs to all the spaces $L^q(\mathbb{R}^N)$ for $1 < q < \frac{2N-1}{2N-3}$, while the function $v^p\nabla v$ belongs to $L^1(\mathbb{R}^N)$ by Theorem 7. Therefore, we can derive from equation (14) the following equation, which holds in all the spaces $L^q(\mathbb{R}^N)$ for $1 < q < \frac{2N-1}{2N-3}$,

$$\nabla v = K_0 * (v^p\nabla v). \tag{123}$$

In particular, it yields for every real number $\beta \geq 0$ and almost every $x \in \mathbb{R}^N$,

$$\begin{aligned}|x|^\beta|\nabla v(x)| \leq & A\Big(\int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta|v(y)|^p|\nabla v(y)|dy + \int_{\mathbb{R}^N} |x-y|^\beta|K_0(x-y)||v(y)|^p \\ & |\nabla v(y)|dy\Big).\end{aligned} \tag{124}$$

However, if $0 \leq \beta \leq N$, by Theorem 3 and Corollary 1, we compute for every positive real number $1 < q < \frac{2N-1}{2N-3}$,

$$\begin{aligned}&\int_{\mathbb{R}^N} |x-y|^\beta|K_0(x-y)||v(y)|^p|\nabla v(y)|dy \\ \leq & A\Big(\int_{B(x,1)^c} |x-y|^{\beta-N}|v(y)|^p|\nabla v(y)|dy + \Big(\int_{B(0,1)} |K_0(z)|^q dz\Big)^{\frac{1}{q}}\Big(\int_{B(x,1)} |v(y)|^{pq'} \\ & |\nabla v(y)|^{q'} dy\Big)^{\frac{1}{q'}}\Big).\end{aligned}$$

Hence, by Theorem 7, there exists some real number A such that for every $0 \leq \beta \leq N$,

$$\int_{\mathbb{R}^N} |x-y|^\beta|K_0(x-y)||v(y)|^p|\nabla v(y)|dy \leq A. \tag{125}$$

On the other hand, by Theorem 3 and Corollary 1, we compute for every positive real numbers $1 < q < \frac{2N-1}{2N-3}$ and $r > 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta |v(y)|^p |\nabla v(y)| dy \\
\leq & A\Bigg(\Big(\int_{B(0,1)^c} |y|^{-Nr} dy \Big)^{\frac{1}{r}} \Big(\int_{B(x,1)^c} |y|^{\beta r'} |v(y)|^{pr'} |\nabla v(y)|^{r'} dy \Big)^{\frac{1}{r'}} \\
& + \Big(\int_{B(0,1)} |K_0(z)|^q dz \Big)^{\frac{1}{q}} \Big(\int_{B(x,1)} |y|^{\beta r'} |v(y)|^{pq'} |\nabla v(y)|^{q'} dy \Big)^{\frac{1}{q'}} \Bigg).
\end{aligned}$$

However, by Theorem 7, the function v is bounded on $\mathbb{R}^N$, so, by Proposition 6, for every positive real numbers $s > 1$ and $0 < \beta < \min\{1, \frac{2}{s}\}$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |y|^{\beta s} |v(y)|^{ps} |\nabla v(y)|^s dy \leq & A\Big(\int_{\mathbb{R}^N} |y|^2 |\nabla v(y)|^2 dy \Big)^{\frac{\beta s}{2}} \Big(\int_{\mathbb{R}^N} |\nabla v(y)|^{\frac{2(1-\beta)s}{2-\beta s}} dy \Big)^{1-\frac{\beta s}{2}} \\
\leq & A,
\end{aligned}$$

so, invoking once more Theorem 7 in case $\beta = 0$, for every $0 \leq \beta < \min\{1, \frac{2}{r'}, \frac{2}{q'}\}$,

$$\int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta |v(y)|^p |\nabla v(y)| dy \leq A. \tag{126}$$

Finally, by equations (124), (125) and (126), there is also some positive real number α_2 such that the function ∇v belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $0 \leq \beta \leq \alpha_2$. It then only remains to set $\alpha_0 = \min\{\alpha_1, \alpha_2\}$ to complete the proof of Step 1.

Remark. In particular, by Step 1, the function v is lipschitzian on the whole space $\mathbb{R}^N$. Indeed, its gradient ∇v is bounded on $\mathbb{R}^N$.

We now improve the algebraic decay of the function v by applying the inductive argument mentioned in the introduction to equation (14).

Step 2. *Consider some positive real number α and assume that the function v belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta \in [0, \alpha]$. Then, it belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta \in [0, N] \cap [0, (p+1)\alpha[$.*

Indeed, equation (14) yields for every positive real number β and for almost every $x \in \mathbb{R}^N$,

$$|x|^\beta |v(x)| \leq A\Big(\int_{\mathbb{R}^N} |x-y|^\beta |K_0(x-y)||v(y)|^{p+1} dy + \int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta |v(y)|^{p+1} dy \Big). \tag{127}$$

However, by Theorem 3 and Corollary 1,

$$\begin{aligned}
\int_{\mathbb{R}^N} |x-y|^\beta |K_0(x-y)||v(y)|^{p+1} dy \leq & A\Big(\int_{B(x,1)^c} |x-y|^{\beta-N} |v(y)|^{p+1} dy \\
& + \int_{B(x,1)} |K_0(x-y)||v(y)|^{p+1} dy \Big),
\end{aligned}$$

so, by Theorem 7, for every $0 \leq \beta \leq N$,

$$\int_{\mathbb{R}^N} |x-y|^\beta |K_0(x-y)||v(y)|^{p+1} dy \leq A\Big(\|v\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \|K_0\|_{L^1(B(0,1))} \|v\|^{p+1}_{L^\infty(\mathbb{R}^N)} \Big) \leq A. \tag{128}$$

Likewise, Theorem 3 and Corollary 1 give for every real number $q > 1$,

$$\int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta |v(y)|^{p+1} dy \leq A\Bigg(\Big(\int_{B(0,1)^c} \frac{dz}{|z|^{Nq}}\Big)^{\frac{1}{q}} \Big(\int_{B(x,1)^c} |y|^{\beta q'} |v(y)|^{(p+1)q'} dy\Big)^{\frac{1}{q'}} + \|K_0\|_{L^1(B(0,1))} \|v\|^{p+1}_{M^\infty_{\frac{\beta}{p+1}}(\mathbb{R}^N)}\Bigg).$$

However, by the assumption of Step 2, there is some real number $q > 1$ such that for every $\beta \in [0, (p+1)\alpha[$, the function $y \mapsto |y|^\beta v(y)^{p+1}$ belongs to the space $L^{q'}(\mathbb{R}^N)$. Hence, we obtain

$$\int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta |v(y)|^{p+1} dy \leq A\Bigg(\||.|^\beta v^{p+1}\|_{L^{q'}(\mathbb{R}^N)} + \|v\|^{p+1}_{M^\infty_{\frac{\beta}{p+1}}(\mathbb{R}^N)}\Bigg) \leq A. \tag{129}$$

Thus, by equations (127), (128) and (129), the function v belongs to all the spaces $M^\infty_\beta(\mathbb{R}^N)$ for $\beta \in [0, N] \cap [0, (p+1)\alpha[$, which is the desired result.

Finally, we deduce the rate of decay of the function v given by Theorem 8.

Step 3. *The function v belongs to the space $M^\infty_N(\mathbb{R}^N)$.*

Since $p + 1 > 1$, the geometric sequence given by $u_0 = \alpha_0$ and $u_{n+1} = (p+1)u_n$ tends to $+\infty$ when n tends to $+\infty$. Thus, by a straightforward inductive argument, it follows from Steps 1 and 2 that the function v belongs to the space $M^\infty_N(\mathbb{R}^N)$.

We now turn to the algebraic decay of the gradient of v. In particular, we improve the rate of decay given by Step 1 by applying the inductive argument mentioned in the introduction to equation (44).

Step 4. *Consider some positive real number α and assume that the function ∇v belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta \in [0, \alpha]$. Then, it belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta \in [0, \min\{N + 1, (p+1)N, pN + \alpha\}]$.*

Indeed, by Theorem 7, the function v^{p+1} is bounded and continuous on $\mathbb{R}^N$. Moreover, by Steps 1 and 3, this function belongs to $M^\infty_{(p+1)N}(\mathbb{R}^N)$ and its gradient to $L^\infty(\mathbb{R}^N)$. Therefore, by Lemma 3, the following equality holds for every $k \in \{1, \ldots, N\}$ and every $x \in \mathbb{R}^N$,

$$\partial_k v(x) = i\int_{B(0,1)^c} K_k(y) v(x-y)^{p+1} dy + i\int_{B(0,1)} K_k(y)(v(x-y)^{p+1} - v(x)^{p+1}) dy + \Big(\int_{\mathbb{S}^{N-1}} y_k K_0(y) dy\Big) v(x)^{p+1}.$$

In particular, by Theorem 3 and Corollary 1, it yields for every positive real number β,

$$|x|^\beta |\partial_k v(x)| \leq |x|^\beta \int_{B(0,1)^c} |K_k(y)||v(x-y)|^{p+1} dy + |x|^\beta \int_{B(0,1)} |K_k(y)||v(x-y)^{p+1} - v(x)^{p+1}| dy + A|x|^\beta |v(x)|^{p+1}. \tag{130}$$

However, Step 3 yields for every $\beta \in [0, (p+1)N]$,

$$|x|^\beta |v(x)|^{p+1} \leq \|v\|^{p+1}_{M^\infty_{\frac{\beta}{p+1}}(\mathbb{R}^N)} \leq A. \tag{131}$$

On the other hand, by Theorem 3,

$$
\begin{aligned}
|x|^\beta \int_{B(0,1)^c} |K_k(y)||v(x-y)|^{p+1}dy \leq & A\Big(\int_{B(0,1)^c} |y|^\beta |K_k(y)||v(x-y)|^{p+1}dy \\
& + \int_{B(0,1)^c} |K_k(y)||x-y|^\beta |v(x-y)|^{p+1}dy\Big) \\
\leq & A\Big(\int_{B(0,1)^c} |y|^{\beta-N-1}|v(x-y)|^{p+1}dy \\
& + \int_{B(0,1)^c} |y|^{-N-1}|x-y|^\beta |v(x-y)|^{p+1}dy\Big),
\end{aligned}
$$

so, by Theorem 7 and Step 3, for every $\beta \in [0, \min\{N+1, (p+1)N\}]$,

$$
|x|^\beta \int_{B(0,1)^c} |K_k(y)||v(x-y)|^{p+1}dy \leq A\Big(\|v\|_{L^{p+1}(\mathbb{R}^N)} + \|v\|^{p+1}_{M^\infty_{\frac{\beta}{p+1}}(\mathbb{R}^N)}\Big) \leq A. \tag{132}
$$

Finally, by Theorem 7 and Step 1, the function v is continuous on $\mathbb{R}^N$, while its gradient ∇v is bounded on $\mathbb{R}^N$. Hence, we compute for every $(x,y) \in (\mathbb{R}^N)^2$,

$$
v(x-y)^{p+1} - v(x)^{p+1} = -(p+1)\int_0^1 v(x-ty)^p \nabla v(x-ty).ydt,
$$

which gives by Theorem 7, Step 3 and the assumption of Step 4,

$$
\begin{aligned}
& |x|^\beta \int_{B(0,1)} |K_k(y)||v(x-y)^{p+1} - v(x)^{p+1}|dy \\
\leq & A \int_{B(0,1)} \sum_{l=1}^N |y_l||K_k(y)|\Big(\int_0^1 |x|^\beta |v(x-ty)|^p |\partial_l v(x-ty)|dt\Big)dy \\
\leq & A \int_{B(0,1)} \sum_{l=1}^N |y_l||K_k(y)|\Big(\int_0^1 \frac{|x|^\beta dt}{1+|x-ty|^{pN+\alpha}}\Big)dy \\
\leq & A \frac{|x|^\beta}{1+|x|^{pN+\alpha}} \int_{B(0,1)} \sum_{l=1}^N |y_l||K_k(y)|dy.
\end{aligned}
$$

Corollary 1 then yields for every $\beta \in [0, pN+\alpha]$,

$$
|x|^\beta \int_{B(0,1)} |K_k(y)||v(x-y)^{p+1} - v(x)^{p+1}|dy \leq A. \tag{133}
$$

Thus, by equations (130), (131), (132) and (133), the function ∇v belongs to $M^\infty_\beta(\mathbb{R}^N)$ for $0 \leq \beta \leq \min\{N+1, (p+1)N, pN+\alpha\}$, which ends the proof of Step 4.

In conclusion, we infer the rate of decay of the function ∇v given by Theorem 8.

Step 5. *The function ∇v belongs to the space $M^\infty_{\min\{N+1,(p+1)N\}}(\mathbb{R}^N)$.*

Indeed, the arithmetic sequence given by $u_0 = \alpha_0$ and $u_{n+1} = u_n + pN$ tends to $+\infty$ when n tends to $+\infty$. Thus, by Steps 1 and 4, the function ∇v belongs to the space $M^\infty_{\min\{N+1,(p+1)N\}}(\mathbb{R}^N)$. This completes the proofs of Step 5 and of Theorem 8. □

2.3 Asymptotic expansion of the solitary waves.

In this section, we complete the proof of Theorem 1. Indeed, in the previous section, we proved Theorem 8 which describes the algebraic decay of the solitary waves and of their gradient. In order to show Theorem 1, it then remains to establish the existence of a first order asymptotic expansion of any solitary wave v, i.e. to compute the limit when $|x|$ tends to $+\infty$ of the function $x \mapsto |x|^N v(x)$. This is the aim of Propositions 7 and 8. Indeed, in Proposition 7, we compute the pointwise limit when R tends to $+\infty$ of the functions v_R given by formula (48). Our argument follows from Theorems 3, 5 and 8, Corollary 1 and a standard application of the dominated convergence theorem. In Proposition 8, we then deduce from Theorem 8 and a standard application of Ascoli-Arzela's theorem that this convergence is uniform in the case $p \geq \frac{1}{N}$. Finally, Theorem 1 follows from Theorem 8 and from Propositions 7 and 8.

However, let us first write the proof of Proposition 7.

Proof of Proposition 7. Let $\sigma \in \mathbb{S}^{N-1}$. By formula (48), we compute for every positive real number R,

$$v_R(\sigma) = \frac{1}{p+1}\Big(\int_{B(R\sigma,\frac{R}{2})} R^N K_0(R\sigma - y) v(y)^{p+1} dy + \int_{B(R\sigma,\frac{R}{2})^c} R^N K_0(R\sigma - y) v(y)^{p+1} dy\Big). \tag{134}$$

However, Theorem 5 yields for every $y \in \mathbb{R}^N$,

$$R^N K_0(R\sigma - y) v(y)^{p+1} \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(1 - N\sigma_1^2) v(y)^{p+1},$$

while by Theorems 3 and 8,

$$|R^N K_0(R\sigma - y) v(y)^{p+1} 1_{|R\sigma - y| \geq \frac{R}{2}}| \leq A \frac{R^N}{(R\sigma - y)^N (1 + |y|^{N(p+1)})} \leq \frac{A}{1 + |y|^{N(p+1)}}.$$

Therefore, the dominated convergence theorem yields

$$\int_{B(R\sigma,\frac{R}{2})^c} R^N K_0(R\sigma - y) v(y)^{p+1} dy \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(1 - N\sigma_1^2) \int_{\mathbb{R}^N} v(y)^{p+1} dy. \tag{135}$$

On the other hand, by Theorem 8,

$$\begin{aligned}\Big|\int_{B(R\sigma,\frac{R}{2})} R^N K_0(R\sigma - y) v(y)^{p+1} dy\Big| \leq & A \int_{B(R\sigma,\frac{R}{2})} R^N |y|^{-N(p+1)} |K_0(R\sigma - y)| dy \\ \leq & \frac{A}{R^{Np}} \int_{B(0,\frac{R}{2})} |K_0(z)| dz,\end{aligned}$$

so, by Theorem 3 and Corollary 1,

$$\begin{aligned}\Big|\int_{B(R\sigma,\frac{R}{2})} R^N K_0(R\sigma - y) v(y)^{p+1} dy\Big| \leq & \frac{A}{R^{Np}}\Big(\int_{B(0,1)} |K_0(z)| dz + \int_{1<|z|<\frac{R}{2}} \frac{dz}{|z|^N}\Big) \\ \leq & \frac{A}{R^{Np}}(1 + \ln(R)) \underset{R\to+\infty}{\to} 0.\end{aligned}$$

Thus, equations (134) and (135) yield

$$v_R(\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(p+1)}(1 - N\sigma_1^2) \int_{\mathbb{R}^N} v(y)^{p+1} dy,$$

which is exactly assertion (49). □

We then establish the uniformity of the pointwise limit computed above in the case $p \geq \frac{1}{N}$. This follows from Proposition 8 whose proof is mentioned below.

Proof of Proposition 8. Assume by contradiction that $(v_R)_{R>0}$ does not converge uniformly to v_∞ when R tends to $+\infty$. There is then some real number $\varepsilon > 0$ and a sequence of positive real numbers $(R_n)_{n\in\mathbb{N}}$ tending to $+\infty$, such that

$$\forall n \in \mathbb{N}, \|v_{R_n} - v_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \geq \varepsilon. \tag{136}$$

However, since $p \geq \frac{1}{N}$, we deduce from Theorem 8 that [4]

$$\forall n \in \mathbb{N}, \begin{cases} \|v_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq A, \\ \|\nabla^{\mathbb{S}^{N-1}} v_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq AR_n^{N+1}\|\nabla v(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})} \leq AR_n^{1-\min\{1,pN\}} \leq A. \end{cases} \tag{137}$$

Therefore, by equation (137) and Ascoli-Arzela's theorem, up to a subsequence, $(v_{R_n})_{n\in\mathbb{N}}$ converges in the space $L^\infty(\mathbb{S}^{N-1})$. By Proposition 8, its limit is necessary equal to v_∞, which leads to a contradiction with assertion (136). Thus, $(v_R)_{R>0}$ uniformly converges to v_∞ when R tends to $+\infty$, which is the desired result. □

Finally, we conclude the proof of Theorem 1 by invoking Theorem 8 and Propositions 7 and 8.

Proof of Theorem 1. Indeed, by Theorem 8, the function $x \mapsto |x|^N v(x)$ is bounded on $\mathbb{R}^N$. Moreover, by Proposition 7, assertion (11) holds for every $\sigma \in \mathbb{S}^{N-1}$, and by Proposition 8, the convergence given by this assertion is actually uniform when $\frac{1}{N} \leq p < \frac{4}{2N-3}$. This concludes the proof of Theorem 1. □

2.4 Link between the asymptotic behaviour at infinity and the energy of solitary waves for the standard Kadomtsev-Petviashvili equation.

This last section deals with the standard Kadomtsev-Petviashvili equation. In particular, we link the asymptotic expansion of a solitary wave to its energy and its action. As mentioned in the introduction, this link stated in Theorem 2 follows from the standard Pohozaev identities of Lemma 4, which were derived by A. de Bouard and J.C. Saut in [13].

Proof of Theorem 2. We split the proof in two steps. We first give an expression of the integrals $\int_{\mathbb{R}^N} \partial_1 v(x)^2 dx$, $\int_{\mathbb{R}^N} v(x)^{p+2} dx$ and $\int_{\mathbb{R}^N} v_k(x)^2 dx$ in function of the integral $\int_{\mathbb{R}^N} v(x)^2 dx$ for every $p \in]0, \frac{4}{2N-3}[$. We then complete the proof of Theorem 2 by linking the energy and the action of v to the integral $\int_{\mathbb{R}^N} v(x)^2 dx$ in the case $p = 1$.

[4] Here, the notation $\nabla^{\mathbb{S}^{N-1}}$ denotes the gradient on the sphere $\mathbb{S}^{N-1}$ immersed in the space $\mathbb{R}^N$. More precisely, if we consider some index $i \in \{1, \dots, N\}$ and some function $f \in C^\infty(\mathbb{S}^{N-1}, \mathbb{C})$, the notation $\partial_i^{\mathbb{S}^{N-1}}$ is defined by

$$\forall x \in \mathbb{S}^{N-1}, \partial_i^{\mathbb{S}^{N-1}} f(x) = \lim_{t\to 0} \frac{f(\frac{x+te_i}{|x+te_i|}) - f(x)}{t},$$

where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^N$. The gradient $\nabla^{\mathbb{S}^{N-1}} f$ of the function f is then given by

$$\forall x \in \mathbb{S}^{N-1}, \nabla^{\mathbb{S}^{N-1}} f(x) = (\partial_1^{\mathbb{S}^{N-1}} f(x), \dots, \partial_N^{\mathbb{S}^{N-1}} f(x)).$$

Step 1. *Let $p \in]0, \frac{4}{2N-3}[$. We have for every $k \in \{2, \ldots, N\}$,*

$$\int_{\mathbb{R}^N} v(x)^{p+2} dx = \frac{2(p+1)(p+2)}{4+p(3-2N)} \int_{\mathbb{R}^N} v(x)^2 dx, \tag{138}$$

$$\int_{\mathbb{R}^N} \partial_1 v(x)^2 dx = \frac{pN}{4+p(3-2N)} \int_{\mathbb{R}^N} v(x)^2 dx, \tag{139}$$

$$\int_{\mathbb{R}^N} v_k(x)^2 dx = \frac{p}{4+p(3-2N)} \int_{\mathbb{R}^N} v(x)^2 dx. \tag{140}$$

Indeed, let us denote

$$I_0 = \int_{\mathbb{R}^N} v(x)^2 dx,$$

$$I_p = \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^N} v(x)^{p+2} dx,$$

$$J_1 = \int_{\mathbb{R}^N} \partial_1 v(x)^2 dx,$$

$$J = \sum_{k=2}^{N} \int_{\mathbb{R}^N} v_k(x)^2 dx.$$

By formula (51), we obtain

$$J = I_0 - 2(p+1)I_p + 3J_1. \tag{141}$$

We then substitute this relation in formula (53) to compute

$$J_1 = -\frac{1}{2}I_0 + \frac{3p+4}{4}I_p. \tag{142}$$

On the other hand, by summing equations (52) for all $k \in \{2, \ldots, N\}$, we find

$$(N-1)(I_0 - 2I_p + J_1) + (N-3)J = 0,$$

which gives by relations (141) and (142),

$$I_0 = \frac{4+p(3-2N)}{2} I_p. \tag{143}$$

Finally, since $0 < p < \frac{4}{2N-3}$, this yields formula (138). Moreover, by substituting equation (143) in equation (142), we compute

$$J_1 = \frac{Np}{4+p(3-2N)} I_0,$$

which is exactly formula (139). Likewise, by substituting equations (139) and (143) in equation (141), we get

$$J = \frac{(N-1)p}{4+p(3-2N)} I_0.$$

However, by formulae (52), we have for every $(j,k) \in \{2, \ldots, N\}$,

$$\int_{\mathbb{R}^N} v_j(x)^2 dx = \frac{1}{2}I_0 - I_p + \frac{1}{2}J_1 + \frac{1}{2}J = \int_{\mathbb{R}^N} v_k(x)^2 dx.$$

Therefore, all the integrals $\int_{\mathbb{R}^N} v_k(x)^2 dx$ for $2 \leq k \leq N$ are equal. By definition of J, this yields

$$\int_{\mathbb{R}^N} v_k(x)^2 dx = \frac{1}{N-1} J = \frac{p}{4+p(3-2N)} I_0,$$

which is exactly formula (140).

Step 2. *Assume now that $p = 1$ and $N = 2$ or $N = 3$. Then, the function v_∞ is given by formula* (12).

Indeed, if $p = 1$, formula (10) becomes

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{4\pi^{\frac{N}{2}}}(1 - N\sigma_1^2) \int_{\mathbb{R}^N} v(x)^2 dx. \tag{144}$$

However, by formula (3), the energy of the function v is given by

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^N} (\partial_1 u(x)^2 + \sum_{j=2}^{N} v_j(x)^2) dx - \frac{1}{6} \int_{\mathbb{R}^N} u(x)^3 dx.$$

Therefore, by formulae (138), (139) and (140), it is equal to

$$E(v) = \frac{2N - 5}{2(7 - 2N)} \int_{\mathbb{R}^N} v(x)^2 dx, \tag{145}$$

so, by equation (144),

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \frac{(7 - 2N)\Gamma(\frac{N}{2})}{2(2N - 5)\pi^{\frac{N}{2}}}(1 - N\sigma_1^2) E(v),$$

which is exactly formula (12). Likewise, by equation (5), the action of the function v is given by

$$S(v) = E(v) + \frac{1}{2} \int_{\mathbb{R}^N} v(x)^2 dx,$$

so, by equation (145),

$$S(v) = \frac{1}{7 - 2N} \int_{\mathbb{R}^N} v(x)^2 dx,$$

and by equation (144),

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \frac{(7 - 2N)\Gamma(\frac{N}{2})}{4\pi^{\frac{N}{2}}}(1 - N\sigma_1^2) S(v),$$

which completes the proof of formula (12). □

Finally, for sake of completeness, we complete this subsection by the proof of Corollary 2, which mentions another straightforward consequence of the identities of Lemma 4: the non-existence of non-trivial solutions of equation (6) in Y if $p \geq \frac{4}{2N-3}$ in dimension $N \geq 4$.

Proof of Corollary 2. Indeed, Lemma 4 holds for any real number $p > 0$. In particular, formulae (141), (142) and (143) of Step 1 of the proof of Theorem 2 also hold for every positive real number p. However, formulae (142) and (143) yield

$$(4 + p(3 - 2N)) \int_{\mathbb{R}^N} \partial_1 v(x)^2 dx = Np \int_{\mathbb{R}^N} v(x)^2 dx.$$

Therefore, if $p \geq \frac{4}{2N-3}$,

$$\int_{\mathbb{R}^N} v(x)^2 dx \leq 0.$$

Thus, v is identically equal to 0, which ends the proof of Corollary 2. □

Appendix. First order asymptotic expansion of travelling waves for the Gross-Pitaevskii equation.

This appendix yields an important application of Theorem 6. Indeed, this theorem enables to solve a conjecture formulated in [26] in the context of the travelling waves for the Gross-Pitaevskii equation. The Gross-Pitaevskii equation is a non-linear Schrödinger equation which writes

$$i\partial_t u = \Delta u + u(1 - |u|^2), \tag{146}$$

where u is a function from $\mathbb{R} \times \mathbb{R}^N$ ($N \geq 2$ here) to $\mathbb{C}$. It conserves at least formally two quantities: the so-called Ginzburg-Landau energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2, \tag{147}$$

and the momentum

$$\vec{P}(u) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla u.u. \tag{148}$$

The travelling waves v for the Gross-Pitaevskii equation are the solutions of equation (146) of finite energy which are of the form

$$u(t, x) = v(x_1 - ct, x_2, \ldots, x_n).$$

The equation for v, which we will consider now, writes

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{149}$$

Here, the parameter $c \geq 0$ represents the speed of the travelling wave v, which moves in direction x_1. In this appendix, we will always assume that

$$0 < c < \sqrt{2},$$

i.e. the travelling waves are subsonic. Indeed, for this equation, the speed of the sound waves near the constant solution $u = 1$ is $\sqrt{2}$. Moreover, the travelling waves which are not subsonic are much less interesting. Indeed, F. Béthuel and J.C. Saut proved in [4] that they are constant when their speed is 0, while we proved the same property in [23] when their speed is strictly more than $\sqrt{2}$. On the other hand, it is also commonly conjectured that they are constant when their speed is $\sqrt{2}$ (we actually proved this result in dimension two in [25], but it is still an open question in dimension $N \geq 3$). Thus, since the only non constant travelling waves seem to be subsonic, we will focus on the subsonic case in the following.

Our aim will be to specify the asymptotic behaviour of subsonic travelling waves at infinity. Indeed, in a previous paper [26], we proved the existence of a first order asymptotic expansion at infinity of a travelling wave.

Theorem ([26]). *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of speed $0 < c < \sqrt{2}$. There exist a complex number λ_∞ of modulus one and a smooth function v_∞ defined from the sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}$ such that*

$$|x|^{N-1}(v(x) - \lambda_\infty) - i\lambda_\infty v_\infty\left(\frac{x}{|x|}\right) \underset{|x|\to+\infty}{\to} 0 \text{ uniformly}.$$

Moreover, we computed explicitly the value of the function v_∞ in dimension $N = 2$, and for axisymmetric travelling waves which only depend on the variables x_1 and

$$|x_\perp| = \sqrt{\sum_{i=2}^{N} x_i^2}.$$

Theorem ([26]). *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of speed $0 < c < \sqrt{2}$. Then, if $N = 2$, there exist some constants α and β such that the function v_∞ is given by*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}},$$

while if v is axisymmetric around axis x_1, the function v_∞ is given by

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}.$$

Moreover, in both cases, the constant α is equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \left(1 - \frac{c^2}{2}\right)^{\frac{N-3}{2}} \left(\frac{4-N}{2} cE(v) + \left(2 + \frac{N-3}{2} c^2\right) p(v)\right),$$

while the constant β is equal to

$$\beta = \frac{\sqrt{1 - \frac{c^2}{2}}}{\pi} P_2(v).$$

However, we were not able to compute explicitly the value of the function v_∞ in the general case. We only conjectured its value in Conjecture 1 of [26]. Here, we will fill this gap by proving this conjecture thanks to Theorem 6.

Theorem 9. *Let v be a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$. Then, there exist some constants α, β_2, ..., β_N such that the function v_∞ is equal to*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + \sum_{j=2}^{N} \beta_j \frac{\sigma_j}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}. \tag{150}$$

Moreover, the constants α and β_j are equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \left(1 - \frac{c^2}{2}\right)^{\frac{N-3}{2}} \left(\frac{4-N}{2} cE(v) + \left(2 + \frac{N-3}{2} c^2\right) p(v)\right), \tag{151}$$

$$\beta_j = \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}} \left(1 - \frac{c^2}{2}\right)^{\frac{N-1}{2}} P_j(v). \tag{152}$$

Remark. There is a difficulty in the definition of $\vec{P}(v)$. Indeed, the integral which appears in definition (148) is not always convergent. In order to state formulae (151) and (152) rigorously, we define the momentum $\vec{P}(v)$ as

$$\vec{P}(v) = \frac{1}{2} \int_{\mathbb{R}^N} i\nabla v.(v - 1), \tag{153}$$

and the scalar momentum in direction x_1 by

$$p(v) = P_1(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\partial_1 v.(v-1). \tag{154}$$

By [24], all those integrals are well-defined in the subsonic case.

Proof. The proof is similar to the proof of Theorem 1. Indeed, in [26], we derive a new formulation of equation (149), which is an equivalent of equation (8). This formulation relies on a polar form of the function v. Indeed, there is some positive real number R_0 and some functions $\rho := |v|$ and θ in $C^\infty(B(0,R_0)^c, \mathbb{R})^2$ such that

$$v = \rho e^{i\theta},$$

on the open set $B(0,R_0)^c$. If we then introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N, [0,1])$ such that

$$\begin{cases} \psi = 0 \text{ on } \mathrm{B}(0, 2\mathrm{R}_0), \\ \psi = 1 \text{ on } \mathrm{B}(0, 3\mathrm{R}_0)^{\mathrm{c}}, \end{cases}$$

we deduce new equations for the variables $\eta := 1 - \rho^2$ and θ,

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}), \tag{155}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{156}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta), \tag{157}$$

and

$$G = i\nabla v.v + \nabla(\psi\theta). \tag{158}$$

We then transform equations (155) and (156) in convolution equations,

$$\eta = K_0 * F + 2c\sum_{j=1}^N K_j * G_j, \tag{159}$$

where K_0 and K_j are the kernels of Fourier transform,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{160}$$

respectively

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{161}$$

and for every $j \in \{1, \dots, N\}$,

$$\partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^N L_{j,k} * G_k + \sum_{k=1}^N R_{j,k} * G_k, \tag{162}$$

where $L_{j,k}$ are the kernels of Fourier transform,

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{163}$$

and $R_{j,k}$ are the composed Riesz kernels given by formula (40). In particular, equations (159) and (162) are very similar to equations (13) and (14). Thus, by the argument of the proof of Theorem 1, we proved in [26] that there exist some functions $(\eta_\infty, v_\infty) \in C^1(\mathbb{S}^{N-1})^2$ and $\theta_\infty \in C^2(\mathbb{S}^{N-1})$ such that

$$R^N \eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty(\sigma) \text{ in } C^1(\mathbb{S}^{N-1}),$$
$$R^{N-1}\theta(R\sigma) \underset{R\to+\infty}{\to} \theta_\infty(\sigma) \text{ in } C^2(\mathbb{S}^{N-1}),$$
$$R^{N-1}(v(R\sigma)-1) \underset{R\to+\infty}{\to} v_\infty(\sigma) \text{ in } C^1(\mathbb{S}^{N-1})$$

(see Proposition 5 of [26]). Moreover, by equations (70) and (72) of [26], the functions η_∞, θ_∞ and v_∞ satisfy for every $\sigma \in \mathbb{S}^{N-1}$,

$$\eta_\infty(\sigma) = K_{0,\infty}(\sigma)\int_{\mathbb{R}^N} F(x)dx + 2c\sum_{j=1}^N K_{j,\infty}(\sigma)\int_{\mathbb{R}^N} G_j(x)dx, \tag{164}$$

and

$$\theta_\infty(\sigma) = v_\infty(\sigma) = -\frac{1}{N-1}\Bigg(\frac{c}{2}\Big(\sum_{j=1}^N \sigma_j K_{j,\infty}(\sigma)\Big)\int_{\mathbb{R}^N} F(x)dx + \sum_{k=1}^N\Big(c^2\sum_{j=1}^N \sigma_j L_{j,k,\infty}(\sigma) - \frac{(N-1)\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\sigma_k\Big)\int_{\mathbb{R}^N} G_k(x)dx\Bigg). \tag{165}$$

Here, the functions $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ denote the limits at infinity of the kernels K_0, K_j and $L_{j,k}$ given by Theorem 5 of [26],

$$\forall\sigma\in\mathbb{S}^{N-1}, \begin{cases} R^N K_0(R\sigma-y) \underset{R\to+\infty}{\to} K_{0,\infty}(\sigma), \\ R^N K_j(R\sigma-y) \underset{R\to+\infty}{\to} K_{j,\infty}(\sigma), \\ R^N L_{j,k}(R\sigma-y) \underset{R\to+\infty}{\to} L_{j,k,\infty}(\sigma). \end{cases}$$

The existence of such limits follow from the same argument as in the proof of Theorem 5 of the present paper. In particular, equation (61) of [26] gives an explicit integral expression for the functions $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$, which is similar to equation (39) of the present paper. More precisely, denoting for every $\xi \in \mathbb{R}^N$ and $1 \leq j,k \leq N$,

$$R^c_{j,k}(\xi) = \frac{\xi_j\xi_k}{2|\xi|^2 - c^2\xi_1^2}, \tag{166}$$

equation (61) of [26] yields for every $\sigma \in \mathbb{S}^{N-1}$ (with a choice of $j \in \{1,\dots,N\}$ such that $\sigma_j \neq 0$),

$$K_{0,\infty}(\sigma) = \frac{i^N}{(2\pi)^N}\sum_{k=1}^N I^c_{j,k,k}(\sigma), \tag{167}$$

$$K_{k,\infty}(\sigma) = \frac{i^N}{(2\pi)^N} I^c_{j,1,k}(\sigma), \tag{168}$$

and

$$L_{k,l,\infty}(\sigma) = \frac{i^N}{(2\pi c^2)^N}\Bigg(2I^c_{j,k,l}(\sigma) - \frac{1}{\sigma_j^N}\Big(\frac{i}{\sigma_j}\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R_{k,l}}(\xi)e^{i\sigma.\xi}d\xi + \frac{i}{\sigma_j}\int_{\mathbb{S}^{N-1}}\xi_j e^{i\sigma.\xi} \partial_j^N\widehat{R_{k,l}}(\xi)d\xi + \int_{\mathbb{S}^{N-1}}\xi_j\partial_j^{N-1}\widehat{R_{k,l}}(\xi)d\xi + \int_{B(0,1)}\partial_j^N\widehat{R_{k,l}}(\xi)(e^{i\sigma.\xi}-1)d\xi\Big)\Bigg). \tag{169}$$

where

$$I^c_{j,k,l}(\sigma) := \frac{1}{\sigma_j^N}\bigg(\frac{i}{\sigma_j}\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R^c_{k,l}}(\xi)e^{i\sigma.\xi}d\xi + \frac{i}{\sigma_j}\int_{\mathbb{S}^{N-1}}\xi_j\partial_j^N\widehat{R^c_{k,l}}(\xi)e^{i\sigma.\xi}d\xi + \int_{\mathbb{S}^{N-1}}\xi_j\partial_j^{N-1}\widehat{R^c_{k,l}}(\xi)d\xi + \int_{B(0,1)}\partial_j^N\widehat{R^c_{k,l}}(\xi)(e^{i\sigma.\xi}-1)d\xi\bigg). \tag{170}$$

Actually, in [26], we did not state equation (61) on the form of equations (167), (168) and (169). Indeed, consider for instance the case of the kernel K_0. Equation (61) of [26] states that the function $K_{0,\infty}$ writes

$$K_{0,\infty}(\sigma) = \frac{i^N}{(2\pi\sigma_j)^N}\Bigg(\int_{B(0,1)}R_1(\xi)(e^{i\xi.\sigma}-1)d\xi + \int_{\mathbb{S}^{N-1}}\xi_j R_0(\xi)e^{i\xi.\sigma}d\xi - \frac{1}{i\sigma_j}\bigg(\int_{B(0,1)^c}R_2(\xi)e^{i\xi.\sigma}d\xi + \int_{\mathbb{S}^{N-1}}\xi_j R_1(\xi)e^{i\xi.\sigma}d\xi\bigg)\Bigg). \tag{171}$$

By equation (56) of [26], the functions R_i, $i \in \{1,2,3\}$, are rational fractions whose numerator are the homogeneous term of lowest degree of the numerator of the rational fraction $\partial_j^{N-1+i}K_0$, and whose denominator is the homogeneous term of lowest degree of the denominator of the rational fraction $\partial_j^{N-1+i}K_0$. However, by equation (160), the kernel K_0 writes

$$K_0(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2},$$

so, by the argument of the proof of Theorem 5 of the present paper, the function R_i is equal to

$$R_i(\xi) = \sum_{k=1}^N \partial_j^{N-1+i}\widehat{R^c_{k,k}}(\xi).$$

Equation (167) then follows from substituting this expression in equation (171). Likewise, equations (168) and (169) follow from equation (61) of [26].

To compute an entirely explicit expression of the functions $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$, it now remains to compute the right members of equations (167), (168) and (169) by the argument of Theorem 6. Indeed, consider the kernels $R^c_{j,k}$. By formula (166), they write

$$\widehat{R^c_{j,k}}(\xi) = \frac{1}{2(1-\frac{c^2}{2})^{\frac{\delta_{j,1}+\delta_{k,1}}{2}}}\widehat{R_{j,k}}\bigg(\Big(1-\frac{c^2}{2}\Big)\xi_1, \xi_\perp\bigg).$$

Therefore, for every $\sigma \in \mathbb{S}^{N-1}$ (with a choice of $j \in \{1,\dots,N\}$ such that $\sigma_j \neq 0$),

$$\begin{aligned}\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R^c_{k,l}}(\xi)e^{i\sigma.\xi}d\xi =& \frac{(1-\frac{c^2}{2})^{\frac{N\delta_{j,1}-\delta_{k,1}}{2}}}{2}\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R_{k,l}}\bigg(\sqrt{1-\frac{c^2}{2}}\xi_1, \xi_\perp\bigg)e^{i\sigma.\xi}d\xi \\ =& \frac{(1-\frac{c^2}{2})^{\frac{N\delta_{j,1}-\delta_{k,1}-1}{2}}}{2}\int_{|\xi|^2-\frac{c^2}{2}|\xi_\perp|^2>1-\frac{c^2}{2}}\partial_j^{N+1}\widehat{R_{k,l}}(\xi)e^{ir_\sigma\sigma'.\xi}d\xi,\end{aligned}$$

where

$$r_\sigma = \sqrt{\frac{2-c^2+c^2\sigma_1^2}{2-c^2}},$$

and

$$\sigma' = \frac{1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{1}{2}}}\Big(\sigma_1, \sqrt{1-\frac{c^2}{2}}\sigma_\perp\Big).$$

Finally, since the function $\partial_j^{N+1}\widehat{R_{k,l}}$ is a homogeneous rational fraction of degree $-N-1$, we deduce from the change of variables $u = r_\sigma \xi$,

$$\frac{1}{\sigma_j^{N+1}}\int_{B(0,1)^c} \partial_j^{N+1}\widehat{R_{k,l}^c}(\xi)e^{i\sigma.\xi}d\xi = \frac{1}{2r_\sigma^N {\sigma'_j}^{N+1}(1-\frac{c^2}{2})^d}\int_{\Omega_{c,\sigma}} \partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du, \quad (172)$$

where $\Omega_{c,\sigma} = \{u \in \mathbb{R}^N, |u|^2 - \frac{c^2}{2}|u_\perp|^2 > r_\sigma^2(1-\frac{c^2}{2})\}$ and $d = \frac{\delta_{j,1}+\delta_{k,1}+1}{2}$. By the same changes of variables, we compute

$$\frac{1}{\sigma_j^{N+1}}\int_{\mathbb{S}^{N-1}} \xi_j\partial_j^{N}\widehat{R_{k,l}^c}(\xi)e^{i\sigma.\xi}d\xi = \frac{1}{2r_\sigma^N {\sigma'_j}^{N+1}(1-\frac{c^2}{2})^d}\int_{\Lambda_{c,\sigma}} \nu_j(u)\partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du, \quad (173)$$

where $\Lambda_{c,\sigma} = \{u \in \mathbb{R}^N, |u|^2 - \frac{c^2}{2}|u_\perp|^2 = r_\sigma^2(1-\frac{c^2}{2})\}$, and ν_j is the j^{th}-component of the outward normal of the hypersurface $\Lambda_{c,\sigma}$. Likewise, we obtain

$$\frac{1}{\sigma_j^{N}}\int_{\mathbb{S}^{N-1}} \xi_j\partial_j^{N-1}\widehat{R_{k,l}^c}(\xi)e^{i\sigma.\xi}d\xi = \frac{1}{2r_\sigma^N {\sigma'_j}^{N}(1-\frac{c^2}{2})^d}\int_{\Lambda_{c,\sigma}} \nu_j(u)\partial_j^{N-1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du, \quad (174)$$

and

$$\frac{1}{\sigma_j^{N}}\int_{B(0,1)} \partial_j^{N}\widehat{R_{k,l}^c}(\xi)(e^{i\sigma.\xi}-1)d\xi = \frac{1}{2r_\sigma^N {\sigma'_j}^{N}(1-\frac{c^2}{2})^d}\int_{\Omega_{c,\sigma}^c} \partial_j^{N}\widehat{R_{k,l}}(u)(e^{i\sigma'.u}-1)du. \quad (175)$$

In particular, it follows from equations (172), (173), (174) and (175) that

$$\begin{aligned} I_{j,k,l}^c(\sigma) =& \frac{1}{2r_\sigma^N {\sigma'_j}^{N}(1-\frac{c^2}{2})^d}\Big(\frac{i}{\sigma'_j}\int_{\Omega_{c,\sigma}} \partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du + \frac{i}{\sigma'_j}\int_{\Lambda_{c,\sigma}} \nu_j(u)\partial_j^{N+1}\widehat{R_{k,l}}(u) \\ & e^{i\sigma'.u}du + \int_{\Lambda_{c,\sigma}} \nu_j(u)\partial_j^{N-1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du + \int_{\Omega_{c,\sigma}^c} \partial_j^{N}\widehat{R_{k,l}}(u)(e^{i\sigma'.u}-1)du\Big), \end{aligned}$$

so, by integrating by parts,

$$\begin{aligned} I_{j,k,l}^c(\sigma) =& \frac{1}{2r_\sigma^N {\sigma'_j}^{N}(1-\frac{c^2}{2})^d}\Big(\frac{i}{\sigma'_j}\int_{B(0,1)^c} \partial_j^{N+1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du + \frac{i}{\sigma'_j}\int_{\mathbb{S}^{N-1}} \nu_j(u)\partial_j^{N+1}\widehat{R_{k,l}}(u) \\ & e^{i\sigma'.u}du + \int_{\mathbb{S}^{N-1}} \nu_j(u)\partial_j^{N-1}\widehat{R_{k,l}}(u)e^{i\sigma'.u}du + \int_{B(0,1)} \partial_j^{N}\widehat{R_{k,l}}(u)(e^{i\sigma'.u}-1)du\Big). \end{aligned}$$

Finally, by Theorem 6, this yields

$$\begin{aligned} \frac{i^N}{(2\pi)^N}I_{j,k,l}^c(\sigma) =& \frac{\Gamma(\frac{N}{2})}{4r_\sigma^N\pi^{\frac{N}{2}}(1-\frac{c^2}{2})^d}(\delta_{k,l}-\sigma'_k\sigma'_l) \\ =& \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-1-\delta_{k,1}-\delta_{l,1}}{2}}}{4\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Big(\delta_{k,l} - \Big(1-\frac{c^2}{2}\Big)^{1-\frac{\delta_{k,1}+\delta_{l,1}}{2}}\frac{\sigma_k\sigma_l}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\Big). \end{aligned} \quad (176)$$

We then deduce from equations (167), (168) and (169) the formulae

$$K_{0,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-3}{2}}c^2}{8\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\left(1-\frac{N\sigma_1^2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\right), \quad (177)$$

$$K_{j,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-1}{2}}}{4\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Bigg(\delta_{j,1}\Big(1-\frac{c^2}{2}\Big)^{-\frac{\delta_{j,1}+1}{2}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}}\sigma_1\sigma_j}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\Bigg), \quad (178)$$

and

$$L_{j,k,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})}{2c^2\pi^{\frac{N}{2}}}\Bigg(\Big(1-\frac{c^2}{2}\Big)^{\frac{N}{2}}\Big(\frac{\delta_{j,k}(1-\frac{c^2}{2})^{-\frac{\delta_{j,1}+\delta_{k,1}+1}{2}}}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}-\delta_{k,1}+\frac{1}{2}}\sigma_j\sigma_k}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big)$$
$$- \delta_{j,k} + N\sigma_j\sigma_k\Bigg). \quad (179)$$

On the other hand, we already computed in [26] that

$$\int_{\mathbb{R}^N} F(x)dx = 2((4-N)E(v) + c(N-3)p(v)), \quad (180)$$

and

$$\int_{\mathbb{R}^N} G_k(x)dx = 2P_k(v) \quad (181)$$

(see the remark of Subsection 2.3 of [26]). Therefore, by formulae (164), (165), (177), (178), (179), (180) and (181), we conclude that

$$\eta_\infty(\sigma) = \frac{c\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Bigg(\Big(\frac{4-N}{2}cE(v) + \Big(2+\frac{N-3}{2}c^2\Big)p(v)\Big)\Big(\frac{1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}$$
$$- \frac{N\sigma_1^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big) - 2\Big(1-\frac{c^2}{2}\Big)\sum_{j=2}^{N} P_j(v)\frac{N\sigma_1\sigma_j}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Bigg),$$

and

$$v_\infty(\sigma) = \theta_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Bigg(\Big(\frac{4-N}{2}cE(v) + \Big(2+\frac{N-3}{2}c^2\Big)p(v)\Big)$$
$$\frac{\sigma_1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + 2\Big(1-\frac{c^2}{2}\Big)\sum_{j=2}^{N} P_j(v)\frac{\sigma_j}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Bigg),$$

which yields formula (150) by equations (151) and (152). □

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