Asymptotic stability of the black soliton for the Gross-Pitaevskii equation

Philippe Gravejat[1] and Didier Smets[2]

May 25, 2015

Abstract

We introduce a new framework for the analysis of the stability of solitons for the one-dimensional Gross-Pitaevskii equation. In particular, we establish the asymptotic stability of the black soliton with zero speed.

1 Introduction

We pursue our analysis of the one-dimensional Gross-Pitaevskii equation

$$i\partial_t\Psi + \partial_{xx}\Psi + \Psi\big(1 - |\Psi|^2\big) = 0, \tag{GP}$$

for a function $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, supplemented with the boundary condition at infinity

$$|\Psi(x,t)| \to 1, \quad \text{as} \quad |x| \to +\infty. \tag{1}$$

The Gross-Pitaevskii equation was introduced in [40, 25] as a model for the Bose-Einstein condensation. In nonlinear optics, it appears as an envelope equation in optical fibers [27]. In dimension one, it gives account of the propagation of dark pulses in slab waveguides. The boundary condition in (1) corresponds to the non-zero background.

On a mathematical level, the Gross-Pitaevskii equation is a defocusing nonlinear Schrödinger equation. Its Hamiltonian is the Ginzburg-Landau energy defined by

$$E(\Psi) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x\Psi|^2 + \frac{1}{4}\int_{\mathbb{R}} (1 - |\Psi|^2)^2.$$

In the sequel, we only consider the solutions Ψ to (GP) with finite Ginzburg-Landau energy, i.e. in the energy space

$$\mathcal{E}(\mathbb{R}) := \big\{\Psi : \mathbb{R} \to \mathbb{C}, \text{ s.t. } \Psi' \in L^2(\mathbb{R}) \text{ and } 1 - |\Psi|^2 \in L^2(\mathbb{R})\big\}.$$

Under this assumption, the boundary condition in (1) is fulfilled due to the Sobolev embedding theorem.

The constant functions with unitary modulus are the simplest examples of finite energy solutions. A linearisation around these constants provides the dispersion relation

$$\omega^2 = k^4 + 2k^2. \tag{2}$$

[1]Centre de Mathématiques Laurent Schwartz, École polytechnique, 91128 Palaiseau Cedex, France. E-mail: philippe.gravejat@polytechnique.edu

[2]Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte Courrier 187, 75252 Paris Cedex 05, France. E-mail: didier.smets@upmc.fr

For high wave numbers, this relation is similar to the dispersion relation of the linear Schrödinger equation. In contrast, for low wave numbers, it matches with the dispersion relation of the linear wave equation with speed $c_s = \sqrt{2}$. The characteristic speed c_s is called the sound speed. As a consequence of (2), the absolute value of the group velocity is always strictly larger than c_s. Roughly speaking, dispersion has at least speed c_s.

The Gross-Pitaevskii equation also owns travelling-wave solutions. The solitons with speed c are special solutions of the form

$$\Psi(x,t) := U_c(x - ct).$$

Their profile U_c are solutions to the ordinary differential equation

$$- icU_c' + U_c'' + U_c\big(1 - |U_c|^2\big) = 0. \tag{3}$$

The solutions to (3) with finite energy are explicitly known. For $|c| \geq \sqrt{2}$, there are no non-constant solutions. In other words, there is no common speed for solitons and dispersion. In contrast, for $|c| < \sqrt{2}$, the non-constant solutions are uniquely given by the formula

$$U_c(x) := \Big(\frac{2 - c^2}{2}\Big)^{\frac{1}{2}} \tanh\Big(\frac{(2 - c^2)^{\frac{1}{2}}}{2} x\Big) + i\frac{c}{\sqrt{2}}, \tag{4}$$

up to the invariances of the problem, i.e. multiplication by a constant of modulus one and translation. Solitons U_c with speed $c \neq 0$ do not vanish on $\mathbb{R}$. They are called dark solitons, with reference to nonlinear optics where $|\Psi|^2$ refers to the intensity of light. Instead, U_0 is called the black soliton.

In dimension one, the Gross-Pitaevskii equation is integrable by means of the inverse scattering method [48]. At least formally, this method provides a description of the long-time dynamics, which is governed by solitons and dispersion. More precisely, the solutions are expected to behave as a chain of solitons plus a dispersive part (see e.g. [45, 44]). A first step in order to derive rigorously this long-time description is to establish the stability of single solitons and chains of solitons.

This issue was mostly solved in a series of recent papers. The orbital stability of dark solitons was derived in [28] (see also [1]), whereas the case of the black soliton was solved in [3, 22]. More recently, the asymptotic stability of dark solitons was proved in [5]. Concerning chains of solitons, their orbital stability was established in [4], when the solitons in the chain have non-zero speed, are well-separated at initial time, and are ordered according to their speed.

Most of these results rely deeply on an alternative formulation of the Gross-Pitaevskii equation. Provided a solution Ψ does not vanish, it may be lifted as $\Psi := \varrho \exp i\varphi$, where $\varrho := |\Psi|$. The functions $\eta := 1 - \varrho^2$ and $v := -\partial_x\varphi$ are solutions, at least formally, to the hydrodynamical system

$$\begin{cases} \partial_t\eta = \partial_x\big(2\eta v - 2v\big), \\ \partial_t v = \partial_x\Big(v^2 - \eta + \partial_x\Big(\dfrac{\partial_x\eta}{2(1-\eta)}\Big) - \dfrac{(\partial_x\eta)^2}{4(1-\eta)^2}\Big). \end{cases}$$

In this hydrodynamical framework, the Ginzburg-Landau energy is equal to

$$E(\eta, v) = \frac{1}{8}\int_{\mathbb{R}} \frac{(\partial_x\eta)^2}{1-\eta} + \frac{1}{2}\int_{\mathbb{R}} (1-\eta)v^2 + \frac{1}{4}\int_{\mathbb{R}} \eta^2.$$

It is natural to define the energy space for the hydrodynamical Gross-Pitaevskii equation as the open subset of $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ given by

$$\mathcal{NV}(\mathbb{R}) := \Big\{(\eta, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{\mathbb{R}} \eta < 1\Big\}.$$

The corresponding functional framework is substantially simpler than the one corresponding to the energy space $\mathcal{E}(\mathbb{R})$. Moreover, it is straightforward to define another conserved quantity, the momentum P given by

$$P(\eta, v) = \frac{1}{2}\int_{\mathbb{R}} \eta v, \tag{5}$$

which plays a crucial role in the analysis of the stability of the solitons.

On the other hand, a major drawback of the hydrodynamical formulation lies in the fact that it only describes non-vanishing solutions. In particular, the black soliton U_0 is excluded from the analysis.

The main goal of this paper is to revisit the stability of solitons getting rid of the hydrodynamical formulation. In particular, this makes possible the proof of the asymptotic stability of the black soliton.

1.1 Statement of the main results

Our first concern lies in introducing a suitable functional setting to handle with the stability of solitons. In this direction, a natural metric structure is provided by the distance

$$d(\Psi_1, \Psi_2) := \big\|\Psi_1 - \Psi_2\big\|_{L^\infty} + \big\|\Psi_1' - \Psi_2'\big\|_{L^2} + \big\|\eta_1 - \eta_2\big\|_{L^2},$$

where we have set $\eta_j := 1 - |\Psi_j|^2$ for $j \in \{1, 2\}$. For this distance, the Ginzburg-Landau energy is continuous on $\mathcal{E}(\mathbb{R})$. Moreover, Zhidkov [49] proved the existence of a unique global solution $\Psi \in \mathcal{C}^0(\mathbb{R}, \mathcal{E}(\mathbb{R}))$ to (GP) for any initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R})$ (see also [21]).

However, the distance d does not seem well-adapted for our purpose. In the hydrodynamical formulation, orbital stability is quantified by an H^1-control on the perturbation of the variable η and an L^2-control on the perturbation of v in the functional framework provided by the space $\mathcal{NV}(\mathbb{R})$. In this space, the maps (η_n, v_n) corresponding to the functions Ψ_n given by $\Psi_n(x) = \exp(i\,(1 + x^2)^{1/8}/n)$ converge to the zero pair $(0, 0)$, as $n \to +\infty$. This does not remain for the distance d. The first term in the definition of d prevents the maps Ψ_n from tending to a constant map of modulus one when $n \to +\infty$. Roughly speaking, the distance d provides too much control on the slow oscillations at infinity.

In view of the hydrodynamical situation, we do not expect such a control concerning the stability of solitons. This is why we introduce an alternative metric structure on $\mathcal{E}(\mathbb{R})$. Given a number $c \in (-\sqrt{2}, \sqrt{2})$, we first consider the weighted Sobolev space

$$\mathcal{H}_c(\mathbb{R}) := \big\{f \in \mathcal{C}^0(\mathbb{R}, \mathbb{C}),\ \text{s.t. } f' \in L^2(\mathbb{R}) \text{ and } (1 - |U_c|^2)^{1/2} f \in L^2(\mathbb{R})\big\},$$

which we endow with the Hilbert structure corresponding to the norm

$$\|f\|_{\mathcal{H}_c} := \Big(\int_{\mathbb{R}} \big(|f'|^2 + (1 - |U_c|^2)|f|^2\big)\Big)^{\frac{1}{2}}.$$

Due to the exponential decay of the functions $1-|U_c|^2$ on one hand, and the $1/2$-Hölder continuity of the maps in $\mathcal{E}(\mathbb{R})$ on the other hand, all the norms $\|\cdot\|_{\mathcal{H}_c}$ are equivalent. As a consequence, the space $\mathcal{H}_c(\mathbb{R})$ does not depend on c. For simplicity, we set $\mathcal{H}(\mathbb{R}) := \mathcal{H}_c(\mathbb{R})$.

Moreover, the energy space $\mathcal{E}(\mathbb{R})$ appears as the subset of $\mathcal{H}(\mathbb{R})$ given by

$$\mathcal{E}(\mathbb{R}) = \big\{\Psi \in \mathcal{H}(\mathbb{R}),\ \text{s.t. } \eta := 1 - |\Psi|^2 \in L^2(\mathbb{R})\big\}.$$

In particular, we can endow it with the metric structure corresponding to the distances

$$d_c(\Psi_1, \Psi_2) := \Big(\big\| \Psi_1 - \Psi_2 \big\|_{\mathcal{H}_c}^2 + \big\| \eta_1 - \eta_2 \big\|_{L^2}^2 \Big)^{\frac{1}{2}}.$$

Note that the corresponding topology is weaker than the one provided by the distance d. As a consequence, the Ginzburg-Landau energy remains continuous on $\mathcal{E}(\mathbb{R})$, and the Cauchy problem remains globally well-posed. More precisely, the unique global solution Ψ to (GP) corresponding to an initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R})$ remains continuous from $\mathbb{R}$ to $\mathcal{E}(\mathbb{R})$ endowed with the metric structure induced by the distances d_c.

Our motivation for introducing the distances d_c originates in the Taylor formula for the energy E around a soliton U_c. Given a function $\varepsilon \in \mathcal{E}(\mathbb{R})$, we check that

$$\begin{aligned} E(U_c + \varepsilon) = & E(U_c) + \int_{\mathbb{R}} \Big(\langle U_c', \varepsilon' \rangle_{\mathbb{C}} - (1 - |U_c|^2) \langle U_c, \varepsilon \rangle_{\mathbb{C}} \Big) \\ & + \frac{1}{2} \int_{\mathbb{R}} \Big(\big|\varepsilon'\big|^2 - (1 - |U_c|^2)|\varepsilon|^2 + \frac{1}{2} \eta_\varepsilon^2 \Big), \end{aligned} \tag{6}$$

where we have set $\eta_\varepsilon := |U_c + \varepsilon|^2 - |U_c| = 2\langle U_c, \varepsilon \rangle_{\mathbb{C}} + |\varepsilon|^2$. An important step in order to establish the orbital stability of solitons is to provide some coercivity for the quantity

$$Q_c(\varepsilon) = \frac{1}{2} \int_{\mathbb{R}} \Big(\big|\varepsilon'\big|^2 - (1 - |U_c|^2)|\varepsilon|^2 + \frac{1}{2} \eta_\varepsilon^2 \Big),$$

in the right-hand side of (6) (see Proposition 1 for a more precise statement). This can only be done for a metric structure which respects the special form of Q_c. In this respect, the natural structure is given by the distances d_c.

We now come to our main results. The first one gives some further details concerning the orbital stability of the black soliton.

Theorem 1. *Given a map $\Psi^0 \in \mathcal{E}(\mathbb{R})$, we consider the unique solution Ψ to* (GP) *with initial datum Ψ^0. There exist two positive numbers α_* and A_* such that, if*

$$\alpha^0 := d_0(\Psi^0, U_0) < \alpha_*, \tag{7}$$

then there exist two functions $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$\big|a'(t)\big| + \big|\theta'(t)\big| < A_* \alpha^0, \tag{8}$$

and

$$d_0\big(e^{-i\theta(t)}\, \Psi(\cdot + a(t), t), U_0\big) < A_* \alpha^0, \tag{9}$$

for any $t \in \mathbb{R}$.

As mentioned previously in this introduction, the orbital stability of the black soliton U_0 was first proved, on one hand, in [3] by applying the variational method introduced in [11], on the other hand, in [22] by making use of the integrability by the inverse scattering transform of the one-dimensional Gross-Pitaevskii equation.

The proof of Theorem 1 relies on a third approach, which was introduced in [46, 47] and then generalized in [23, 24]. The main ingredient in the proof is to establish the coercivity of the quantity $Q_c(\varepsilon)$, when the function ε satisfies suitable orthogonality conditions (see Proposition 1 below). These orthogonality conditions are guaranteed by the introduction of modulation parameters (see Proposition 2 below). This third approach presents the advantage to provide a better control on the perturbation with respect to the soliton. Such a control is very useful in order to tackle the asymptotic stability of the black soliton.

An important difficulty in applying rigorously this third strategy lies in the property that the functional $Q_c(\varepsilon)$ does not depend quadratically on the variable ε. As a consequence, it does not seem possible to derive its coercivity from standard spectral theory. We refer to Subsection 1.2 for more details concerning this issue, and more generally, regarding the main elements in the proof of Theorem 1.

Our second result concerns the asymptotic stability of the black soliton.

Theorem 2. *Given a map $\Psi^0 \in \mathcal{E}(\mathbb{R})$, we consider the unique solution Ψ to* (GP) *with initial datum Ψ^0. There exists a positive number $\beta_* \leq \alpha_*$ such that, if*

$$d_0(\Psi^0, U_0) < \beta_*,$$

then there exist a number $c_* \in (-\sqrt{2}, \sqrt{2})$, and two functions $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$a'(t) \to c_*, \quad \text{and} \quad \theta'(t) \to 0, \tag{10}$$

as $t \to +\infty$, and for which we have

$$e^{-i\theta(t)}\,\Psi(\cdot + a(t), t) \rightharpoonup U_{c_*} \text{ in } \mathcal{H}(\mathbb{R}), \quad \text{and} \quad 1 - \big|\Psi(\cdot + a(t), t)\big|^2 \rightharpoonup 1 - \big|U_{c_*}\big|^2 \text{ in } L^2(\mathbb{R}), \tag{11}$$

as $t \to +\infty$. In particular, we have

$$e^{-i\theta(t)}\,\Psi(\cdot + a(t), t) \to U_{c_*} \text{ in } L^\infty_{\text{loc}}(\mathbb{R}), \tag{12}$$

as $t \to +\infty$.

Remark. A natural question about the position a and the phase θ concerns the existence of possible limits for the quantities $a(t) - c_*t$ and $\theta(t)$, when $t \to +\infty$. We believe that the answer to this question is negative, unless additional assumptions are made on the initial perturbation. As a matter of fact, this property has been proved to be false in the context of the Korteweg-de Vries equation (see [33, Theorem 2]).
Concerning the limit speed c_*, it is controlled by the initial distance $d_0(\Psi^0, U_0)$ between the initial datum Ψ^0 and the black soliton U^0. This property is a direct consequence of orbital stability (see (33) and (36) below). In particular, the limit speed c_* converges to 0 as the initial perturbation tends to 0.

In contrast with orbital stability, which expresses the fact that the solution remains close to the family of black solitons corresponding to the geometric invariances of the Gross-Pitaevskii equation, asymptotic stability provides the convergence of the solution towards a special orbit in this family.

A crucial issue when dealing with this further notion of stability lies in the nature of the convergence. As a matter of fact, it is not possible to prove a strong convergence in the energy space. Indeed, orbital stability, since it holds both forward and backward in time, would then guarantee that the solution is exactly a black soliton. As a consequence, one has to weaken the notion of convergence in order to establish asymptotic stability.

In this direction, a natural choice is to show a weak convergence in the energy space. This is exactly the main statement of Theorem 2. As a consequence of the Sobolev embedding theorem, the convergence is also locally uniform in the reference frame of the limit soliton. Due to the possible presence of additional small solitons, this local uniform convergence cannot be improved into a global uniform convergence. On the other hand, it is possible that the solution converges in $H^1_{\mathrm{loc}}(\mathbb{R})$ towards a soliton. Martel and Merle [33] proved such a convergence in the context of the Korteweg-de Vries equation. To our knowledge, this question still remains open for the Gross-Pitaevskii equation.

Asymptotic stability originates in the property that the perturbation of the soliton disperses at infinity in the reference frame of the limit soliton. As a consequence, a natural strategy in order to prove it is to establish dispersive estimates for the linearized equation around the soliton, and then to implement a fixed point argument in suitable function spaces. In the context of nonlinear Schrödinger equations with potential, this first strategy was implemented by Soffer and Weinstein in [41, 42, 43] for proving the asymptotic stability of ground states (see also [20]). It was then extended to various equations including the generalized Korteweg-de Vries equations [39, 38], and the nonlinear Schrödinger equations without potential (see e.g. [8, 9, 10, 14]). We refer to [15] for a detailed survey about those and related works.

This first strategy describes the limit behaviour of the solution as the superposition of a soliton and a dispersive perturbation. In general, the long-time dynamics is more complicated. In particular, it is well-known that multi-soliton solutions play a major role in the long-time dynamics (see e.g [36, 37, 4]). This limitation originates in a priori spectral assumptions, or the use of weighted spaces for the initial perturbation, in order to perform this first strategy rigorously.

In a series of papers, Martel and Merle [31, 32, 30, 33, 35, 34] proposed an alternative strategy in order to establish the asymptotic stability of solitons for the generalized Korteweg-de Vries equations. They rely on monotonicity formulae to establish the compactness of a limit profile, and then classify the compact solutions to the Korteweg-de Vries equations in the neighbourhood of solitons. This second strategy presents the advantages not to require additional a priori assumptions, and to apply to multi-soliton solutions (see e.g. [36]). It was extended to various equations including the Benjamin-Bona-Mahony equation [17] and the Benjamin-Ono equation [26].

In [5], we relied on this second strategy in order to prove the asymptotic stability of the dark solitons for the Gross-Pitaevskii equation. As mentioned previously, this was performed in the hydrodynamical setting, so that we were not able to handle with the black soliton. Here, we by-pass this limitation by working directly in the Schrödinger setting, to the expense of a functional setting which involve completely nonlinear quantities.

Another motivation for working in the Schrödinger setting comes from the focusing nonlinear Schrödinger equations. In general, they also own solitons which are supposed to play a major role in their long-time dynamics. However, in this case, the description of the long-time dynamics is certainly more intricate, due to the existence of breathers which prevent asymptotic stability in the energy space.

In another direction, note that the Gross-Pitaevskii equation also owns travelling waves in higher dimension (see e.g. [7, 6, 2, 29]). Likewise, they are supposed to play an important role in the long-time dynamics. Their orbital stability was investigated in [13]. To our knowledge, their asymptotic stability still remains an open problem.

Finally, it would be of interest to investigate further the global basin of attraction of solitons or multi-solitons (in dimension one or higher), in particular having in mind the so-called "soliton resolution conjecture". Some results in that direction have been obtained for the integrable one dimensional Gross-Pitaevskii equation using Deift-Zhou's steepest descent method [45, 44, 16]. Let us emphasize that our proof of asymptotic stability does not rely on the integrability by means of the inverse scattering transform of the one-dimensional Gross-Pitaevskii equation. As a consequence, our result presumably extends to nonlinearities for which the equation does not remain integrable (see e.g. [12] for examples of possible nonlinearities).

In the remaining part of this introduction, we present the main ingredients leading to the proof of Theorems 1 and 2.

1.2 Main elements in the proof of Theorem 1

Orbital stability results from a variational characterization for U_0. The function U_0 is the unique minimizer of the variational problem

$$E(U_0) = \min \Big\{ E(\Psi),\ \Psi \in \mathcal{E}(\mathbb{R}) \text{ s.t. } [P](\Psi) = \frac{\pi}{2} \text{ mod } \pi \Big\}, \tag{13}$$

up to the geometric invariances of the equations, i.e. translations and multiplication by constants of modulus one. In this expression, the functional $[P]$ refers to a renormalized version of the momentum P. In (5), the quantity $P(\Psi)$ is defined in the hydrodynamical framework, where the function Ψ does not vanish. Extending the definition to functions which possibly vanish is not so immediate. In [3], we introduced the following renormalized version of the momentum

$$[P](\Psi) := \lim_{R_1, R_2 \to +\infty} \frac{1}{2} \int_{-R_1}^{R_2} \langle i\Psi, \Psi' \rangle_{\mathbb{C}} - \frac{1}{2}\big(\varphi(R_2) - \varphi(-R_1)\big) \quad \text{mod } \pi.$$

In this definition, φ stands for a phase function at infinity for Ψ. Indeed, when Ψ lies in $\mathcal{E}(\mathbb{R})$, the Sobolev embedding theorem implies that $|\Psi(x)| \to 1$ as $|x| \to +\infty$. As a consequence, the phase of Ψ is well-defined modulo 2π on intervals of the form $(-\infty, -R_1)$ and $(R_2, +\infty)$ for R_1 and R_2 large enough. In particular, the renormalized momentum $[P](\Psi)$ is only defined modulo π due to the ambiguity on the phase function φ.

We will provide more details on the momentum in Propositions 4 and 5. We first go on with the orbital stability of U_0.

As a consequence of the variational characterization in (13), the soliton U_0 is a critical point of the functional E. Due to the minimizing nature of U_0, this quantity actually provides a control on a large class of small perturbations of U_0. More precisely, we can establish

Proposition 1. *Let $\varepsilon \in \mathcal{H}(\mathbb{R})$, with $U_0 + \varepsilon \in \mathcal{E}(\mathbb{R})$, and set $\eta_\varepsilon := 2\langle U_0, \varepsilon \rangle_{\mathbb{C}} + |\varepsilon|^2$. There exists a universal positive number Λ_0 such that*

$$E(U_0 + \varepsilon) - E(U_0) \geq \Lambda_0 \left(\|\varepsilon\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon\|_{L^2}^2 \right) - \frac{1}{\Lambda_0} \|\varepsilon\|_{\mathcal{H}_0}^3, \tag{14}$$

as soon as

$$\int_{\mathbb{R}} \langle \varepsilon, U_0' \rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, iU_0' \rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, iU_0 \rangle_{\mathbb{C}} \big(1 - |U_0|^2\big) = 0. \tag{15}$$

The orbital stability of U_0 is a consequence of the coercivity inequality in (14). As a matter of fact, consider a solution Ψ to (GP) and decompose it as $\Psi(\cdot, t) = U_0 + \varepsilon(\cdot, t)$ for any $t \in \mathbb{R}$. Due to the conservation of the energy, the quantity $E(U_0 + \varepsilon(\cdot, t)) - E(U_0)$ remains small at any time if the initial datum Ψ^0 is close to the soliton U_0. In view of (14), the quantity $\|\varepsilon\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon\|_{L^2}^2$ remains small for all time, which gives the orbital stability of U_0.

In order to apply this argument, we first have to guarantee the orthogonality conditions in (15). As usual in such a situation, we introduce modulation parameters. Given a function $\Psi \in \mathcal{H}(\mathbb{R})$, which lies in a neighbourhood of the orbit of U_0 of the form

$$\mathcal{U}_0(\alpha) = \Big\{ \Psi \in \mathcal{H}(\mathbb{R}),\ \text{s.t. } \inf_{(a,\theta) \in \mathbb{R}^2} \big\| e^{-i\theta} \Psi(\cdot + a) - U_0 \big\|_{\mathcal{H}_0} < \alpha \Big\},$$

for some positive number α, we decompose it as

$$e^{-i\theta}\Psi(\cdot + a) = U_c + \varepsilon = R_c + iI_c + \varepsilon,$$

and we make the choice of the modulation parameters $(c, a, \theta) \in (-\sqrt{2}, \sqrt{2}) \times \mathbb{R}^2$ such that the remainder ε satisfies the orthogonality conditions

$$\int_{\mathbb{R}} \langle \varepsilon, U_c' \rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, iU_c' \rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, iR_c \rangle_{\mathbb{C}} \big(1 - |U_c|^2\big) = 0. \tag{16}$$

Proposition 2. *There exist two positive numbers α_0 and A_0, and three continuously differentiable functions $\boldsymbol{c} \in \mathcal{C}^1(\mathcal{U}_0(\alpha_0), (-\sqrt{2}, \sqrt{2}))$, $\boldsymbol{\theta} \in \mathcal{C}^1(\mathcal{U}_0(\alpha_0), \mathbb{R}/2\pi\mathbb{Z})$ and $\boldsymbol{a} \in \mathcal{C}^1(\mathcal{U}_0(\alpha_0), \mathbb{R})$ such that for any $\Psi \in \mathcal{U}_0(\alpha_0)$, the function*

$$\varepsilon := e^{-i\boldsymbol{\theta}(\Psi)}\Psi(\cdot + \boldsymbol{a}(\Psi)) - U_{\boldsymbol{c}(\Psi)}, \tag{17}$$

satisfies the orthogonality conditions in (16)*. Moreover, if*

$$\big\| \Psi - e^{i\theta} U_0(\cdot - a) \big\|_{\mathcal{H}_0} \leq \alpha \leq \alpha_0,$$

for some $(a, \theta) \in \mathbb{R}^2$, then,

$$\big\| \varepsilon \big\|_{\mathcal{H}_0} + \big| \boldsymbol{c}(\Psi) \big| + \big| \boldsymbol{a}(\Psi) - a \big| + \big| e^{i\boldsymbol{\theta}(\Psi)} - e^{i\theta} \big| \leq A_0 \alpha. \tag{18}$$

Concerning Proposition 2, we observe that the orthogonality conditions in (16) are generalizations of the ones in (15) through the introduction of a modulation parameter related to the speed c. For that reason, we have to extend the coercivity estimates in (14) to this new framework. In this direction, the following will be derived from Proposition 1.

Corollary 1. *Let $c \in (-\sqrt{2}, \sqrt{2})$. For $\varepsilon \in \mathcal{H}(\mathbb{R})$, with $U_c + \varepsilon \in \mathcal{E}(\mathbb{R})$, we set $\eta_\varepsilon := 2\langle U_c, \varepsilon \rangle_{\mathbb{C}} + |\varepsilon|^2$. Given any number $\sigma \in (0, \sqrt{2})$, there exists a positive number Λ_σ, depending only on σ, such that*

$$E(U_c + \varepsilon) - E(U_0) \geq \Lambda_\sigma \left(\|\varepsilon\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon\|_{L^2}^2 \right) - \frac{1}{\Lambda_\sigma} \left(c^2 + \|\varepsilon\|_{\mathcal{H}_0}^3 \right), \tag{19}$$

as soon as $|c| \leq \sigma$, and ε satisfies the orthogonality conditions in (16)*.*

Another remark regarding Proposition 2 lies in the property that the modulation parameters for a solution $\Psi(\cdot, t)$ necessarily depend on time. In particular, we need to control their evolution along the flow of the Gross-Pitaevskii equation. In this direction, we rely on a standard continuation argument.

We first invoke the continuity of the (GP) flow in $\mathcal{E}(\mathbb{R})$. We choose a positive number α to be fixed later. When the initial datum Ψ^0 satisfies the condition $\alpha^0 := d_0(\Psi^0, U_0) < \alpha$, we can find a positive time T such that $\Psi(\cdot, t)$ lies in the set

$$\mathcal{V}_0(\alpha) := \Big\{ \Psi \in \mathcal{E}(\mathbb{R}), \text{ s.t. } \inf_{(a,\theta) \in \mathbb{R}^2} d_0\big(e^{-i\theta}\Psi(\cdot + a), U_0\big) < \alpha \Big\}, \tag{20}$$

for any $t \in (-T, T)$. Our final goal is to establish that we can fix α small enough such that the solution $\Psi(\cdot, t)$ remains in $\mathcal{V}_0(\alpha)$ for any $t \in \mathbb{R}$.

We first assume that $\alpha < \alpha_0$, where α_0 is defined in Proposition 2. In this case, we can define modulation parameters for $\Psi(\cdot, t)$ by setting $(c(t), a(t), \theta(t)) = (\boldsymbol{c}(\Psi(\cdot, t)), \boldsymbol{a}(\Psi(\cdot, t)), \boldsymbol{\theta}(\Psi(\cdot, t)))$ for any $t \in (-T, T)$. In this definition, the function θ is a priori valued in $\mathbb{R}/2\pi\mathbb{Z}$. However, the map $t \mapsto \boldsymbol{\theta}(\Psi(\cdot, t))$ is continuous from $(-T, T)$ to $\mathbb{R}/2\pi\mathbb{Z}$. As a consequence, we can define the function θ as a continuous real valued function, up to the choice of a constant in $2\pi\mathbb{Z}$. We fix this choice such that $\theta(0)$ lies in $[0, 2\pi)$. In the sequel, the function θ only appears through the function $e^{i\theta}$, or the derivative θ', so that this special choice does not affect our proofs.

We next check the continuous differentiability of the modulation parameters with respect to time (see Proposition 3 below). In particular, we are allowed to write the equation satisfied by the perturbation $\varepsilon(\cdot, t) := e^{-i\theta(t)}\Psi(\cdot + a(t)) - U_{c(t)}$, which is given by

$$\begin{aligned}\partial_t \varepsilon = & - c'(t)\partial_c U_{c(t)} - i\theta'(t)\big(U_{c(t)} + \varepsilon\big) + \big(a'(t) - c(t)\big)\big(\partial_x U_{c(t)} + \partial_x \varepsilon\big) \\ & + i\Big(\partial_{xx}\varepsilon - ic(t)\partial_x\varepsilon + \eta_{c(t)}\varepsilon - \eta_\varepsilon\big(U_{c(t)} + \varepsilon\big)\Big),\end{aligned} \tag{21}$$

with $\eta_\varepsilon(\cdot, t) := 2\langle U_{c(t)}, \varepsilon(\cdot, t)\rangle_{\mathbb{C}} + |\varepsilon(\cdot, t)|^2$. Differentiating the orthogonality conditions in (16), we derive from (21) the following control on the modulation parameters. This control eventually provides the estimates on the time derivatives of a and θ in Theorem 1.

Proposition 3. *There exist two positive numbers $\alpha_1 < \alpha_0$ and A_1 such that, if the solution $\Psi(\cdot, t)$ lies in $\mathcal{V}_0(\alpha_1)$ for any $t \in (-T, T)$, then, the functions c, a and θ are of class $\mathcal{C}^1$ on $(-T, T)$, and their derivatives satisfy*

$$|c'(t)| + |a'(t) - c(t)|^2 + |\theta'(t)|^2 \leq A_1 \|\varepsilon(\cdot, t)\|_{\mathcal{H}_0}^2, \tag{22}$$

for any $t \in (-T, T)$.

We next assume that $\alpha < \alpha_1$ so that the estimates in Proposition 3 are available on $(-T, T)$, and we come back to inequality (19). We assume that $A_0\alpha < 1$, where A_0 is defined in Proposition 2, so that $|c(t)| < 1$ for any $t \in (-T, T)$. When α satisfies the further condition $2A_0\alpha < \Lambda_1^2$, we are allowed to use the conservation of the energy and (18) to rephrase (19) as

$$\|\varepsilon(\cdot, t)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot, t)\|_{L^2}^2 \leq \frac{2}{\Lambda_1^2}\Big(\Lambda_1\big(E(\Psi^0) - E(U_0)\big) + c(t)^2\Big), \tag{23}$$

for any $t \in (-T, T)$. On the other hand, we can use the property that U_0 is a critical point of E in order to infer from (6) the existence of a positive number K_0 such that

$$E(\Psi^0) - E(U_0) \leq K_0\big(\alpha^0\big)^2. \tag{24}$$

As a consequence, it only remains to estimate the remainder quantity $c(t)^2$ so as to complete the proof of Theorem 1. In order to bound this term, we rely on the conservation of the momentum, and a Taylor expansion of this quantity in the neighbourhood of the solitons U_c.

In order to compute this expansion, we introduce an alternative definition of the momentum. Given an arbitrary function Ψ in $\mathcal{V}_0(\alpha)$, we can rely on the modulation decomposition provided by Proposition 2 to define the function

$$\Psi_{\rm mod}(x) := e^{-i\theta}\,\Psi(x + a),$$

with $\theta := \boldsymbol{\theta}(\Psi)$ and $a := \boldsymbol{a}(\Psi)$. Combining estimates (18) and definition (20) with the Sobolev embedding theorem, we can also assume that the number α is sufficiently small so that we have

$$\big|\Psi_{\rm mod}(x)\big| \geq \frac{1}{2}, \tag{25}$$

for any $x \in \mathbb{R} \setminus [-1, 1]$. As a consequence, we can define a continuous phase function $\varphi_{\rm mod}$ from the two simply connected components of the set $\mathbb{R} \setminus [-1, 1]$ into $\mathbb{R}$ such that $\Psi_{\rm mod}(x) =$

$\big|\Psi_{\rm mod}(x)\big| \exp i\varphi_{\rm mod}(x)$ for any $x \in \mathbb{R} \setminus [-1,1]$. Decreasing α further if necessary, we may additionally impose (and this makes the choice unique) that

$$\big|\varphi_{\rm mod}(x) - \pi\big| < \frac{\pi}{2} \quad \text{for } x \in [-2,-1], \quad \text{and} \quad \big|\varphi_{\rm mod}(x)\big| < \frac{\pi}{2} \quad \text{for } x \in [1,2]. \tag{26}$$

We now fix a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R},[0,1])$ such that $\chi = 1$ on $[-1,1]$ and $\chi = 0$ outside $(-2,2)$, and we define the momentum as

$$\mathcal{P}(\Psi) := \frac{1}{2} \int_{\mathbb{R}} \Big(\langle i\Psi_{\rm mod}, \partial_x \Psi_{\rm mod} \rangle_{\mathbb{C}} - \partial_x \big((1-\chi)\varphi_{\rm mod} \big) \Big). \tag{27}$$

Similarly to $[P]$, the quantity $\mathcal{P}$ is invariant by translation and multiplication by a complex number of modulus one. Contrary to $[P]$, it is well-defined as an element of $\mathbb{R}$ rather than $\mathbb{R}/\pi\mathbb{Z}$. On the other hand, it is only defined for functions in a tubular neighbourhood of the family of black solitons. As a matter of fact, we have

Proposition 4. *There exists a positive number $\alpha_2 < \alpha_1$ such that the map $\mathcal{P}$ is well-defined from $\mathcal{V}_0(\alpha_2)$ to $\mathbb{R}$, and it satisfies*

$$\mathcal{P}(\Psi) = [P](\Psi) \mod \pi, \tag{28}$$

for any $\Psi \in \mathcal{V}_0(\alpha_2)$.

Coming back to decomposition (17), we can expand the quantity $\mathcal{P}(\Psi)$ with respect to $\varepsilon = \Psi_{\rm mod} - U_{\boldsymbol{c}(\Psi)}$ for ε small enough. More precisely, we show

Proposition 5. *Let $\Psi \in \mathcal{V}_0(\alpha_2)$. Set $\varepsilon := \Psi_{\rm mod} - U_c$, with $c := \boldsymbol{c}(\Psi)$, and $\eta_\varepsilon := 2\langle U_c, \varepsilon\rangle_{\mathbb{C}} + |\varepsilon|^2$. There exists a positive number A_2 such that the momentum $\mathcal{P}(\Psi)$ may be written as*

$$\mathcal{P}(\Psi) = \mathcal{P}(U_c) - \int_{\mathbb{R}} \langle iU_c', \varepsilon\rangle_{\mathbb{C}} + R_c(\varepsilon),$$

with

$$\big|R_c(\varepsilon)\big| \leq A_2 \Big(\|\varepsilon\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon\|_{L^2}^2 \Big). \tag{29}$$

We are now in position to conclude the proof of Theorem 1.

End of the proof of Theorem 1. Recall that so far we have obtained (23) and (24), so that it only remains to control the quantity $c(t)$. Assume that $\alpha < \alpha_2$. In this case, we are allowed to apply Proposition 5 in order to write

$$\mathcal{P}(\Psi(\cdot,t)) = \mathcal{P}(U_{c(t)}) - \int_{\mathbb{R}} \langle iU_{c(t)}', \varepsilon(\cdot,t)\rangle_{\mathbb{C}} + R_{c(t)}(\varepsilon(\cdot,t)),$$

for any $t \in (-T,T)$. In view of the second orthogonality condition in (16), and estimate (29), we conclude that

$$\big|\mathcal{P}(\Psi(\cdot,t)) - \mathcal{P}(U_{c(t)})\big| \leq A_2 \Big(\|\varepsilon(\cdot,t)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot,t)\|_{L^2}^2 \Big).$$

We now combine this estimate with the conservation of the renormalized momentum (see [3, Proposition 1.16]), and with identity (28). Since the left hand side of this identity is continuous in time and well-defined as an element of $\mathbb{R}$, we obtain

$$\mathcal{P}(\Psi(\cdot,t)) = \mathcal{P}(\Psi^0),$$

for any $t \in (-T, T)$. This leads to the inequality

$$\big|\mathcal{P}(U_{c(t)}) - \mathcal{P}(U_{c(0)})\big| \leq A_2\Big(\|\varepsilon(\cdot,t)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot,t)\|_{L^2}^2 + \|\varepsilon(\cdot,0)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot,0)\|_{L^2}^2\Big).$$

In view of (18), we have

$$|c(0)| \leq A_0 \alpha^0, \tag{30}$$

so that it follows from (23) and (24) that

$$\|\varepsilon(\cdot,0)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot,0)\|_{L^2}^2 \leq \frac{2}{\Lambda_1^2}\Big(\Lambda_1 K_0 + A_0^2\Big)\big(\alpha^0\big)^2. \tag{31}$$

On the other hand, we can use the explicit formula for the soliton U_c in (4) to compute

$$\mathcal{P}(U_c) = \frac{\pi}{2} - \arctan\Big(\frac{c}{\sqrt{2-c^2}}\Big) - \frac{c}{2}\sqrt{2-c^2}.$$

Since $|c(t)| < 1$, we are led to

$$|\mathcal{P}(U_{c(t)}) - \mathcal{P}(U_{c(0)})| \geq |c(t) - c(0)|.$$

In view of (30) and (31), this provides

$$|c(t)| \leq A_2\Big(\|\varepsilon(\cdot,t)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot,t)\|_{L^2}^2\Big) + A_0\alpha^0 + \frac{2A_2}{\Lambda_1^2}\Big(\Lambda_1 K_0 + A_0^2\Big)\big(\alpha^0\big)^2.$$

Inserting this inequality, and estimate (24) into (23), we deduce the existence of a positive number A such that

$$d_0\big(e^{-i\theta(t)}\Psi(\cdot + a(t)), U_{c(t)}\big) = \big(\|\varepsilon(\cdot,t)\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon(\cdot,t)\|_{L^2}^2\big)^{\frac{1}{2}} \leq A\alpha^0, \tag{32}$$

for any $t \in (-T, T)$. In particular, we have

$$|c(t)| \leq A\alpha^0, \tag{33}$$

for a further positive number A. It now remains to check that

$$d_0\big(U_0, U_{c(t)}\big) \leq A|c(t)| \leq A\alpha^0,$$

to obtain the final estimate

$$d_0\big(e^{-i\theta(t)}\Psi(\cdot + a(t)), U_0\big) + |c(t)| \leq A\alpha^0, \tag{34}$$

for any $t \in (-T, T)$.

All this is available for a given choice of the number α that we now fix. We next set $\alpha_* := \alpha/A$, where A is the number in (34). When $\alpha^0 < \alpha_*$, we deduce from (34) that the solution Ψ remains in $\mathcal{V}_0(\alpha)$ for any time $t \in (-T, T)$. Applying a standard continuation argument, we conclude that it remains in this set for any time. In particular, estimate (34) is available for any $t \in \mathbb{R}$. This is exactly statement (9) in Theorem 1. Statement (8) is then a consequence of (22), (32) and (34). This concludes the proof of Theorem 1. □

1.3 Main elements in the proof of Theorem 2

1.3.1 Construction of a limit profile

Let Ψ^0 be as in the statement of Theorem 2. Since $\beta_* \leq \alpha_*$ in the assumptions of Theorem 2, we may apply Theorem 1 to the unique globally defined solution Ψ to (GP) with initial datum Ψ^0. As in the proof of Theorem 1, we decompose the solution Ψ as

$$\Psi(x,t) = e^{i\theta(t)}\big(U_{c(t)}(x-a(t)) + \varepsilon(x-a(t),t)\big),$$

according to Proposition 2, and we therefore have estimates (32) and (33) for any $t \in \mathbb{R}$.

We fix an arbitrary sequence of times $(t_n)_{n\in\mathbb{N}}$ tending to $+\infty$. In view of (32) and (33), we may assume, going to a subsequence if necessary, that there exist $\varepsilon_0^* \in \mathcal{H}(\mathbb{R})$ and $c_0^* \in (-\sqrt{2}, \sqrt{2})$ such that

$$\varepsilon(\cdot, t_n) = e^{-i\theta(t_n)}\Psi(\cdot + a(t_n), t_n) - U_{c(t_n)} \rightharpoonup \varepsilon_0^* \quad \text{in } \mathcal{H}(\mathbb{R}), \tag{35}$$

and that

$$c(t_n) \to c_0^*, \tag{36}$$

as $n \to +\infty$. In this situation, since the function $1 - |\Psi(\cdot,t)|^2$ is uniformly bounded in $L^2(\mathbb{R})$ by energy conservation, we can also deduce from the Rellich theorem that

$$1 - \big|e^{-i\theta(t_n)}\Psi(\cdot + a(t_n), t_n)\big|^2 \rightharpoonup 1 - |U_{c_0^*} + \varepsilon_0^*|^2 \quad \text{in } L^2(\mathbb{R}). \tag{37}$$

Our main goal is to obtain the conclusion that necessarily $\varepsilon_0^* \equiv 0$, by establishing smoothness and rigidity properties for the solution Ψ^* to (GP) with initial datum given by $U_{c_0^*} + \varepsilon_0^*$. Also, in order to prove Theorem 2, we will need to show that our conclusions are independent of the choice of the sequence $(t_n)_{n\in\mathbb{N}}$. This will be done in Subsection 1.3.4.

We first deduce from the weak lower semi-continuity of the norm that the function $\Psi_0^* := U_{c_0^*} + \varepsilon_0^*$ satisfies

$$\big\|\Psi_0^* - U_0\big\|_{\mathcal{H}_0} \leq A_*\beta_* + \big\|U_0 - U_{c_0^*}\big\|_{\mathcal{H}_0}.$$

On the other hand, we infer from (33) that we have $|c_0^*| \leq A_*\alpha^0 \leq A_*\beta_*$. Therefore, we can impose a supplementary smallness assumption on β_* so that necessarily

$$d_0(\Psi_0^*, U_0) \leq \alpha_*.$$

Applying Theorem 1 then yields a unique global solution $\Psi^* \in \mathcal{C}^0(\mathbb{R}, (\mathcal{E}(\mathbb{R}), d))$ to (GP) with initial datum Ψ_0^*, and maps $c^* \in \mathcal{C}^1(\mathbb{R}, (-\sqrt{2}, \sqrt{2}))$ and $(a^*, \theta^*) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})^2$ such that the function ε^* defined by

$$\varepsilon^*(\cdot, t) := e^{-i\theta^*(t)}\Psi^*(\cdot + a^*(t), t) - U_{c^*(t)}, \tag{38}$$

satisfies the orthogonality conditions

$$\int_\mathbb{R} \langle \varepsilon^*(\cdot,t), U_{c^*(t)}' \rangle_\mathbb{C} = \int_\mathbb{R} \langle \varepsilon^*(\cdot,t), iU_{c^*(t)}' \rangle_\mathbb{C} = \int_\mathbb{R} \langle \varepsilon^*(t,\cdot), iR_{c^*(t)} \rangle_\mathbb{C} \big(1 - |U_{c^*(t)}|^2\big) = 0. \tag{39}$$

Note that, in view of Proposition 2 and estimate (32), the modulated speed c^* and the perturbation ε^* satisfy the estimate

$$\big|c^*(t)\big| + \|\varepsilon^*(\cdot,t)\|_{\mathcal{H}_0} + \|\eta_{\varepsilon^*}(\cdot,t)\|_{L^2} \leq A_0\beta_*, \tag{40}$$

for any $t \in \mathbb{R}$. Similarly, we deduce from Proposition 3 that

$$\big|(c^*)'(t)\big| + \big|(a^*)'(t) - c^*(t)\big|^2 + \big|(\theta^*)'(t)\big|^2 \leq A_1\|\varepsilon^*(\cdot,t)\|_{\mathcal{H}_0}^2, \tag{41}$$

for any $t \in \mathbb{R}$.

In this situation, we can establish the following weak convergence of the perturbation ε towards the limit perturbation ε^*, as well as of the convergence of the modulation parameters c, a and θ towards the limit parameters c^*, a^* and θ^*. More precisely, we show

Proposition 6. *Let $t \in \mathbb{R}$ be fixed. Then, we have*

$$\begin{aligned} e^{-i\theta(t_n)}\Psi(\cdot + a(t_n), t_n + t) \rightharpoonup \Psi^*(\cdot, t) \quad \text{in } \mathcal{H}(\mathbb{R}), \\ 1 - \left|e^{-i\theta(t_n)}\Psi(\cdot + a(t_n), t_n + t)\right|^2 \rightharpoonup 1 - \left|\Psi^*(\cdot, t)\right|^2 \quad \text{in } L^2(\mathbb{R}), \end{aligned} \tag{42}$$

and

$$a(t_n + t) - a(t_n) \to a^*(t), \quad \theta(t_n + t) - \theta(t_n) \to \theta^*(t) \quad \text{and} \quad c(t_n + t) \to c^*(t), \tag{43}$$

as $n \to +\infty$. In particular, we obtain

$$\begin{aligned} \varepsilon(\cdot, t_n + t) \rightharpoonup \varepsilon^*(\cdot, t) \quad \text{in } \mathcal{H}(\mathbb{R}), \\ 2\langle U_{c(t_n+t)}, \varepsilon(\cdot, t_n + t)\rangle_{\mathbb{C}} + |\varepsilon(\cdot, t_n + t)|^2 \rightharpoonup 2\langle U_{c^*(t)}, \varepsilon^*(\cdot, t)\rangle_{\mathbb{C}} + |\varepsilon^*(\cdot, t)|^2 \quad \text{in } L^2(\mathbb{R}), \end{aligned} \tag{44}$$

as $n \to +\infty$.

Using this limit characterization of the profile Ψ^*, we are able to show its localized and smooth nature.

1.3.2 Localization and smoothness of the limit profile

In this subsection, we consider an arbitrary solution Ψ as in the statement of Theorem 1, which can therefore be uniquely modulated by functions which we denote here again by a, θ and c. By (7) and (9), we have the closeness estimate

$$d_0\big(\Psi_{\rm mod}(\cdot, t), U^0\big) < (A_* + 1)\alpha^0, \tag{45}$$

for any $t \in \mathbb{R}$. Here, we have set, as before, $\Psi_{\rm mod}(x, t) := e^{-i\theta(t)}\, \Psi(x + a(t), t)$. In the sequel, we assume further that the number α^0 (which appears in the statement of Theorem 1) is sufficiently small so that we can write

$$\Psi_{\rm mod}(x, t) = |\Psi_{\rm mod}(x, t)| \, \exp i\varphi_{\rm mod}(x, t),$$

for any $x \in \mathbb{R} \setminus [-1, 1]$ and any $t \in \mathbb{R}$, with phase functions $\varphi_{\rm mod}(\cdot, t)$ which satisfy (26) for any $t \in \mathbb{R}$. As in Proposition 4, we choose and fix a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ such that $\chi = 1$ on $[-1, 1]$ and $\chi = 0$ outside $(-2, 2)$. For arbitrary $R \in \mathbb{R}$ and $t \in \mathbb{R}$, we then define the quantity

$$I_R(t) \equiv I_R^\Psi(t) := \frac{1}{2} \int_{\mathbb{R}} \Big[\langle i\Psi_{\rm mod}, \partial_x \Psi_{\rm mod}\rangle_{\mathbb{C}} - \partial_x\big((1 - \chi)\varphi_{\rm mod}\big)\Big](x, t)\, \Phi(x - R)\, dx,$$

where

$$\Phi(x) := \frac{1}{2}\Big(1 + \tanh\Big(\frac{x}{2}\Big)\Big).$$

The quantity I_R corresponds in rough terms to the amount of momentum starting at a distance R to the right of the soliton. The fact that it is almost increasing in time, as expressed in the following proposition, is a key ingredient for our subsequent analysis (see [5] for an informal discussion regarding the physical interpretation of such a monotonicity).

Proposition 7. *There exist a universal constant K and a number $0 < \alpha_m \leq \alpha_*$ such that, if $\alpha^0 \leq \alpha_m$, then we have*

$$\frac{d}{dt}\big[I_{R+\sigma t}(t)\big] \geq \frac{1}{24}\int_{\mathbb{R}} \big[(\partial_x \Psi)^2 + (1-|\Psi|^2)^2\big](x+a(t),t)\Phi'(x-R-\sigma t)\,dx - Ke^{-|R+\sigma t|}, \quad (46)$$

for any $R \in \mathbb{R}$, any $t \in \mathbb{R}$, and any $\sigma \in [-1/12, 1/12]$. As a consequence, we also have

$$I_R(t_1) \geq I_R(t_0) - Ke^{-|R|}, \quad (47)$$

for any real numbers $t_0 \leq t_1$.

In the sequel, we denote $I_R^*(t) := I_R^{\Psi^*}(t)$ the corresponding quantity for the specific choice of Ψ^* as the solution to the Gross-Pitaevskii equation. Note in particular that

$$\lim_{R \to -\infty} I_R^*(t) = \mathcal{P}(\Psi^*).$$

We deduce from Proposition 7 the following bounds on I_R^*.

Proposition 8. *Given any positive number δ, there exists a positive number R_δ, depending only on δ, such that we have*

$$\big|I_R^*(t)\big| \leq \delta, \quad \forall R \geq R_\delta,$$
$$\big|I_R^*(t) - \mathcal{P}(\Psi^*)\big| \leq \delta, \quad \forall R \leq -R_\delta,$$

for any $t \in \mathbb{R}$.

Proposition 8 guarantees that the momentum density of the solution Ψ^* remains localized for any time. Combining this information with the monotonicity formula in (46), we derive the following weak localization of the energy density.

Corollary 2. *There exists a positive number M_0 such that*

$$\int_t^{t+1}\int_{\mathbb{R}} \Big[|\partial_x \Psi^*|^2 + (1-|\Psi^*|^2)^2\Big](x+a^*(s),s)e^{|x|}\,dx\,ds \leq M_0, \quad (48)$$

for any $t \in \mathbb{R}$.

In order to conclude that Ψ^* is a smooth and localized solution to (GP), we now improve the weak localization of the energy density in (48) by using standard smoothing properties of the linear Schrödinger equation. More precisely, we invoke

Proposition 9 ([5]). *Let $\lambda \in \mathbb{R}$, and consider a solution $u \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}))$ to the linear Schrödinger equation*

$$i\partial_t u + \partial_{xx} u = F,$$

with $F \in L^2(\mathbb{R}, L^2(\mathbb{R}))$. Then, there exists a positive constant K_λ, depending only on λ, such that

$$\lambda^2 \int_{-T}^{T}\int_{\mathbb{R}} |\partial_x u(x,t)|^2 e^{\lambda x}\,dx\,dt \leq K_\lambda \int_{-T-1}^{T+1}\int_{\mathbb{R}} \Big(|u(x,t)|^2 + |F(x,t)|^2\Big)e^{\lambda x}\,dx\,dt, \quad (49)$$

for any positive number T.

The smoothing properties in Proposition 9 were analysed in a more general context in [19]. We refer to [5] for a detailed proof of this proposition.

Arguing as in [5], we next derive from (48) and Proposition 9 the smoothness and exponential decay of the derivatives of the solution Ψ^*.

Proposition 10. *The solution Ψ^* is of class $\mathcal{C}^\infty$ on $\mathbb{R} \times \mathbb{R}$. Moreover, given any integer $k \geq 1$, there exists a positive number M_k such that we have*

$$\sum_{j=1}^{k} |\partial_x^j \Psi^*(x + a^*(t), t)|^2 + (1 - |\Psi^*(x + a^*(t), t)|^2)^2 \leq M_k e^{-|x|}, \tag{50}$$

for any $(x, t) \in \mathbb{R}^2$.

In terms of the perturbation ε^*, this may be rephrased as

Corollary 3. *Set $\eta_{\varepsilon^*}(\cdot, t) := 2\langle U_{c^*(t)}, \varepsilon^*(\cdot, t)\rangle_{\mathbb{C}} + |\varepsilon^*(\cdot, t)|^2$. Given any integer $k \geq 1$, there exists a positive number M_k such that we have*

$$\sum_{j=1}^{k} |\partial_x^j \varepsilon^*(x, t)|^2 + \eta_{\varepsilon^*}(x, t)^2 \leq M_k e^{-|x|}, \tag{51}$$

for any $(x, t) \in \mathbb{R}^2$.

In conclusion, the function Ψ^* is a very special solution to the dispersive Gross-Pitaevskii equation. We now prove that the only solutions with similar localization and smoothness properties, which moreover remain perturbations of the black soliton U_0 along the Gross-Pitaevskii flow, are exact solitons. This rigidity property is sufficient to complete the proof of Theorem 2. As a matter of fact, it guarantees that the limit profile Ψ^* is exactly a soliton, which provides the convergences in (11).

1.3.3 Rigidity of the limit profile

In order to establish this rigidity property, we follow the strategy developed in [5] for the non-vanishing solitons U_c (see also [34] for similar arguments in the context of the Korteweg-de Vries equations). We rely on the combination of two monotonicity formulae. In [5], they were written in the hydrodynamical framework. We rephrase them in the framework corresponding to the original variable Ψ^*. This makes possible to handle with possibly vanishing solutions to (GP).

More precisely, we come back to the equation satisfied by the limit perturbation ε^*, which we write as

$$\begin{aligned} \partial_t \varepsilon^* = & \big(\theta^*\big)'(t) \big(- iU_{c^*(t)} - i\varepsilon^*\big) + \big(\big(a^*\big)'(t) - c^*(t)\big) \big(\partial_x U_{c^*(t)} + \partial_x \varepsilon^*\big) \\ & - \big(c^*\big)'(t)\, \partial_c U_{c^*(t)} - i\mathcal{L}_{c^*(t)}(\varepsilon^*) - i\eta_{\varepsilon^*}\varepsilon^*, \end{aligned} \tag{52}$$

with $\eta_{\varepsilon^*}(\cdot, t) := 2\langle U_{c^*(t)}, \varepsilon^*(\cdot, t)\rangle_{\mathbb{C}} + |\varepsilon^*(\cdot, t)|^2$. In this equation, the functional $\mathcal{L}_c(\varepsilon)$ is defined as

$$\begin{aligned} \mathcal{L}_c(\varepsilon) := \mathcal{L}_c^+(\varepsilon) + i\mathcal{L}_c^-(\varepsilon) := & - \partial_{xx}\varepsilon_1 - c\partial_x \varepsilon_2 - \big(1 - |U_c|^2\big)\varepsilon_1 + R_c \eta_\varepsilon \\ & + i\Big(- \partial_{xx}\varepsilon_2 + c\partial_x \varepsilon_1 - \big(1 - |U_c|^2\big)\varepsilon_2 + \frac{c}{\sqrt{2}}\eta_\varepsilon\Big), \end{aligned} \tag{53}$$

where ε_1 and ε_2 refer to the real and imaginary parts of the complex function ε. The quantity $\mathcal{L}_c(\varepsilon)$ is the first-order term in the expansion of (GP) for small ε and η_ε. In contrast with the classical situation where this quantity is the linearized part of (GP), the functional $\mathcal{L}_c(\varepsilon)$ is not linear with respect to ε due to the nonlinear dependence of η_ε with respect to ε. It is only linear with respect to both ε and η_ε. This complicates deeply the analysis of this first-order term since we cannot, as usual, rely on spectral theory. However, it is possible to by-pass this problem by using monotonicity formulae, which present the advantage to apply to nonlinear situations.

In order to derive these monotonicity formulae properly, we now set

$$\mathcal{T}_c(\varepsilon) := \sqrt{2} R_c \partial_x \varepsilon_2 - c \partial_x \varepsilon_1 - (1 - |U_c|^2) \varepsilon_2 - \frac{c}{\sqrt{2}} \eta_\varepsilon, \tag{54}$$

and we introduce the quantity

$$M_c^{\phi_c}(\varepsilon) := \int_{\mathbb{R}} \phi_c \mathcal{T}_c(\varepsilon) \mathcal{L}_c^+(\varepsilon),$$

where ϕ_c is a smooth real-valued function, possibly depending smoothly on c. The quantity $M_c^{\phi_c}(\varepsilon)$ is related to the amount of momentum localized around the soliton. Its actual definition is motivated mainly by algebraic reasons which we will try to enlighten in the remaining of this subsection.

The important property regarding $M_c^{\phi_c}(\varepsilon)$ is that the derivative with respect to time of the functions

$$\mathcal{M}^*(t) := M_{c^*(t)}^{\phi_{c^*(t)}}(\varepsilon^*(\cdot, t)),$$

is essentially positive for suitable choices of the functions ϕ_c. In order to prove this, we let

$$\begin{aligned} G_c^{\phi_c}(\varepsilon) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \bigg(\Big(\sqrt{2}(1 - |U_c|^2) \phi_c + \partial_x(R_c \phi_c) \Big) \mathcal{L}_c^+(\varepsilon)^2 + \partial_x(R_c \phi_c) \, \mathcal{L}_c^-(\varepsilon)^2 \\ - \sqrt{2}(1 - |U_c|^2) \phi_c \, \mathcal{L}_c^-(\varepsilon) \, \mathcal{T}_c(\varepsilon) + \partial_x(R_c \phi_c) \, \mathcal{T}_c(\varepsilon)^2 \bigg), \end{aligned} \tag{55}$$

and we set $\mathcal{G}^*(t) := G_{c^*(t)}^{\phi_{c^*(t)}}(\varepsilon^*(\cdot, t))$. The latter quantity turns out to be the main contribution to the derivative of $\mathcal{M}^*$. We also introduce a first-order remainder given by

$$\mathcal{R}^* := \int_{\mathbb{R}} \Big(\partial_{xx} \phi_{c^*} \mathcal{T}_{c^*}(\varepsilon^*) \mathcal{L}_{c^*(t)}^-(\varepsilon^*) + \partial_x \phi_{c^*} \partial_x \mathcal{T}_{c^*}(\varepsilon^*) \mathcal{L}_{c^*}^-(\varepsilon^*) + c^* \partial_x \phi_{c^*} \mathcal{T}_{c^*}(\varepsilon^*) \mathcal{L}_{c^*}^+(\varepsilon^*) \Big). \tag{56}$$

With these notations at hand, we next show

Proposition 11. *Assume that the maps $\phi_c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ depend smoothly on $c \in (-1, 1)$, and that there exists a positive number K_ϕ for which*

$$\big|\phi_c(x)\big| + \big|\partial_c \phi_c(x)\big| \leq K_\phi \big(1 + |x|\big), \quad \text{and} \quad \big|\partial_x \phi_c(x)\big| + \big|\partial_{xx} \phi_c(x)\big| \leq K_\phi, \tag{57}$$

for any $c \in (-1, 1)$ and $x \in \mathbb{R}$. The function $\mathcal{M}^*$ is then of class $\mathcal{C}^1$ on $\mathbb{R}$, and there exists a positive number κ_0, depending only on K_ϕ and the numbers M_k in Corollary 3, such that we have

$$\left| \frac{d}{dt} \big(\mathcal{M}^*(t)\big) - \mathcal{G}^*(t) + \mathcal{R}^*(t) \right| \leq \kappa_0 \big(\beta_*\big)^{\frac{1}{4}} \int_{\mathbb{R}} \Big(|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big), \tag{58}$$

for any $t \in \mathbb{R}$.

We now make the choice of the functions ϕ_c so that the quantity $\mathcal{G}^*(t)$ controls the perturbation ε^*. In order to clarify the presentation, we introduce two families of functions ϕ_c. The first one provides a localized control, while the second one gives a control at spatial infinity.

More precisely, we first set $\phi_c \equiv 1$, and we denote by $\mathcal{M}_1^*$, $\mathcal{G}_1^*$ and $\mathcal{R}_1^*$ the quantities appearing in Proposition 11 for this first choice. The assumptions of Proposition 11 are then satisfied. By definition, the functional $\mathcal{R}_1^*$ is identically equal to 0. Moreover, we have

$$\mathcal{G}_1^*(t) = \frac{1}{2} \int_{\mathbb{R}} \big(1 - |U_{c^*(t)}|^2\big) \Big(3 \big(\mathcal{L}_{c^*(t)}^+(\varepsilon^*(\cdot, t))\big)^2 + \big(\mathcal{L}_{c^*(t)}^-(\varepsilon^*(\cdot, t)) - \mathcal{T}_{c^*(t)}(\varepsilon^*(\cdot, t))\big)^2 \Big).$$

We observe that this expression is positive and localized due to the exponential decay of the function $1 - |U_{c^*(t)}|^2$. For β_* small enough, it provides the following bound on the function ε^*.

Proposition 12. *There exist two numbers* $\sigma_1 \in (0, \sqrt{2})$ *and* $\kappa_1 \in (0, +\infty)$ *such that, when* $\beta_* \leq \sigma_1$*, we have*

$$\mathcal{G}_1^*(t) \geq \kappa_1 \int_{\mathbb{R}} (1 - |U_{c^*}(t)|^2)\Big(|\partial_{xx}\varepsilon^*(\cdot, t)|^2 + |\partial_x \varepsilon^*(\cdot, t)|^2 + |\varepsilon^*(\cdot, t)|^2\Big), \tag{59}$$

for any $t \in \mathbb{R}$.

It is next necessary to recover a control at spatial infinity. In this direction, we consider a second family of functions ϕ_c given by

$$\phi_c(x) := \frac{x}{R_c(x)},$$

for any $x \in \mathbb{R}$. Here, the function R_c refers, as before, to the real part of the soliton U_c. We denote by $\mathcal{M}_2^*$, $\mathcal{G}_2^*$ and $\mathcal{R}_2^*$ the quantities appearing in Proposition 11 for this second choice. The functions ϕ_c satisfy the assumptions in (57).

Lemma 1. *The functions* ϕ_c *are of class* $\mathcal{C}^\infty$ *on* $\mathbb{R}$*, depend smoothly on* $c \in (-1, 1)$ *and satisfy assumption* (57)*.*

Moreover, the quantities $\mathcal{G}_2^*$ and $\mathcal{R}_2^*$ provide the following bound on ε^*.

Proposition 13. *There exist two numbers* $\sigma_2 \in (0, \sigma_1)$ *and* $\kappa_2 \in (0, +\infty)$ *such that, when* $\beta_* \leq \sigma_2$*, we have*

$$\begin{aligned} \mathcal{G}_2^*(t) - \mathcal{R}_2^*(t) \geq & \kappa_2 \int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*(\cdot, t)|^2 + |\partial_x \varepsilon^*(\cdot, t)|^2 + \eta_{\varepsilon^*}(\cdot, t)^2\Big) \\ & - \frac{1}{\kappa_2} \int_{\mathbb{R}} (1 - |U_{c^*(t)}|^2)\Big(|\partial_{xx}\varepsilon^*(\cdot, t)|^2 + |\partial_x \varepsilon^*(\cdot, t)|^2 + |\varepsilon^*(\cdot, t)|^2\Big), \end{aligned} \tag{60}$$

for any $t \in \mathbb{R}$.

When $\beta_* \leq \sigma_2$, the combination of Propositions 11, 12 and 13 provides the inequality

$$\begin{aligned} \frac{d}{dt}\Big(2\mathcal{M}_1^*(t) + \kappa_1\kappa_2\mathcal{M}_2^*(t)\Big) \geq & \kappa_1\kappa_2^2 \int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big) + \kappa_1 \int_{\mathbb{R}} (1 - |U_{c^*}|^2)|\varepsilon^*|^2 \\ & - \kappa_0(2 + \kappa_1\kappa_2)\big(\beta_*\big)^{\frac{1}{4}} \int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1 - |U_{c^*}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big). \end{aligned}$$

In particular, we can decrease again, if necessary, the value of β_* in order to obtain

$$\frac{d}{dt}\Big(2\mathcal{M}_1^*(t) + \kappa_1\kappa_2\mathcal{M}_2^*(t)\Big) \geq \kappa \int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1 - |U_{c^*}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big). \tag{61}$$

with $\kappa = \min\{\kappa_1\kappa_2^2, \kappa_1\}/2 > 0$.

In view of the exponential bounds in (51), the quantity $2\mathcal{M}_1^* + \kappa_1\kappa_2\mathcal{M}_2^*$ is uniformly bounded on $\mathbb{R}$. As a consequence, we have

$$\int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1 - |U_{c^*}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x, t)\, dx\bigg)\, dt < +\infty.$$

Hence, we can find two sequences $(t_n^\pm)_{n\in\mathbb{N}}$, with $t_n^\pm \to \pm\infty$, such that

$$\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1 - |U_{c^*}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x, t_n^\pm)\, dx \to 0,$$

as $n \to +\infty$. Relying again on the exponential bounds in (51), this proves that

$$2\mathcal{M}_1^*(t_n^\pm) + \kappa_1\kappa_2\mathcal{M}_2^*(t_n^\pm) \to 0,$$

so that, by (61),

$$\int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1 - |U_{c^*}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x, t)\, dx \bigg)\, dt = 0.$$

In other words, the function ε^* is identically equal to 0. In view of (41), we infer that

$$c^*(t) = c^*(0),$$

for any $t \in \mathbb{R}$. Combining (35) and (36) with Proposition 6, we conclude that

Corollary 4. *We have*

$$\Psi_0^* = U_{c_0^*}.$$

In other words, a solution to (GP), which is smooth and localized according to Proposition 10, and which is moreover a perturbation of the black soliton at initial time, is exactly a soliton. With this rigidity result at hand, we are in position to conclude the proof of Theorem 2.

1.3.4 Proof of Theorem 2 completed

From now on, we have established that, given any sequence of times $(t_n)_{n\in\mathbb{N}}$ tending to $+\infty$, there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ and a number c_0^* such that

$$\begin{aligned} e^{-i\theta(t_{n_k})}\Psi(\cdot + a(t_{n_k}), t_{n_k}) &\rightharpoonup U_{c_0^*} \quad \text{in } \mathcal{H}(\mathbb{R}), \\ 1 - \big|\Psi(\cdot + a(t_{n_k}), t_{n_k})\big|^2 &\rightharpoonup 1 - \big|U_{c_0^*}\big|^2 \quad \text{in } L^2(\mathbb{R}), \end{aligned} \tag{62}$$

as $k \to +\infty$. By a compactness argument, the proof of (11) reduces to show that the speed c_0^* does not depend on the sequence $(t_n)_{n\in\mathbb{N}}$. We argue by contradiction assuming that we are able to find two sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$, and two different speeds c_1^* and c_2^*, for which we have the convergences in (62). Without loss of generality, we can assume that $c_1^* < c_2^*$, and that

$$t_n \leq s_n \leq t_{n+1}, \tag{63}$$

for any $n \in \mathbb{N}$.

In order to provide a contradiction, we rely on the monotonicity formula in Proposition 7. We set $\delta := \mathcal{P}(U_{c_1^*}) - \mathcal{P}(U_{c_2^*}) > 0$, and we apply (47) for a positive number R such that

$$K e^{-|R|} \leq \frac{\delta}{8},$$

where K refers to the universal constant in Proposition 7, and

$$\bigg| \frac{1}{2} \int_{\mathbb{R}} \big(\Phi(x + R) - \Phi(x - R)\big)\Big(\langle iU_{c_j^*}, \partial_x U_{c_j^*}\rangle_{\mathbb{C}} - \partial_x\big((1 - \chi)\varphi_{c_j^*}\big)\Big)\, dx - \mathcal{P}(U_{c_j^*}) \bigg| \leq \frac{\delta}{8}, \tag{64}$$

for $j = 1$ and $j = 2$. Here, $\varphi_{c_j^*}$ refers to the unique phase function for $U_{c_j^*}$ on $\mathbb{R} \setminus [-1, 1]$, which satisfies (26). In this situation, we first deduce from Proposition 7 and (63) that

$$I_{\pm R}(s_n) \geq I_{\pm R}(t_n) - \frac{\delta}{8}, \quad \text{and} \quad I_{\pm R}(t_{n+1}) \geq I_{\pm R}(s_n) - \frac{\delta}{8}, \tag{65}$$

for any $n \in \mathbb{N}$. Combining (62) with (64), we also have

$$\big|I_{-R}(s_n) - I_R(s_n) - \mathcal{P}(U_{c_1^*})\big| \leq \frac{\delta}{4}, \quad \text{and} \quad \big|I_{-R}(t_n) - I_R(t_n) - \mathcal{P}(U_{c_2^*})\big| \leq \frac{\delta}{4},$$

for n large enough. In view of (65), we are led to

$$I_R(s_n) \geq I_R(t_n) + \frac{3\delta}{8},$$

so that, by (65) again,

$$I_R(t_{n+1}) \geq I_R(t_n) + \frac{\delta}{4}.$$

As a consequence, the sequence $(I_R(t_n))_{n\in\mathbb{N}}$ is unbounded, which provides the desired contradiction.

In conclusion, the convergences in (62) are independent of the choice of the sequence $(t_n)_{n\in\mathbb{N}}$. Statement (11) follows with $c_* := c_0^*$, and (12) is then a consequence of the Sobolev embedding theorem. Coming back to (35), (36) and (37), we observe that

$$c(t) \to c_*, \quad \varepsilon(\cdot, t) \rightharpoonup 0 \quad \text{in } \mathcal{H}(\mathbb{R}), \quad \text{and} \quad \eta_\varepsilon(\cdot, t) \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \tag{66}$$

as $t \to +\infty$, where $\eta_\varepsilon(\cdot, t) := 2\langle U_{c(t)}, \varepsilon(\cdot, t)\rangle_{\mathbb{C}} + |\varepsilon(\cdot, t)|^2$.

In order to complete the proof of Theorem 2, it remains to establish the convergences in (10). We rely on the formulae for the time derivatives of the modulation parameters a, c and θ, which appear in the proof of Proposition 3 below. According to (2.23), the derivatives $a'(t)$, $c'(t)$ and $\theta'(t)$ are indeed given by

$$\begin{pmatrix} a'(t) - c(t) \\ c'(t) \\ \theta'(t) \end{pmatrix} = \mathfrak{M}(c(t), \varepsilon(\cdot, t))^{-1}\big(\mathfrak{F}(c(t), \varepsilon(\cdot, t))\big), \tag{67}$$

where the matrix $\mathfrak{M}(c, \varepsilon)$ is defined in (2.24), and the vector $\mathfrak{F}(c, \varepsilon)$ is equal to

$$\mathfrak{F}(c, \varepsilon) = \begin{pmatrix} \langle i\partial_x U_c, \eta_\varepsilon(U_c + \varepsilon)\rangle_{L^2} - \langle iU_c, (\partial_x \eta_c)\varepsilon\rangle_{L^2} \\ \langle \partial_x U_c, \eta_\varepsilon \varepsilon + |\varepsilon|^2 U_c\rangle_{L^2} \\ -2\langle \partial_x U_c, (\partial_x \eta_c)\varepsilon\rangle_{L^2} + \langle R_c, 2(|\partial_x U_c|^2 - \eta_c R_c^2)\varepsilon + \eta_c \eta_\varepsilon (U_c + \varepsilon)\rangle_{L^2} - c\langle iR_c, \eta_c \varepsilon\rangle_{L^2} \end{pmatrix},$$

with $U_c = R_c + ic/\sqrt{2}$ and $\eta_c = 1 - |U_c|^2$.

In order to take the limit $t \to +\infty$ in (67), we invoke the weak convergences in (66). Concerning the variable ε, they may be rephrased as

$$\partial_x \varepsilon(\cdot, t) \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \quad \text{and} \quad (1 - |U_\sigma|^2)^{\frac{1}{2}} \varepsilon(\cdot, t) \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}),$$

as $t \to +\infty$, for any $\sigma \in (-\sqrt{2}, \sqrt{2})$. As a consequence of the Rellich compactness theorem, we also have the local uniform convergence

$$\varepsilon(\cdot, t) \to 0 \quad \text{in } L^\infty_{\rm loc}(\mathbb{R}),$$

as $t \to +\infty$. Applying all these convergences to (2.24), we first obtain

$$\mathfrak{M}(c(t), \varepsilon(\cdot, t)) \to \begin{pmatrix} \frac{1}{3}\big(2 - c_*^2\big)^{\frac{3}{2}} & 0 & c_*\big(2 - c_*^2\big)^{\frac{1}{2}} \\ 0 & -\big(2 - c_*^2\big)^{\frac{1}{2}} & 0 \\ 0 & 0 & -\frac{1}{3}\big(2 - c_*^2\big)^{\frac{3}{2}} \end{pmatrix}, \tag{68}$$

as $t \to +\infty$. Concerning the vector $\mathfrak{F}(c,\varepsilon)$, we derive from the previous convergences that

$$(1-|U_\sigma|^2)^{\frac{1}{2}}\eta_\varepsilon(\cdot,t)\varepsilon(\cdot,t) \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \quad \text{and} \quad (1-|U_\sigma|^2)^{\frac{1}{2}}|\varepsilon(\cdot,t)|^2 \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}),$$

as $t \to +\infty$, for any $\sigma \in (-\sqrt{2}, \sqrt{2})$. This is enough to conclude that

$$\mathfrak{F}(c(t),\varepsilon(\cdot,t)) \to 0,$$

as $t \to +\infty$. The convergences in (10), as well as the property that $c'(t) \to 0$ as $t \to +\infty$, then follows from (66), (67) and (68). This ends the proof of Theorem 2. □

1.4 Outline of the paper

The remaining part of this paper is devoted to the proofs of all the results, which we have used in the introduction in order to establish the orbital and asymptotic stability of the black soliton.

In Section 2, we gather the results concerning the derivation of orbital stability: the minimizing properties of the black soliton in Subsections 2.1 and 2.2, the construction of the modulation parameters in Subsection 2.3, and the analysis of their evolution in Subsection 2.4.

Section 3 is devoted to the statements used in the proof of asymptotic stability. The construction of the limit profile is detailed in Subsection 3.1. The derivation of its smoothness and localization is performed in Subsection 3.2, while its rigidity properties are investigated in Subsection 3.3.

In a separate appendix, we finally provide the proofs of Proposition 4 and Proposition 5 regarding the definition and properties of the momentum.

2 Orbital stability of the black soliton

2.1 Proof of Proposition 1

We split the proof into three steps. We first consider the quadratic form

$$Q_0(f) := \frac{1}{2}\int_{\mathbb{R}} \big((f')^2 - (1-U_0^2)f^2\big). \tag{2.1}$$

Here, f refers to a *real-valued* function in $\mathcal{H}(\mathbb{R})$. We denote by $\mathfrak{H}(\mathbb{R})$ this Euclidean subspace of $\mathcal{H}(\mathbb{R})$, and we endow it with the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{H}_0}$ corresponding to the norm $\|\cdot\|_{\mathcal{H}_0}$. We claim that

Step 1. *There exist a positive number Λ_0 such that*

$$Q_0(f) \geq \Lambda_0 \int_{\mathbb{R}} \big((f')^2 + (1-U_0^2)f^2\big), \tag{2.2}$$

for any function $f \in \mathfrak{H}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} f\, U_0' = \int_{\mathbb{R}} f\, U_0\,(1-U_0^2) = 0. \tag{2.3}$$

Moreover, the quantity $Q_0(f)$ remains non-negative if only the first orthogonality condition in (2.3) *is satisfied.*

In view of (2.1), the quadratic form Q_0 is well-defined and continuous on $\mathfrak{H}(\mathbb{R})$. We claim that the corresponding self-adjoint operator $\mathcal{Q}_0$ (on $\mathfrak{H}(\mathbb{R})$) may be written as

$$\mathcal{Q}_0 = \frac{1}{2} I - T_0, \tag{2.4}$$

where T_0 is a compact self-adjoint operator on $\mathfrak{H}(\mathbb{R})$.

As a matter of fact, given a function $f \in \mathfrak{H}(\mathbb{R})$, the linear mapping $g \mapsto \int_{\mathbb{R}}(1 - U_0^2) f g$ is continuous on $\mathfrak{H}(\mathbb{R})$. As a consequence of the Riesz theorem, there exists a bounded, self-adjoint operator T_0 on $\mathfrak{H}(\mathbb{R})$ such that

$$\langle T_0(f), g \rangle_{\mathfrak{H}_0} = \int_{\mathbb{R}} (1 - U_0^2) f g, \tag{2.5}$$

for any $g \in \mathfrak{H}(\mathbb{R})$. In view of (2.1), the operator $\mathcal{Q}_0$ writes according to identity (2.4).

We next check the compactness of T_0. Given a uniformly bounded sequence of functions $f_n \in \mathfrak{H}(\mathbb{R})$, we can assume, up to a subsequence, that it weakly converges to a function $f_\infty \in \mathfrak{H}(\mathbb{R})$, as $n \to +\infty$. Since the functions $(1 - U_0^2)^{1/2} f_n$ are uniformly bounded in $H^1(\mathbb{R})$, we can also assume that

$$(1 - U_0^2)^{\frac{1}{2}} f_n \to (1 - U_0^2)^{\frac{1}{2}} f_\infty \text{ in } \mathcal{C}^0_{\rm loc}(\mathbb{R}). \tag{2.6}$$

Recall now that

$$|f_n(x)| \leq |f_n(0)| + |x|^{\frac{1}{2}} \|f_n'\|_{L^2}.$$

Since $1 - U_0^2$ has exponential decay at infinity, the functions $(1 - U_0^2)^{1/4} f_n$ are also uniformly bounded in $L^2(\mathbb{R})$. Combining with (2.6), and again the exponential decay of $1 - U_0^2$, we obtain

$$\int_{\mathbb{R}} (1 - U_0^2)(f_n - f_\infty)^2 \to 0,$$

as $n \to +\infty$. It remains to observe that

$$\|T_0(f)\|_{\mathfrak{H}_0} \leq \int_{\mathbb{R}} (1 - U_0^2) f^2,$$

by (2.5) in order to deduce the compactness of T_0 on $\mathfrak{H}(\mathbb{R})$.

Coming back to (2.4), we apply the spectral theorem to find a sequence of eigenvalues μ_n for $\mathcal{Q}_0$, with $\mu_n \to 1/2$ as $n \to +\infty$, as well as an Hilbert basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{H}(\mathbb{R})$ such that

$$\mathcal{Q}_0(e_n) = \mu_n e_n,$$

for any $n \in \mathbb{N}$. Since $\mathcal{Q}_0 \leq 1/2\, I$ as a self-adjoint operator, the numbers μ_n actually belong to $(-\infty, 1/2]$. We can furthermore assume that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is non-decreasing. We next compute the ordinary differential equation for an eigenfunction f with eigenvalue $\mu \in (-\infty, 1/2)$, which writes as

$$-f'' - (1 - U_0^2) f = \frac{4\mu}{1 - 2\mu} (1 - U_0^2) f.$$

The constant function 1 solves this equation for $\mu = -1/2$. The function U_0 solves this equation for $\mu = 0$, and it owns exactly one zero. By Sturm-Liouville theory (see e.g. [18]), the operator $\mathcal{Q}_0$ has exactly one negative direction, and its kernel reduces to the function U_0. In other words,

$$\mu_0 = -\frac{1}{2} < \mu_1 = 0 < \mu_2, \quad \mathrm{Ker}\Big(\mathcal{Q}_0 + \frac{I}{2}\Big) = \mathbb{R}\,1, \quad \text{and} \quad \mathrm{Ker}\, \mathcal{Q}_0 = \mathbb{R}\, U_0.$$

As a consequence, estimate (2.2) holds for $\Lambda_0 = \mu_2$ under the two orthogonality conditions $\langle f, 1\rangle_{\mathfrak{H}_0} = \langle f, U_0\rangle_{\mathfrak{H}_0} = 0$. Since

$$\langle f, 1\rangle_{\mathfrak{H}_0} = \sqrt{2}\int_{\mathbb{R}} fU_0', \quad \text{and} \quad \langle f, U_0\rangle_{\mathfrak{H}_0} = \int_{\mathbb{R}} f\, U_0\, (1 - U_0^2),$$

due to (3), this achieves the proof of (2.2). Finally, since U_0 spans the kernel of $\mathcal{Q}_0$, the quadratic form Q_0 remains non-negative if we omit the second orthogonality condition in (2.3).

We now recall that

$$E(U_0 + \varepsilon) - E(U_0) = Q_0(\varepsilon_1) + Q_0(\varepsilon_2) + \frac{1}{4}\int_{\mathbb{R}} \eta_\varepsilon^2, \tag{2.7}$$

due to (6), where $\varepsilon := \varepsilon_1 + i\varepsilon_2$ and $\eta_\varepsilon = 2U_0\varepsilon_1 + |\varepsilon|^2$. As a consequence of Step 1, we obtain

Step 2. *When $\varepsilon \in \mathcal{E}(\mathbb{R})$ satisfies the three orthogonality conditions in* (15)*, we have*

$$\int_{\mathbb{R}} \eta_\varepsilon^2 \leq 4\big(E(U_0 + \varepsilon) - E(U_0)\big), \quad \text{and} \quad \Lambda_0\|\varepsilon_2\|_{\mathfrak{H}(\mathbb{R})}^2 \leq E(U_0 + \varepsilon) - E(U_0).$$

We indeed deduce from Step 1 the two inequalities

$$Q_0(\varepsilon_1) \geq 0, \quad \text{and} \quad Q_0(\varepsilon_2) \geq \Lambda_0\|\varepsilon_2\|_{\mathfrak{H}(\mathbb{R})}^2,$$

so that Step 2 follows from identity (2.7).

We are now in position to provide the

Step 3. *End of the proof of Proposition 1.*

The first element is to observe that

$$\frac{1}{4}\int_{\mathbb{R}}(1 - U_0^2)\eta_\varepsilon^2 = \frac{1}{4}\int_{\mathbb{R}}(1 - U_0^2)|\varepsilon|^4 + \int_{\mathbb{R}}(1 - U_0^2)U_0^2\varepsilon_1^2 + \int_{\mathbb{R}}(1 - U_0^2)U_0\varepsilon_1|\varepsilon|^2. \tag{2.8}$$

Going back to (3), and applying the Sobolev embedding theorem to the map $(1 - U_0)^{1/2}\varepsilon$, we compute

$$\int_{\mathbb{R}}(1 - U_0^2)U_0\varepsilon_1|\varepsilon|^2 = \frac{1}{\sqrt{2}}\int_{\mathbb{R}}(1 - U_0^2)\Big(\varepsilon_1'(3\varepsilon_1^2 + \varepsilon_2^2) + 2\varepsilon_2'\varepsilon_1\varepsilon_2\Big) \leq K\|\varepsilon\|_{\mathcal{H}_0}^3,$$

where K refers, here as in the sequel, to a universal constant. In particular, we deduce from (2.8) that

$$\frac{1}{4}\int_{\mathbb{R}}(1 - U_0^2)\eta_\varepsilon^2 \geq \int_{\mathbb{R}}(1 - U_0^2)U_0^2\varepsilon_1^2 - K\|\varepsilon\|_{\mathcal{H}_0}^3.$$

Under the three orthogonality conditions in (15), we can now combine Steps 1 and 2 with identity (2.7) to obtain

$$E(U_0 + \varepsilon) - E(U_0) \geq Q_0(\varepsilon_1) + \frac{1}{4}\int_{\mathbb{R}}(1 - U_0^2)\eta_\varepsilon^2 \geq Q_0(\varepsilon_1) + \int_{\mathbb{R}}(1 - U_0^2)U_0^2\varepsilon_1^2 - K\|\varepsilon\|_{\mathcal{H}_0}^3.$$

At this stage, we apply the Cauchy-Schwarz inequality to obtain

$$\langle \varepsilon_1, U_0\rangle_{\mathfrak{H}_0}^2 \leq 2\sqrt{2}\int_{\mathbb{R}}(1 - U_0^2)U_0^2\varepsilon_1^2,$$

so that

$$E(U_0+\varepsilon)-E(U_0)\geq Q_0(\varepsilon_1)+\frac{1}{2\sqrt{2}}\langle\varepsilon_1,U_0\rangle_{\mathfrak{H}_0}^2-K\|\varepsilon\|_{\mathcal{H}_0}^3.$$

Under the first orthogonality condition in (15), it follows from Step 1 that

$$Q_0(\varepsilon_1)\geq\Lambda_0\big\|\varepsilon_1-\langle\varepsilon_1,e_1\rangle_{\mathfrak{H}_0}e_1\big\|_{\mathfrak{H}_0}^2,$$

where we have set $e_1:=U_0/\|U_0\|_{\mathfrak{H}_0}$, so that

$$E(U_0+\varepsilon)-E(U_0)\geq\Lambda_0\|\varepsilon_1\|_{\mathfrak{H}_0}^2-K\|\varepsilon\|_{\mathcal{H}_0}^3,$$

for a possible further choice of the number Λ_0. In view of Step 2, this is enough to conclude the proof of Proposition 1.

2.2 Proof of Corollary 1

The proof relies on the expansion in (6). In view of (3), we can rewrite this identity as

$$E(U_c+\varepsilon)=E(U_c)-c\int_{\mathbb{R}}\langle iU_c',\varepsilon\rangle_{\mathbb{C}}+\frac{1}{2}\int_{\mathbb{R}}\Big(\big|\varepsilon'\big|^2-(1-|U_c|^2)|\varepsilon|^2+\frac{1}{2}\eta_\varepsilon^2\Big). \tag{2.9}$$

Note here that the function η_ε is defined with respect to U_c, and not U_0. In other terms, it is equal to $\eta_\varepsilon=2\langle U_c,\varepsilon\rangle_{\mathbb{C}}+|\varepsilon|^2$.

The first-order term in the right-hand side of (2.9) is equal to 0 due to the second orthogonality conditions in (16). Concerning the first one, we derive from (4) the formula

$$E(U_c)=\frac{1}{3}(2-c^2)^{\frac{3}{2}}.$$

A first-order expansion of this quantity yields

$$E(U_c)-E(U_0)\geq-4\sqrt{2}c^2,$$

when $c\in(-\sqrt{2},\sqrt{2})$. For $|c|\leq\sigma$, we also check that

$$\big|\eta_c(x)-\eta_0(x)\big|\leq A_\sigma c^2\big(1+|x|^2\big)\eta_\sigma(x), \tag{2.10}$$

where, here as in the sequel, A_σ refers to a constant depending only on σ. As a consequence of the inequality

$$|\varepsilon(x)|\leq|\varepsilon(0)|+|x|^{\frac{1}{2}}\|\varepsilon'\|_{L^2},$$

we deduce that

$$\int_{\mathbb{R}}\big||U_c|^2-|U_0|^2\big||\varepsilon|^2\leq A_\sigma c^2\|\varepsilon\|_{\mathcal{H}_0}^2.$$

Therefore, identity (2.9) reduces to

$$Q_0(\varepsilon_1)+Q_0(\varepsilon_2)+\frac{1}{4}\int_{\mathbb{R}}\eta_\varepsilon^2\leq E(U_c+\varepsilon)-E(U_0)+A_\sigma c^2\big(1+\|\varepsilon\|_{\mathcal{H}_0}^2\big). \tag{2.11}$$

We next expand the orthogonality conditions in (16). Arguing as in (2.10), we deduce from (4) that

$$\|R_c-U_0\|_{L^\infty}\leq Ac^2.$$

Since $U_c' = \eta_c/\sqrt{2}$ by (4), we derive from (16), and again (2.10), that

$$\big|\langle \varepsilon, \partial_x U_0 \rangle_{L^2}\big| + \big|\langle \varepsilon, i\partial_x U_0 \rangle_{L^2}\big| + \big|\langle \varepsilon, iU_0(1 - |U_0|^2) \rangle_{L^2}\big| \leq A_\sigma c^2 \|\varepsilon\|_{\mathcal{H}_0}.$$

In particular, we can decompose ε_1 and ε_2 along the form

$$\varepsilon_1 = \epsilon_1 + \mu_1 U_0', \quad \text{and} \quad \varepsilon_2 = \epsilon_2 + \mu_2 U_0' + \nu_2 U_0(1 - |U_0|^2),$$

where $\epsilon = (\epsilon_1, \epsilon_2)$ satisfies the orthogonality conditions in (15), and the coefficients μ_1, μ_2 and ν_2 satisfy the estimate

$$|\mu_1| + |\mu_2| + |\nu_2| \leq K c^2 \|\varepsilon\|_{\mathcal{H}_0}.$$

As a consequence of Step 1 in the proof of Proposition 1, we obtain

$$Q_0(\varepsilon_1) \geq -A_\sigma c^2 \|\varepsilon\|_{\mathcal{H}_0}^2, \quad \text{and} \quad Q_0(\varepsilon_2) \geq \big(\Lambda_0 - A_\sigma c^2\big) \|\varepsilon\|_{\mathcal{H}_0}^2. \tag{2.12}$$

Inserting these inequalities into (2.11), we are led to

$$\Lambda_0 \|\varepsilon_2\|_{\mathfrak{H}_0}^2 + \frac{1}{4} \int_{\mathbb{R}} \eta_\varepsilon^2 \leq E(U_c + \varepsilon) - E(U_0) + A_\sigma c^2 \big(1 + \|\varepsilon\|_{\mathcal{H}_0}^2\big). \tag{2.13}$$

Finally, we argue as in Step 3 of the proof of Proposition 1. Since ϵ_1 is orthogonal to U_0' in $L^2(\mathbb{R})$, we have

$$Q_0(\epsilon_1) + \int_{\mathbb{R}} (1 - U_0^2) U_0^2 \epsilon_1^2 \geq \Lambda_0 \|\epsilon_1\|_{\mathfrak{H}_0}^2,$$

so that

$$Q_0(\varepsilon_1) + \int_{\mathbb{R}} (1 - U_0^2) U_0^2 \varepsilon_1^2 \geq \Lambda_0 \|\varepsilon_1\|_{\mathfrak{H}_0}^2 - A_\sigma c^2 \|\varepsilon\|_{\mathcal{H}_0}^2. \tag{2.14}$$

On the other hand, we check that

$$\frac{1}{4} \int_{\mathbb{R}} (1 - U_0^2) \eta_\varepsilon^2 \geq \int_{\mathbb{R}} (1 - U_0^2) \langle U_c, \varepsilon \rangle_{\mathbb{C}}^2 - K \|\varepsilon\|_{\mathcal{H}_0}^3 \geq \int_{\mathbb{R}} (1 - U_0^2) U_0^2 \varepsilon_1^2 - A_\sigma \|\varepsilon\|_{\mathcal{H}_0}^2 \big(|c| + \|\varepsilon\|_{\mathcal{H}_0}\big).$$

Combining with (2.11), (2.12) and (2.14), we are led to

$$\Lambda_0 \|\varepsilon_1\|_{\mathfrak{H}_0}^2 \leq E(U_c + \varepsilon) - E(U_0) + A_\sigma \big(c^2 + |c| \|\varepsilon\|_{\mathcal{H}_0}^2 + \|\varepsilon\|_{\mathcal{H}_0}^3\big).$$

In view of (2.13), this completes the proof of Corollary 1. □

2.3 Proof of Proposition 2

The construction of the modulation parameters is standard. For sake of completeness, we recall the following details. We first establish

Lemma 2.1. *Let $(c, a, \theta) \in (-\sqrt{2}, \sqrt{2}) \times \mathbb{R}^2$, and set $U_{c,a,\theta} := e^{i\theta} U_c(\cdot - a)$. Given a positive number δ, there exists a positive number β such that, if*

$$\big\|U_{c,b_1,\vartheta_1} - U_{c,b_2,\vartheta_2}\big\|_{\mathcal{H}_c} < \beta,$$

then,

$$\big|b_2 - b_1\big| + \big|e^{i\vartheta_2} - e^{i\vartheta_1}\big| < \delta.$$

Proof. The proof is by contradiction. Assume that the conclusions of Lemma 2.1 are false. Then, there exist a positive number δ and two sequences $(b_n)_{n\in\mathbb{N}}$ and $(\vartheta_n)_{n\in\mathbb{N}}$ such that

$$\big\|e^{i\vartheta_n}U_c(\cdot - a_n) - U_c\big\|_{\mathcal{H}_c} \to 0, \tag{2.15}$$

as $n \to +\infty$, with $|a_n| + |e^{i\vartheta_n} - 1| \geq \delta$ for any $n \in \mathbb{N}$. Up to a subsequence, we can assume that $e^{i\vartheta_n} \to e^{i\vartheta}$, as $n \to +\infty$. On the other hand, if the sequence $(a_n)_{n\in\mathbb{N}}$ were unbounded, then a subsequence would tend to either $+\infty$ or $-\infty$. In any case, the left-hand side of (2.15) would tend to $2\|U_c\|_{\mathcal{H}_c}$, which is not possible. Therefore, the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, and we can extract a further subsequence, which converges to a. In view of (2.15), we have $e^{i\vartheta}U_c(\cdot - a) = U_c$, so that $a = 0$ and $e^{i\vartheta} = 1$. This gives the desired contradiction with the fact that $|a| + |e^{i\vartheta} - 1| \geq \delta$. □

We are now in position to provide the

Proof of Proposition 2. Set, as before, $U_\sigma := R_\sigma + iI_\sigma$, and consider the map Ξ given by

$$\Xi(\Psi, \sigma, b, \vartheta) = \Big(\langle i\partial_x U_\sigma, \varepsilon\rangle_{L^2}, \langle \partial_x U_\sigma, \varepsilon\rangle_{L^2}, \langle iR_\sigma(1 - |U_\sigma|^2), \varepsilon\rangle_{L^2}\Big), \tag{2.16}$$

with $\varepsilon := e^{-i\vartheta}\Psi(\cdot + b) - U_\sigma$. The map Ξ is well-defined and smooth from $\mathcal{H}(\mathbb{R}) \times (-\sqrt{2}, \sqrt{2}) \times \mathbb{R}^2$ to $\mathbb{R}^3$. Moreover, we can apply the implicit function theorem in order to obtain

Step 1. *Let $(c, a, \theta) \in (-\sqrt{2}, \sqrt{2}) \times \mathbb{R}^2$. There exist two positive numbers ρ and Λ, depending continuously on c, for which there exists a map $\gamma_{c,a,\theta} \in \mathcal{C}^1(B_{\mathcal{H}_0}(U_{c,a,\theta}, \rho), (-\sqrt{2}, \sqrt{2}) \times \mathbb{R}^2)$ such that, given any $\Psi \in B_{\mathcal{H}_0}(U_{c,a,\theta}, \rho)$, $(\sigma, b, \vartheta) = \gamma_{c,a,\theta}(\Psi)$ is the unique solution in $B((c, a, \theta), \Lambda\rho)$ of the equation*

$$\Xi(\Psi, \sigma, b, \vartheta) = 0.$$

Moreover, the map $\gamma_{c,a,\theta}$ is Lipschitz on $B_{\mathcal{H}_0}(U_{c,a,\theta}, \rho)$ with Lipschitz constant at most Λ.

In view of (2.16), we have $\Xi(U_{c,a,\theta}, c, a, \theta) = 0$. Moreover, we derive from (4) that

$$\begin{aligned}
\partial_\sigma\Xi(U_{c,a,\theta}, c, a, \theta) &= -\big(\langle i\partial_x U_c, \partial_c U_c\rangle_{L^2}, 0, 0\big) = -\big((2 - c^2)^{\frac{1}{2}}, 0, 0\big),\\
\partial_b\Xi(U_{c,a,\theta}, c, a, \theta) &= \big(0, \|\partial_x U_c\|_{L^2}^2, 0\big) = \frac{1}{3}\big(0, (2 - c^2)^{\frac{3}{2}}, 0\big),\\
\partial_\vartheta\Xi(U_{c,a,\theta}, c, a, \theta) &= -\big(0, \langle \partial_x U_c, iU_c\rangle_{L^2}, \langle R_c(1 - |U_c|^2), U_c\rangle_{L^2}\big) = \frac{1}{3}\big(0, 3c(2 - c^2)^{\frac{1}{2}}, -(2 - c^2)^{\frac{3}{2}}\big).
\end{aligned}$$

Therefore, the differential $D_c := d_{\sigma,b,\vartheta}\Xi(U_{c,a,\theta}, c, a, \theta)$ is a continuous isomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$, with operator norm bounded from below by $\tau_c := (2 - c^2)^{3/2}/3$. In particular, the differential

$$\begin{aligned}
&d_\Psi\Xi(U_{c,a,\theta}, c, a, \theta)(\phi)\\
&\quad = \Big(\langle i\partial_x U_c, e^{-i\theta}\phi(\cdot + a)\rangle_{L^2}, \langle \partial_x U_c, e^{-i\theta}\phi(\cdot + a)\rangle_{L^2}, \langle iR_c(1 - |U_c|^2), e^{-i\theta}\phi(\cdot + a)\rangle_{L^2}\Big),
\end{aligned}$$

may be written as

$$d_\Psi\Xi(U_{c,a,\theta}, c, a, \theta) = D_c T_{c,a,\theta},$$

where $T_{c,a,\theta}$ is a continuous linear mapping from $\mathcal{H}(\mathbb{R})$ to $\mathbb{R}^3$, with operator norm depending continuously on τ_c. Finally, given any number $0 < \tau < \sqrt{2}$, the operator norm of the second order differential $d^2\Xi(\Psi, \sigma, b, \vartheta)$ is bounded by a constant A_τ, depending only on τ, when $(\Psi, \sigma, b, \vartheta) \in \mathcal{H}(\mathbb{R}) \times (-\sigma(\tau/2), \sigma(\tau/2)) \times \mathbb{R}^2$, with $\sigma(x) = (2 - x^2)^{1/2}$. It then remains to note that Assumption *(iv)* of [4, Proposition A.1] is satisfied when $U = (-\sigma(\tau), \sigma(\tau)) \times \mathbb{R}^2$ and $V = (-\sigma(\tau/2), \sigma(\tau/2)) \times \mathbb{R}^2$, and to apply this proposition to the map Ξ in order to establish the statements in Step 1.

Step 2. *End of the proof.*

Let ρ_0 and Λ_0 be the constants in Step 1 corresponding to the case $c = 0$. Without loss of generality, we can assume that $\Lambda_0\rho_0 < 1$. Consider the number β_0 provided by Lemma 2.1 for $\delta_0 = \Lambda_0\rho_0/16$, and set $\alpha_0 := \min\{\rho_0/2, \beta_0/4\}$. When $\Psi \in \mathcal{U}_0(\alpha_0)$, there exist numbers b and ϑ such that $\Psi \in B_{\mathcal{H}_0}(U_{0,b,\vartheta}, \rho_0/2)$. By Step 1, we can define the numbers $\boldsymbol{c}(\Psi)$, $\boldsymbol{a}(\Psi)$ and $\boldsymbol{\theta}(\Psi)$ by setting

$$\big(\boldsymbol{c}(\Psi), \boldsymbol{a}(\Psi), \boldsymbol{\theta}(\Psi)\big) = \gamma_{0,b,\vartheta}(\Psi).$$

We claim that the definition of $\boldsymbol{c}(\Psi)$ and $\boldsymbol{a}(\Psi)$ does not depend on the choice of b and ϑ. Concerning the number $\boldsymbol{\theta}(\Psi)$, it is also independent of b and ϑ modulo 2π. In particular, the map $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{\theta})$ is well-defined from $\mathcal{U}_0(\alpha_0)$ with values in $\mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}$.

Indeed, assume that $\Psi \in B_{\mathcal{H}_0}(U_{0,b_1,\vartheta_1}, \rho_0/2)$ for other numbers b_1 and ϑ_1. Then, we have

$$\|U_{0,b_1,\vartheta_1} - U_{0,b,\vartheta}\|_{\mathcal{H}_0} < 4\alpha_0 \leq \beta_0,$$

so that, by Lemma 2.1,

$$\big|b_1 - b\big| + |e^{i\vartheta_1} - e^{i\vartheta}| < \frac{\Lambda_0\rho_0}{16}. \tag{2.17}$$

Since $\lambda_0\rho_0 < 1$, and $|e^{it} - 1| \geq 5\pi|t|/6$ when $t \in (-\pi, \pi)$, with $|e^{it} - 1| < 1$, there exists an integer $k \in \mathbb{Z}$ such that

$$|\vartheta_1 - \vartheta - 2\pi k| < \frac{5\pi\Lambda_0\rho_0}{96} < \frac{\Lambda_0\rho_0}{4}. \tag{2.18}$$

On the other hand, the map $\gamma_{0,b,\vartheta}$ is Lipschitz on $B(U_{0,b,\vartheta}, 2\alpha_0)$, with Lipschitz constant at most Λ_0. Hence,

$$\big|(\boldsymbol{c}(\Psi), \boldsymbol{a}(\psi), \boldsymbol{\theta}(\Psi)) - (0, b, \vartheta)\big| = \big|\gamma_{0,b,\vartheta}(\Psi) - \gamma_{0,b,\vartheta}(U_{0,b,\vartheta})\big| \leq \frac{\Lambda_0\rho_0}{2}.$$

Combining with (2.17) and (2.18), we obtain

$$\big|(\boldsymbol{c}(\Psi), \boldsymbol{a}(\Psi), \boldsymbol{\theta}(\Psi) + 2k\pi) - (0, b_1, \vartheta_1)\big| < \Lambda_0\rho_0.$$

Since $\Xi(\Psi, \boldsymbol{c}(\Psi), \boldsymbol{a}(\Psi), \boldsymbol{\theta}(\Psi) + 2k\pi) = 0$, we deduce from Step 1 that $(\boldsymbol{c}(\Psi), \boldsymbol{a}(\Psi), \boldsymbol{\theta}(\Psi) + 2k\pi) = \gamma_{0,b_1,\vartheta_1}(\Psi)$. In conclusion, $\boldsymbol{c}(\Psi)$, $\boldsymbol{a}(\Psi)$ and $\boldsymbol{\theta}(\Psi)$ (modulo 2π) do not depend on the choice of b and ϑ such that $\Psi \in B_{\mathcal{H}_0}(U_{0,b,\vartheta}, \rho_0/2)$.

We next turn to the smoothness of $\boldsymbol{c}$, $\boldsymbol{a}$ and $\boldsymbol{\theta}$. For $\Phi \in \mathcal{U}_0(\alpha_0)$ such that $\|\Phi - \Psi\|_{\mathcal{H}_0} < \alpha_0$, and $(b, \vartheta) \in \mathbb{R}^2$ such that $\Psi \in B_{\mathcal{H}_0}(U_{0,b,\vartheta}, \rho_0/2)$, we have $\|\Phi - U_{0,b,\vartheta}\|_{\mathcal{H}_0} < \rho_0$. By Step 1, we obtain

$$\big(\boldsymbol{c}(\Psi), \boldsymbol{a}(\Psi), \boldsymbol{\theta}(\Psi)\big) = \gamma_{0,b,\vartheta}(\Psi), \quad \text{and} \quad \big(\boldsymbol{c}(\Phi), \boldsymbol{a}(\Phi), \boldsymbol{\theta}(\Phi)\big) = \gamma_{0,b,\vartheta}(\Phi).$$

Since $\gamma_{0,b,\vartheta}$ is of class $\mathcal{C}^1$ on $B_{\mathcal{H}_0}(U_{0,b,\vartheta}, \rho_0)$, the maps $\boldsymbol{c}$, $\boldsymbol{a}$ and $\boldsymbol{\theta}$ are in turn of class $\mathcal{C}^1$ on $\mathcal{U}_0(\alpha_0)$.

Concerning estimate (18), recall that

$$\big(\boldsymbol{c}(\Psi), \boldsymbol{a}(\Psi), \boldsymbol{\theta}(\Psi)\big) = \gamma_{0,b,\vartheta}(\Psi)$$

when $\Psi \in B_{\mathcal{H}_0}(U_{0,b,\vartheta}, \alpha)$, with $\alpha \leq \alpha_0$. In view of the Lipschitz continuity on $B_{\mathcal{H}_0}(U_{0,b,\vartheta}, \rho_0)$ of the map $\gamma_{0,b,\vartheta}$, this provides

$$|\boldsymbol{c}(\Psi)| + |\boldsymbol{a}(\Psi) - b| + |\boldsymbol{\theta}(\Psi) - \vartheta| \leq \Lambda_0\|\Psi - U_{0,b,\vartheta}\|_{\mathcal{H}_0} \leq \Lambda_0\alpha. \tag{2.19}$$

On the other hand, we derive from (4) the existence of a universal constant K such that

$$\|U_{\boldsymbol{c}(\Psi),\boldsymbol{a}(\Psi),\boldsymbol{\theta}(\Psi)} - U_{0,b,\vartheta}\|_{\mathcal{H}_0} \leq K\big(|\boldsymbol{c}(\Psi)| + |\boldsymbol{a}(\Psi) - b| + |\boldsymbol{\theta}(\Psi) - \vartheta|\big).$$

This leads to

$$\|\varepsilon\|_{\mathcal{H}_0} = \|\Psi - U_{\boldsymbol{c}(\Psi),\boldsymbol{a}(\Psi),\boldsymbol{\theta}(\Psi)}\|_{\mathcal{H}_0} \leq \alpha\big(1 + K\Lambda_0\big).$$

Estimate (18) follows combining with (2.19).

Finally, conditions (16) are direct consequences of the definitions of the maps $\gamma_{0,b,\vartheta}$. This completes the proof of Proposition 2. □

2.4 Proof of Proposition 3

As mentioned previously in the introduction, the proof relies on differentiating with respect to time the orthogonality conditions in (16). In order to justify the computations, we first assume that $\partial_x\Psi^0$ belongs to $H^2(\mathbb{R})$. In this situation, it was proved in [49] that the derivative $\partial_x\Psi$ of the corresponding solution is in $\mathcal{C}^0(\mathbb{R}, H^2(\mathbb{R}))$. As a consequence of (GP), the solution Ψ belongs to $\mathcal{C}^1(\mathbb{R}, \mathcal{H}(\mathbb{R}))$.

When it moreover lies in the set $\mathcal{U}_0(\alpha_0)$ for any $t \in (-T, T)$, we can invoke Proposition 2 to define the modulation parameters $(c(t), a(t), \theta(t)) \in (-1, 1) \times \mathbb{R}^2$ for any $t \in (-T, T)$. The corresponding functions c, a and θ are of class $\mathcal{C}^1$ on $(-T, T)$ due to the chain rule theorem. Note also that the remainder ε is in $\mathcal{C}^1((-T, T), \mathcal{H}(\mathbb{R}))$, so that we are allowed to write (21).

Recall here that the function θ is not valued in the torus $\mathbb{R}/2\pi\mathbb{Z}$, but instead, is a continuous real valued function. Moreover, the number $\theta(0)$ is fixed so that it belongs to $[0, 2\pi)$ (see Subsection 1.2 for more details)

As a consequence, we can differentiate the first orthogonality condition in (16) to obtain

$$\begin{aligned}(a' - c)\,\big(\|\partial_x U_c\|_{L^2}^2 + \langle \partial_x U_c, \partial_x\varepsilon\rangle_{L^2}\big) + c'\,\big(\langle \partial_c\partial_x U_c, \varepsilon\rangle_{L^2} - \langle \partial_c U_c, \partial_x U_c\rangle_{L^2}\big)\\ + \theta'\,\langle i\partial_x U_c, U_c + \varepsilon\rangle_{L^2} = \langle i\partial_x U_c, \partial_{xx}\varepsilon - ic\partial_x\varepsilon + \eta_c\varepsilon - \eta_\varepsilon(U_c + \varepsilon)\rangle_{L^2}.\end{aligned} \tag{2.20}$$

At this stage, we use (4) to compute

$$\|\partial_x U_c\|_{L^2}^2 = \frac{(2 - c^2)^{\frac{3}{2}}}{3}, \quad \langle \partial_c U_c, \partial_x U_c\rangle_{L^2} = 0, \quad \text{and} \quad \langle i\partial_x U_c, U_c\rangle_{L^2} = c(2 - c^2)^{\frac{1}{2}}.$$

We also derive from (3) that

$$\langle i\partial_x U_c, \partial_{xx}\varepsilon - ic\partial_x\varepsilon + \eta_c\varepsilon\rangle_{L^2} = -\langle iU_c, (\partial_x\eta_c)\varepsilon\rangle_{L^2}.$$

Inserting these identities into (2.20), and combining with (16), we obtain

$$\begin{aligned}\Big(\frac{(2 - c^2)^{\frac{3}{2}}}{3} + \langle \partial_x U_c, \partial_x\varepsilon\rangle_{L^2}\Big)\,(a' - c) + \langle \partial_c\partial_x U_c, \varepsilon\rangle_{L^2}\,c' + c(2 - c^2)^{\frac{1}{2}}\,\theta'\\ = \langle i\partial_x U_c, \eta_\varepsilon(U_c + \varepsilon)\rangle_{L^2} - \langle iU_c, (\partial_x\eta_c)\varepsilon\rangle_{L^2}.\end{aligned} \tag{2.21}$$

We next differentiate the second and third conditions in (16), and we derive from similar computations the identities

$$\langle i\partial_x U_c, \partial_x\varepsilon\rangle_{L^2}\,(a' - c) + \big(-(2 - c^2)^{\frac{1}{2}} + \langle i\partial_c\partial_x U_c, \varepsilon\rangle_{L^2}\big)\,c' = \langle \partial_x U_c, \eta_\varepsilon\varepsilon + |\varepsilon|^2 U_c\rangle_{L^2}, \tag{2.22}$$

and

$$\begin{aligned}&\langle i\eta_c R_c, \partial_x\varepsilon\rangle_{L^2}\,(a' - c) + \big(\langle i\partial_c R_c, \eta_c\varepsilon\rangle_{L^2} + \langle iR_c, (\partial_x\eta_c)\varepsilon\rangle_{L^2}\big)\,c' - \Big(\frac{(2 - c^2)^{\frac{3}{2}}}{3} + \langle R_c, \eta_c\varepsilon\rangle_{L^2}\Big)\,\theta'\\ &= -2\langle \partial_x U_c, (\partial_x\eta_c)\varepsilon\rangle_{L^2} + 2\langle R_c, (|\partial_x U_c|^2 - \eta_c R_c^2)\varepsilon\rangle_{L^2} - c\langle iR_c, \eta_c\varepsilon\rangle_{L^2} + \langle R_c, \eta_c\eta_\varepsilon(U_c + \varepsilon)\rangle_{L^2}.\end{aligned}$$

Combining with (2.21), this gives a system of the form

$$\mathfrak{M}(c,\varepsilon)\begin{pmatrix} a'-c \\ c' \\ \theta' \end{pmatrix} = \mathfrak{F}(c,\varepsilon), \tag{2.23}$$

where the matrix $\mathfrak{M}(c,\varepsilon)$ is equal to

$$\mathfrak{M}(c,\varepsilon) := \begin{pmatrix} \frac{(2-c^2)^{\frac{3}{2}}}{3} + \langle \partial_x U_c, \partial_x \varepsilon\rangle_{L^2} & \langle \partial_c \partial_x U_c, \varepsilon\rangle_{L^2} & c(2-c^2)^{\frac{1}{2}} \\ \langle i\partial_x U_c, \partial_x \varepsilon\rangle_{L^2} & -(2-c^2)^{\frac{1}{2}} + \langle i\partial_c \partial_x U_c, \varepsilon\rangle_{L^2} & 0 \\ \langle i\eta_c R_c, \partial_x \varepsilon\rangle_{L^2} & \langle i\partial_c R_c, \eta_c \varepsilon\rangle_{L^2} + \langle iR_c, (\partial_x \eta_c)\varepsilon\rangle_{L^2} & -\frac{(2-c^2)^{\frac{3}{2}}}{3} - \langle R_c, \eta_c \varepsilon\rangle_{L^2} \end{pmatrix}. \tag{2.24}$$

When $\Psi(\cdot,t)$ actually lies in $\mathcal{U}_0(\alpha_1)$, with $\alpha_1 < \alpha_0$, for any $t \in (-T,T)$, we derive from (18) that

$$\|\varepsilon(\cdot,t)\|_{\mathcal{H}_0} + |c(t)| \leq A_0 \alpha_1. \tag{2.25}$$

In particular, we can fix α_1 such that the matrix $\mathfrak{M}(c,\varepsilon)$ is invertible, and the operator norm of its inverse is bounded by some positive number A_1, depending only on α_1.

Similarly, we can check that the right-hand side of (2.23) satisfies

$$\|\mathfrak{F}(c,\varepsilon)\|_{\mathbb{R}^3} \leq A_1 \|\varepsilon\|_{\mathcal{H}_0},$$

for a further choice of the constant A_1. In view of (2.23), this provides

$$|a'(t) - c(t)| + |c'(t)| + |\theta'(t)| \leq A_1^2 \|\varepsilon(\cdot,t)\|_{\mathcal{H}_0}, \tag{2.26}$$

for any $t \in (-T,T)$.

In order to complete the proof of (22), we rewrite (2.22) as

$$c' = \frac{1}{(2-c^2)^{\frac{1}{2}}}\Big(\langle i\partial_x U_c, \partial_x \varepsilon\rangle_{L^2}\,(a'-c) + \langle i\partial_c \partial_x U_c, \varepsilon\rangle_{L^2}\big)\,c' - \langle \partial_x U_c, \eta_\varepsilon \varepsilon + |\varepsilon|^2 U_c\rangle_{L^2}\Big),$$

Combining with (2.25) and (2.26), this yields the quadratic estimate in (22) for a possible further choice of A_1.

At this stage, we only have to prove that this estimate remains available for a general initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R})$. This results from a density argument. Indeed, the (GP) flow is continuous with respect to the initial datum in $\mathcal{E}(\mathbb{R})$ (see e.g. [49]). Moreover, we observe that the matrix $\mathfrak{M}(c,\varepsilon)$, as well as the quantity $\mathfrak{F}(c,\varepsilon)$, depend continuously on $\Psi \in \mathcal{H}(\mathbb{R})$. This follows in particular from the continuity of the modulation maps $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{\theta})$. Since the matrix $\mathfrak{M}(c,\varepsilon)$ is invertible with an operator norm of its inverse depending only on α_1, we can use a density argument to extend (2.23) to a general solution. This establishes the continuous differentiability of the maps (c, a, θ), and the estimates in (22) again result from (2.23). We refer to [4], where a similar density argument is performed, for more details. □

3 Asymptotic stability of the black soliton

3.1 First properties of the limit profile

In order to prove the convergences towards the limit profile and limit modulation parameters in Proposition 6, we rely on the following result proved in [5, Proposition A.3], which we only rephrase according to the terminology and topologies of the present paper.

Proposition 3.1 ([5]). *Consider a sequence* $(\Psi_{n,0})_{n\in\mathbb{N}} \in \mathcal{E}(\mathbb{R})^{\mathbb{N}}$*, and a function* $\Psi_0 \in \mathcal{E}(\mathbb{R})$ *such that*

$$\Psi_{n,0} \rightharpoonup \Psi_0 \quad \text{in } \mathcal{H}(\mathbb{R}), \quad \text{and} \quad 1 - |\Psi_{n,0}|^2 \rightharpoonup 1 - |\Psi_0|^2 \quad \text{in } L^2(\mathbb{R}),$$

as $n \to +\infty$*. Denote by* Ψ_n*, respectively* Ψ*, the unique global solutions to* (GP) *with initial datum* $\Psi_{n,0}$*, respectively* Ψ_0*. Given any fixed* $t \in \mathbb{R}$*, we have*

$$\Psi_n(\cdot, t) \rightharpoonup \Psi(\cdot, t) \quad \text{in } \mathcal{H}(\mathbb{R}), \quad \text{and} \quad 1 - |\Psi_n(\cdot, t)|^2 \rightharpoonup 1 - |\Psi(\cdot, t)|^2 \quad \text{in } L^2(\mathbb{R}),$$

when $n \to +\infty$*.*

We are now in position to provide the

Proof of Proposition 6. First, we have $U_{c(t_n)} \to U_{c_0^*}$ in $\mathcal{H}(\mathbb{R})$, as $n \to +\infty$. We therefore derive from (35) that

$$e^{-i\theta(t_n)}\Psi(\cdot + a(t_n), t_n) \rightharpoonup U_{c_0^*} + \varepsilon_0^* \quad \text{in } \mathcal{H}(\mathbb{R}),$$

as $n \to +\infty$. In view of (37), the weak convergences in (42) are then direct consequences of Proposition 3.1.

Concerning (43), it suffices to prove that the only possible accumulation points of the sequences $(a(t_n + t) - a(t_n))_{n\in\mathbb{N}}$, $(\theta(t_n + t) - \theta(t_n))_{n\in\mathbb{N}}$ and $(c(t_n + t))_{n\in\mathbb{N}}$ are given respectively by $a^*(t)$, $\theta^*(t)$ and $c^*(t)$. The convergences in (43) will then follow from (8) and (33) applying a compactness argument. Fix $t \in \mathbb{R}$, and assume that, up to a possible subsequence, we have

$$a(t_n + t) - a(t_n) \to \tilde{a}(t), \quad \theta(t_n + t) - \theta(t_n) \to \tilde{\theta}(t), \quad \text{and} \quad c(t_n + t) \to \tilde{c}(t), \tag{3.1}$$

as $n \to +\infty$. By the weak sequential continuity of translations, we deduce that

$$\begin{aligned} e^{-i\theta(t_n+t)}\Psi(\cdot + a(t_n + t), t_n + t) &\rightharpoonup e^{-i\tilde{\theta}(t)}\Psi^*(\cdot + \tilde{a}(t), t) \quad \text{in } \mathcal{H}(\mathbb{R}),\\ 1 - \big|\Psi(\cdot + a(t_n + t), t_n + t)\big|^2 &\rightharpoonup 1 - \big|\Psi^*(\cdot + \tilde{a}(t), t)\big|^2 \quad \text{in } L^2(\mathbb{R}),\end{aligned}$$

as $n \to +\infty$. Since $U_{c(t_n+t)} \to U_{\tilde{c}(t)}$ in $\mathcal{H}(\mathbb{R})$, and $1 - |U_{c(t_n+t)}|^2 \to 1 - |U_{\tilde{c}(t)}|^2$ in $L^2(\mathbb{R})$ by (3.1), we also obtain

$$\begin{aligned} \varepsilon(\cdot, t_n + t) &\rightharpoonup \tilde{\varepsilon}(\cdot, t) := e^{-i\tilde{\theta}(t)}\Psi^*(\cdot + \tilde{a}(t), t) - U_{\tilde{c}(t)} \quad \text{in } \mathcal{H}(\mathbb{R}),\\ 2\langle U_{c(t_n+t)}, \varepsilon(\cdot, t_n + t)\rangle_{\mathbb{C}} + |\varepsilon(\cdot, t_n + t)|^2 &\rightharpoonup 2\langle U_{\tilde{c}(t)}, \tilde{\varepsilon}(\cdot, t)\rangle_{\mathbb{C}} + |\tilde{\varepsilon}(\cdot, t)|^2 \quad \text{in } L^2(\mathbb{R}),\end{aligned} \tag{3.2}$$

as $n \to +\infty$. Recall that by construction $\varepsilon_n(\cdot, t) := \varepsilon(\cdot, t_n + t)$ satisfies the orthogonality conditions

$$\int_{\mathbb{R}} \langle \varepsilon_n(\cdot, t), U'_{c(t_n+t)}\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon_n(\cdot, t), iU'_{c(t_n+t)}\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon_n(\cdot, t), iR_{c(t_n+t)}\rangle_{\mathbb{C}}\big(1 - |U_{c(t_n+t)}|^2\big) = 0.$$

Passing to the limit $n \to +\infty$, we obtain

$$\int_{\mathbb{R}} \langle \tilde{\varepsilon}(\cdot, t), U'_{\tilde{c}(t)}\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \tilde{\varepsilon}(\cdot, t), iU'_{\tilde{c}(t)}\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \tilde{\varepsilon}(\cdot, t), iR_{\tilde{c}}\rangle_{\mathbb{C}}\big(1 - |U_{\tilde{c}(t)}|^2\big) = 0.$$

The uniqueness of the modulation parameters claimed in Proposition 2 then yields the equalities $\tilde{a}(t) = a^*(t)$, $e^{i\tilde{\theta}(t)} = e^{i\theta^*(t)}$, and $\tilde{c}(t) = c^*(t)$, so that $\varepsilon^*(\cdot, t) = \tilde{\varepsilon}(\cdot, t)$, and (44) then reduces to (3.2).

Finally, observe that the function $\tilde{\theta}$ is continuous on $\mathbb{R}$ due to (8) and (3.1). Since θ^* is also continuous, with $\theta^*(0) \in [0, 2\pi)$, and since $\tilde{\theta}(0) = 0$ by (3.1), we conclude that $\tilde{\theta} = \theta^*$. This completes the proof of Proposition 6. □

3.2 Monotonicity and localization properties of the limit profile

3.2.1 Proof of Proposition 7

Without loss of generality, we may assume that $\partial_x \Psi^0 \in H^2(\mathbb{R})$, so that $\Psi \in \mathcal{C}^1(\mathbb{R}, (\mathcal{E}(\mathbb{R}), d))$ (see e.g [49] for the smoothness of the (GP) flow). The general case follows by an approximation argument, and in that case, inequality (46) has to be understood in the distributional sense, while (47) is unaffected.

We define the function

$$\chi^a(x,t) := \chi(x - a(t)),$$

for any $(x,t) \in \mathbb{R}^2$, and we set

$$\varphi(x,t) := \varphi_{\rm mod}(x - a(t), t) + \theta(t),$$

so that $\varphi(\cdot, t)$ is a phase function for $\Psi(\cdot, t)$ on $\mathbb{R} \setminus [a(t) - 1, a(t) + 1]$ for any $t \in \mathbb{R}$. After a change of variables, we may rewrite

$$I_{R+\sigma t}(t) = \frac{1}{2} \int_{\mathbb{R}} \Big[\langle i\Psi, \partial_x \Psi \rangle_{\mathbb{C}} - \partial_x \big((1 - \chi^a)(\varphi - \theta) \big) \Big] (x,t)\, \Phi(x - a(t) - R - \sigma t)\, dx.$$

Coming back to (GP), we first obtain the identity

$$\begin{aligned} \partial_t \Big(\langle i\Psi, \partial_x \Psi \rangle_{\mathbb{C}} - &\partial_x \big((1 - \chi^a)(\varphi - \theta) \big) \Big) \\ = &\partial_x \bigg(\frac{(1 - \chi^a) - (1 - \eta)}{2(1 - \eta)} \partial_{xx} \eta + \chi^a \eta - a'(t) \partial_x \chi^a (\varphi - \theta) \\ & - \theta'(t)(1 - \chi^a) - \frac{1}{2} \eta^2 + \frac{(1 - \chi^a) - 2(1 - \eta)}{1 - \eta} |\partial_x \Psi|^2 \bigg), \end{aligned}$$

which has the form of a conservation law. It follows that

$$\begin{aligned} &\frac{d}{dt} \big[I_{R+\sigma t}(t) \big] \\ = &\frac{1}{2} \int_{\mathbb{R}} \Big[\frac{1}{2} \eta^2 + \Big(2 - \frac{(1 - \chi^a)}{1 - \eta} \Big) |\partial_x \Psi|^2 + \frac{1 - \chi^a}{(1 - \eta)^2} (\partial_x \eta)^2 \Big] (x + a(t), t)\, \Phi'(x - R - \sigma t)\, dx \\ & - \frac{1}{2} \big(a'(t) + \sigma \big) \int_{\mathbb{R}} \Big[\langle i\Psi, \partial_x \Psi \rangle_{\mathbb{C}} - \partial_x \big((1 - \chi^a)(\varphi - \theta) \big) \Big] (x + a(t), t)\, \Phi'(x - R - \sigma t)\, dx \\ & + \frac{1}{4} \int_{\mathbb{R}} \Big[\eta + (1 - \chi^a) \ln(1 - \eta) \Big] (x + a(t), t)\, \Phi'''(x - R - \sigma t)\, dx \\ & + \frac{1}{2} \int_{\mathbb{R}} \Big[- \chi^a \eta + a'(t) \partial_x \chi^a (\varphi - \theta) + \frac{1}{2} \partial_{xx} \chi^a \ln(1 - \eta) \Big] (x + a(t), t)\, \Phi'(x - R - \sigma t)\, dx \\ & + \frac{1}{2} \theta'(t) \int_{\mathbb{R}} \partial_x \chi^a(x)\, \Phi(x - R - \sigma t)\, dx \\ := &I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.3}$$

In the previous formula, the first integral I_1 on the right-hand side is the favourable term, since it is non-singular, positive and comparable to the energy density, at least when η is sufficiently small, that is at spatial infinity. Note in particular that $\Phi' = (4 \cosh^2(\cdot/2))^{-1}$ is positive and exponentially decaying. The second and third integrals I_2 and I_3 have no definite sign a priori, but they will be controlled in absolute value by a fraction of the first one, at least at spatial infinity. The integral I_4 is a local one, since its integrand vanishes outside $[-1, 1]$. It will just be treated as an error term. Finally, the last integral I_5 is controlled by $|\theta'(t)|$ times an exponentially decreasing term independent of Ψ.

More precisely, we first fix a number $L > 2$ such that $\eta \leq 1/3$ outside $[-L, L]$. In view of (45), such a number L exists, and it only depends on α^0, provided the latter is sufficiently small. We divide the analysis of the integrals in (3.3) by considering separately the cases $x \notin [-L, L]$ and $x \in [-L, L]$.

We begin with the case $x \notin [-L, L]$. Using the fact that $\eta \leq 1/3$, we first have

$$\frac{1}{2}\eta^2 + \Big(2 - \frac{1-\chi^a}{1-\eta}\Big)|\partial_x\Psi|^2 + \frac{1-\chi^a}{(1-\eta)^2}(\partial_x\eta)^2 \geq \frac{1}{2}\big[\eta^2 + |\partial_x\Psi|^2\big].$$

Since $\eta \leq 1/3$ and $L \geq 2$, we also have

$$\Big|\langle i\Psi, \partial_x\Psi\rangle_{\mathbb{C}} - \partial_x\big((1-\chi^a)\varphi\big)\Big| = \Big|\Big\langle i\frac{\Psi}{|\Psi|^2}(|\Psi|^2-1), \partial_x\Psi\Big\rangle_{\mathbb{C}}\Big| \leq \frac{\sqrt{3}}{2\sqrt{2}}\big(\eta^2 + |\partial_x\Psi|^2\big),$$

and

$$\big|\eta + (1-\chi^a)\ln(1-\eta)\big| \leq \frac{2}{3}\eta^2.$$

Direct computations also yield the global inequality $|\Phi'''| \leq \Phi'$. Combining these inequalities, and denoting by $(I_j^{\rm out})_{1\leq j\leq 5}$ the restriction of the integrals I_j to $\mathbb{R} \setminus [-L, L]$, we finally obtain the lower bound

$$I_1^{\rm out} + I_2^{\rm out} + I_3^{\rm out} \geq \Big(\frac{1}{12} - \frac{\sqrt{3}}{4\sqrt{2}}\big|a'(t)+\sigma\big|\Big)\int_{\mathbb{R}\setminus[-L,L]} \big[\eta^2 + |\partial_x\Psi|^2\big](x+a(t),t)\,\Phi'(x-R-\sigma t)\,dx.$$

In view of our restriction on σ, and the estimate $|a'(t)| \leq A_*\alpha^0$ given by (8), we have

$$\frac{1}{12} - \frac{\sqrt{3}}{4\sqrt{2}}|a'(t)+\sigma| \geq \frac{1}{24},$$

provided once more that α^0 is sufficiently small. Finally, note that $I_4^{\rm out} = I_5^{\rm out} = 0$ since the integrands identically vanish there.

It remains to consider the case $x \in [-L, L]$. In view of (8) and the explicit form of Φ, we directly estimate I_5 by

$$|I_5| \leq Ke^{-|R+\sigma t|}.$$

Concerning the other four integrals, we uniformly bound the terms Φ' and Φ''' by $K\exp(-|R+\sigma t|)$, and the remaining integrands are then controlled (pointwise) by a constant plus the energy density. Here, we use in particular the pointwise estimate in (26) on $|\varphi - \theta|$. Conclusion (46) then follows.

It remains to prove (47). For that purpose, we distinguish two cases, depending on the sign of R. If $R \geq 0$, we integrate (46) from $t = t_0$ to $t = (t_0+t_1)/2$ with the choice $\sigma = \frac{1}{12}$ and $R = R - \frac{1}{12}t_0$, and then from $t = (t_0+t_1)/2$ to $t = t_1$ with the choice $\sigma = -\frac{1}{12}$ and $R = R + \frac{1}{12}t_1$. In total, we hence integrate on a broken line starting and ending at a distance R from the origin. If $R \leq 0$, we argue similarly, choosing first $\sigma = -\frac{1}{12}$, and next $\sigma = \frac{1}{12}$. This yields (47), and completes the proof of Proposition 7. □

3.2.2 Proof of Proposition 8

The proof is almost identical to the corresponding one in [5, Proposition 3]. We reproduce it here for the sake of completeness with the minor adaptations. We argue by contradiction and assume that there exists a positive number δ_0 such that, for any positive number R_{δ_0}, there exist two numbers $R \geq R_{\delta_0}$ and $t \in \mathbb{R}$ such that either $|I_R^*(t)| \geq \delta_0$, or $|I_R^*(t) - \mathcal{P}(\Psi^*)| \geq \delta_0$. Since at time $t = 0$, we have $\lim_{R\to+\infty} I_R^*(0) = 0$ and $\lim_{R\to-\infty} I_R^*(0) = \mathcal{P}(\Psi^*)$, we first fix $R_{\delta_0} > 0$ such that

$$\big|I_R^*(0)\big| + \big|I_{-R}^*(0) - \mathcal{P}(\Psi^*)\big| \leq \frac{\delta_0}{4}, \quad \text{and} \quad Ke^{-R} \leq \frac{\delta_0}{32}, \tag{3.4}$$

for any $R \geq R_{\delta_0}$. Here, the notation K refers to the corresponding constant in Proposition 7. We next fix $R > 0$ and $t \in \mathbb{R}$ obtained from the contradiction assumption for that choice of R_{δ_0}, so that either $|I_R^*(t)| \geq \delta_0$ or $|I_{-R}^*(t) - \mathcal{P}(\Psi^*)| \geq \delta_0$. In the sequel, we assume that $I_R^*(t) \geq \delta_0$ holds, the three other cases would follow in a very similar manner. In particular, we infer from (3.4) that

$$I_R^*(t) \geq \delta_0 \geq \frac{\delta_0}{4} + \frac{\delta_0}{16} \geq I_R^*(0) + 2Ke^{-R},$$

and therefore it follows from Proposition 7 applied to Ψ^* that $t > 0$. Finally, we fix $R' \geq R$ such that

$$\big|I_{-R'}^*(t) - \mathcal{P}(\Psi^*)\big| \leq \frac{\delta_0}{4}. \tag{3.5}$$

Since $R' \geq R$, we also deduce from (3.4) that

$$\big|I_{-R'}^*(0) - \mathcal{P}(\Psi^*)\big| \leq \frac{\delta_0}{4}, \quad \text{and} \quad Ke^{-R'} \leq \frac{\delta_0}{32}. \tag{3.6}$$

Combining the inequality $|I_R^*(t)| \geq \delta_0$ with (3.4), (3.5) and (3.6), we obtain

$$\big|I_{-R'}^*(t) - I_R^*(t) - \mathcal{P}(\Psi^*)\big| \geq \frac{3\delta_0}{4}, \quad \text{and} \quad \big|I_{-R'}^*(0) - I_R^*(0) - \mathcal{P}(\Psi^*)\big| \leq \frac{\delta_0}{2},$$

and therefore

$$\Big|\big(I_{-R'}^*(0) - I_R^*(0)\big) - \big(I_{-R'}^*(t) - I_R^*(t)\big)\Big| \geq \frac{\delta_0}{4}.$$

Since the integrands of the expressions between parenthesis are localized in space, we deduce from Proposition 6 that there exists an integer n_0 such that

$$\Big|\big(I_{-R'}(t_n) - I_R(t_n)\big) - \big(I_{-R'}(t_n + t) - I_R(t_n + t)\big)\Big| \geq \frac{\delta_0}{8},$$

for any $n \geq n_0$. Rearranging the terms in the previous inequality yields

$$\max\Big\{\big|I_{-R'}(t_n) - I_{-R'}(t_n + t)\big|, \big|I_R(t_n) - I_R(t_n + t)\big|\Big\} \geq \frac{\delta_0}{16}. \tag{3.7}$$

On the other hand, since $t \geq 0$, by the monotonicity formula in Proposition 7, (3.4) and (3.6), we have

$$I_{-R'}(t_n) - I_{-R'}(t_n + t) \leq \frac{\delta_0}{32}, \quad \text{and} \quad I_R(t_n) - I_R(t_n + t) \leq \frac{\delta_0}{32}.$$

Therefore, we deduce from (3.7) that, given any $n \geq n_0$, we have

$$\text{either} \quad I_{-R'}(t_n + t) - I_{-R'}(t_n) \geq \frac{\delta_0}{16}, \quad \text{or} \quad I_R(t_n + t) - I_R(t_n) \geq \frac{\delta_0}{16}.$$

In particular, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $t_{n_{k+1}} \geq t_{n_k} + t$ for any $k \in \mathbb{N}$, and either

$$I_R(t_{n_k} + t) - I_R(t_{n_k}) \geq \frac{\delta_0}{16}, \tag{3.8}$$

for any $k \in \mathbb{N}$, or

$$I_{-R'}(t_{n_k} + t) - I_{-R'}(t_{n_k}) \geq \frac{\delta_0}{16},$$

for any $k \in \mathbb{N}$. In the sequel, we assume that (3.8) holds. Here also, the other case would follow in a very similar manner. Since $t_{n_{k+1}} \geq t_{n_k} + t$, we obtain by the monotonicity formula of Proposition 7, (3.4) and (3.8), that

$$I_R(t_{n_{k+1}}) \geq I_R(t_{n_k+t}) - \frac{\delta_0}{32} \geq I_R(t_{n_k}) + \frac{\delta_0}{32}, \tag{3.9}$$

for any $k \in \mathbb{N}$. On the other hand, an inspection of the number I_R yields the estimate

$$\big|I_R(t_{n_k})\big| \leq K\big(1 + E(\Psi(\cdot, t_k))\big),$$

where the last term does not depend on k by conservation of the energy. This yields a contradiction with (3.9). □

3.2.3 Proof of Corollary 2

The proof is an adaptation of the proof of [5, Proposition 4]. For sake of completeness, we provide the following details.

We fix a number $s \in \mathbb{R}$. Given any arbitrary positive number R, we integrate (46) for the special choice $\sigma = 1/12$ so as to obtain

$$I_R^*(s) \leq I_{R+\tau/12}^*(s+\tau) + 12Ke^{-R},$$

for any $\tau \in [0, +\infty)$. Invoking Proposition 8, we know that

$$I_{R+\tau/12}^*(s+\tau) \to 0,$$

as $\tau \to +\infty$, which is enough to conclude that

$$I_R^*(s) \leq 12Ke^{-R}.$$

In order to bound the quantity $I_R^*(s)$ from below, we now integrate (46) for the special choice $\sigma = -1/12$. This gives

$$I_R^*(s) \geq I_{R+\tau/12}^*(s-\tau) - 12Ke^{-R},$$

for any $\tau \in [0, +\infty)$. Taking the limit $\tau \to +\infty$, we deduce similarly that

$$I_R^*(s) \geq -12Ke^{-R},$$

which yields the estimate

$$\big|I_R^*(s)\big| \leq 12Ke^{-R}.$$

Similarly, we obtain

$$\big|I_R^*(s) - \mathcal{P}(\Psi^*)\big| \leq 12Ke^{-|R|},$$

for any negative number R. Therefore, we can integrate (46) from t to $t+1$ with the choice $\sigma = 0$ to get

$$\int_t^{t+1} \int_{\mathbb{R}} \Big[\big|\partial_x \Psi^*\big|^2 + \big(1 - |\Psi^*|^2\big)^2\Big](x + a^*(s), s)\Phi'(x - R)\, dx\, ds \leq 25Ke^{-|R|}, \tag{3.10}$$

for any $R \in \mathbb{R}$.

We finally observe that

$$\lim_{R \to \pm\infty} e^{|R|}\Phi'(x - R) = e^{\pm x},$$

for any $x \in \mathbb{R}$. Applying the Fatou lemma to (3.10), and using the inequality

$$e^{|x|} \leq e^{-x} + e^{x},$$

we derive (48). This completes the proof of Corollary 2. □

3.2.4 Proof of Proposition 10

The proof is almost the same as the proof of [5, Proposition 6]. Invoking the smoothing properties of the linear Schrödinger flow in Proposition 9, we first show inductively the existence of positive numbers A_k such that we have

$$\int_t^{t+1} \int_{\mathbb{R}} \big| \partial_x^k \Psi^*(x + a^*(t), s) \big|^2 \, e^{|x|} \, dx \, ds \leq A_k, \tag{3.11}$$

for any $k \geq 1$ and $t \in \mathbb{R}$. In order to initiate the induction, we rely on the inequality

$$\int_t^{t+1} \int_{\mathbb{R}} \Big(\big| \partial_x \Psi^*(x + a^*(t), t) \big|^2 + \big(1 - |\Psi^*(x + a^*(t), t)|^2 \big)^2 \Big) \, e^{|x|} \, dx \leq A_1,$$

which is a consequence of (48), and the property that the function $s \mapsto |a^*(t) - a^*(s)|$ is uniformly bounded on $[t, t+1]$ due to (40) and (41). The proof by induction of (3.11) then follows as in [5].

We next apply the Sobolev embedding theorem with respect to the time variable in order to prove the existence of possibly further positive numbers A_k such that

$$\int_{\mathbb{R}} \Big(\sum_{j=1}^k \big| \partial_x^k \Psi^*(x + a^*(t), t) \big|^2 + \big(1 - |\Psi^*(x + a^*(t), t)|^2 \big)^2 \Big) \, e^{|x|} \, dx \leq A_k,$$

for any $k \geq 1$ and $t \in \mathbb{R}$. Estimate (50) then results from the Sobolev embedding theorem with respect to the space variable. Combining this estimate with the equation for $\partial_t \Psi^*$, we deduce that Ψ^* is of class $\mathcal{C}^\infty$ on $\mathbb{R} \times \mathbb{R}$. This completes the proof of Proposition 10. □

3.2.5 Proof of Corollary 3

In view of definition (38) and Proposition 10, the function ε^* belongs to $\mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Moreover, we have

$$\eta_{\varepsilon^*}(\cdot, t) = 1 - |U_{c^*(t)}|^2 - \big(1 - |\Psi^*(\cdot + a^*(t), t)|^2 \big), \text{ and } \partial_x^k \varepsilon^*(\cdot, t) = e^{-i\theta^*(t)} \partial_x^k \Psi^*(\cdot + a^*(t), t) - \partial_x^k U_{c^*(t)}.$$

Recall that the function $1 - |U_c|^2$ and the derivatives $\partial_x^k U_c$ decay at least as $e^{-\sqrt{2 - c^2}|x|}$ at infinity. Since c^* is uniformly small in view of (40), this decay property is enough to deduce (51) from (50). This concludes the proof of Corollary 3. □

3.3 Rigidity properties of the limit profile

3.3.1 Proof of Proposition 11

Under the conclusions of Corollary 3 and the assumptions of Proposition 11, the map $\mathcal{M}^*$ is of class $\mathcal{C}^1$ on $\mathbb{R}$. Moreover, we are allowed to derive from (52) the identity

$$\frac{d}{dt} \big(\mathcal{M}^*(t) \big) = \big(c^* \big)'(t) \, \mathcal{I}_1(t) + \mathcal{I}_2(t),$$

where we have set

$$\begin{aligned} \mathcal{I}_1 := & \int_{\mathbb{R}} \partial_c \phi_{c^*} \, \mathcal{T}_{c^*}(\varepsilon^*) \, \mathcal{L}_{c^*}^+(\varepsilon^*) + \int_{\mathbb{R}} \phi_{c^*} \, \mathcal{T}_{c^*}(\varepsilon^*) \Big(- \partial_x \varepsilon_2^* + 2 R_{c^*} \partial_c R_{c^*} \varepsilon_1^* + \partial_c R_{c^*} \eta_{\varepsilon^*} \Big) \\ & + \int_{\mathbb{R}} \phi_{c^*} \mathcal{L}_{c^*}^+(\varepsilon^*) \Big(\sqrt{2} \partial_c R_{c^*} \partial_x \varepsilon_2^* - \partial_x \varepsilon_1^* + 2 R_{c^*} \partial_c R_{c^*} \varepsilon_2^* - \frac{1}{\sqrt{2}} \eta_{\varepsilon^*} \Big), \end{aligned} \tag{3.12}$$

and

$$\mathcal{I}_2 := \int_{\mathbb{R}} \phi_{c^*} \Big(\mathcal{T}_{c^*}(\varepsilon^*) \Big(- \partial_{xx}\partial_t \varepsilon_1^* - c^* \partial_x \partial_t \varepsilon_2^* - (1 - |U_{c^*}|^2) \varepsilon_1^* + R_{c^*} \partial_t \eta_{\varepsilon^*} \Big) + \mathcal{L}_{c^*}^+(\varepsilon^*) \Big(\sqrt{2} R_{c^*} \partial_x \partial_t \varepsilon_2^* - c^* \partial_x \partial_t \varepsilon_1^* - (1 - |U_{c^*}|^2) \partial_t \varepsilon_2^* - \frac{c^*}{\sqrt{2}} \partial_t \eta_{\varepsilon^*} \Big) \Big).$$

Concerning the integral $\mathcal{I}_2$, we deduce from (52) that the derivative with respect to time of η_{ε^*} is equal to

$$\begin{aligned} \partial_t \eta_{\varepsilon^*} =& 2\big((a^*)' - c^*\big) \langle U_{c^*} + \varepsilon^*, \partial_x U_c^* + \partial_x \varepsilon^* \rangle_{\mathbb{C}} - 2(c^*)' \langle U_{c^*}, \partial_c U_{c^*} \rangle_{\mathbb{C}} \\ & - 2\langle U_{c^*}, i\mathcal{L}_{c^*}(\varepsilon^*) \rangle_{\mathbb{C}} - 2\langle \varepsilon^*, i\mathcal{L}_{c^*}(\varepsilon^*) \rangle_{\mathbb{C}} - 2\eta_{\varepsilon^*} \langle U_{c^*}, i\varepsilon^* \rangle_{\mathbb{C}}. \end{aligned}$$

Invoking again (52), as well as the identities $\partial_x R_c = (1 - |U_c|^2)/\sqrt{2}$ and $\partial_{xx} R_c = -R_c(1 - |U_c|^2)$, we can decompose the integral $\mathcal{I}_2$ as

$$\mathcal{I}_2 = \big(\theta^*\big)' \mathcal{I}_2^\theta + \big((a^*)' - c^*\big) \mathcal{I}_2^a + \big(c^*\big)' \mathcal{I}_2^c + \mathcal{I}_2^L + \mathcal{I}_2^N,$$

where we denote

$$\begin{aligned} \mathcal{I}_2^\theta := \int_{\mathbb{R}} \phi_{c^*} \Big(& \mathcal{T}_{c^*}(\varepsilon^*) \big(- \partial_{xx} \varepsilon_2^* + c^* \partial_x \varepsilon_1^* - (1 - |U_{c^*}|^2) \varepsilon_2^* \big) \\ & + \mathcal{L}_{c^*}^+(\varepsilon^*) \big(- \sqrt{2} R_{c^*} \partial_x \varepsilon_1^* - c^* \partial_x \varepsilon_2^* + (1 - |U_{c^*}|^2) \varepsilon_1^* \big) \Big), \end{aligned} \tag{3.13}$$

$$\begin{aligned} \mathcal{I}_2^a := \int_{\mathbb{R}} \phi_{c^*} \Big(& \mathcal{T}_{c^*}(\varepsilon^*) \big(- \partial_{xxx} \varepsilon_1^* - c^* \partial_{xx} \varepsilon_2^* - (1 - |U_{c^*}|^2) \partial_x \varepsilon_1^* + 2R_{c^*} \langle U_{c^*}, \partial_x \varepsilon^* \rangle_{\mathbb{C}} \\ & + 2R_{c^*} \langle \varepsilon^*, \partial_x U_{c^*} + \partial_x \varepsilon^* \rangle_{\mathbb{C}} \big) + \mathcal{L}_{c^*}^+(\varepsilon^*) \big(\sqrt{2} R_{c^*} \partial_{xx} \varepsilon_2^* - c^* \partial_{xx} \varepsilon_1^* \\ & - (1 - |U_{c^*}|^2) \partial_x \varepsilon_2^* - \sqrt{2} c^* \langle U_{c^*}, \partial_x \varepsilon^* \rangle_{\mathbb{C}} - \sqrt{2} c^* \langle \varepsilon^*, \partial_x U_{c^*} + \partial_x \varepsilon^* \rangle_{\mathbb{C}} \big) \Big), \end{aligned} \tag{3.14}$$

$$\mathcal{I}_2^c := \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \phi_{c^*} (1 - |U_{c^*}|^2) \mathcal{L}_{c^*}^+(\varepsilon^*), \tag{3.15}$$

$$\begin{aligned} \mathcal{I}_2^L := \int_{\mathbb{R}} \phi_{c^*} \Big(& \mathcal{T}_{c^*}(\varepsilon^*) \big(- \partial_{xx} \mathcal{L}_{c^*}^-(\varepsilon^*) + c^* \partial_x \mathcal{L}_{c^*}^+(\varepsilon^*) - (1 - |U_{c^*}|^2 - 2R_{c^*}^2) \mathcal{L}_{c^*}^-(\varepsilon^*) \\ & - \sqrt{2} c^* R_{c^*} \mathcal{L}_{c^*}^+(\varepsilon^*) \big) + \mathcal{L}_{c^*}^+(\varepsilon^*) \big(- \sqrt{2} R_{c^*} \partial_x \mathcal{L}_{c^*}^+(\varepsilon^*) - c^* \partial_x \mathcal{L}_{c^*}^-(\varepsilon^*) \\ & + \big(1 + (c^*)^2 - |U_{c^*}|^2\big) \mathcal{L}_{c^*}^+(\varepsilon^*) - \sqrt{2} c^* R_{c^*} \mathcal{L}_{c^*}^-(\varepsilon^*) \big) \Big), \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \mathcal{I}_2^N := \int_{\mathbb{R}} \phi_{c^*} \Big(& \mathcal{T}_{c^*}(\varepsilon^*) \big(- \partial_{xx}(\eta_{\varepsilon^*} \varepsilon_2^*) + c^* \partial_x(\eta_{\varepsilon^*} \varepsilon_1^*) - (1 - |U_{c^*}|^2) \eta_{\varepsilon^*} \varepsilon_2^* - 2R_{c^*} \langle \varepsilon^*, i\mathcal{L}_{c^*}(\varepsilon^*) \rangle_{\mathbb{C}} \\ & - 2R_{c^*} \eta_{\varepsilon^*} \langle U_{c^*}, i\varepsilon^* \rangle_{\mathbb{C}} \big) + \mathcal{L}_{c^*}^+(\varepsilon^*) \big(- \sqrt{2} R_{c^*} \partial_x(\eta_{\varepsilon^*} \varepsilon_1^*) - c^* \partial_x(\eta_{\varepsilon^*} \varepsilon_2^*) \\ & + (1 - |U_{c^*}|^2) \eta_{\varepsilon^*} \varepsilon_1^* + \sqrt{2} c^* \langle \varepsilon^*, i\mathcal{L}_{c^*}(\varepsilon^*) \rangle_{\mathbb{C}} + \sqrt{2} c^* \eta_{\varepsilon^*} \langle U_{c^*}, i\varepsilon^* \rangle_{\mathbb{C}} \big) \Big). \end{aligned} \tag{3.17}$$

We are now reduced to estimate all these integrals according to (58). We split the proof into six steps. We first consider the integral $\mathcal{I}_1(t)$.

Step 1. *There exists a positive number A_1, depending only on the constants M_2 in Corollary 3 and K_ϕ in Proposition 11, such that we have*

$$\big|\mathcal{I}_1(t)\big| \leq A_1 \Big(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x, t)\, dx \Big)^{\frac{1}{2}},$$

for any $t \in \mathbb{R}$.

In view of definitions (53) and (54), and assumption (57), we can estimate (3.12) by

$$\big|\mathcal{I}_1(t)\big| \leq A \int_{\mathbb{R}} \big(1 + |x|\big) \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx,$$

where A denotes, here as in the sequel, a positive number depending on K_ϕ. At this stage, we know that there exists a universal constant K such that

$$1 + |x| \leq K e^{\frac{|x|}{8}}, \tag{3.18}$$

for any $x \in \mathbb{R}$, so that we can bound $\mathcal{I}_1(t)$ by

$$\begin{aligned} \big|\mathcal{I}_1(t)\big| \leq & A\bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx\bigg)^{\frac{1}{2}} \\ & \times \bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, e^{\frac{|x|}{4}}\, dx\bigg)^{\frac{1}{2}}, \end{aligned}$$

It remains to invoke (51) to obtain the estimate in Step 1.

We now deal with the integral $\mathcal{I}_2^\theta(t)$.

Step 2. *There exists a positive number A_2, depending only on the constants M_2 in Corollary 3 and K_ϕ in Proposition 11, such that we have*

$$\big|\mathcal{I}_2^\theta(t)\big| \leq A_2\bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx\bigg)^{\frac{3}{4}},$$

for any $t \in \mathbb{R}$.

Similarly, we derive from (53), (54), (57) and (3.13) that

$$\big|\mathcal{I}_2^\theta(t)\big| \leq A \int_{\mathbb{R}} \big(1 + |x|\big) \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx,$$

Arguing as in the proof of Step 1 gives

$$\begin{aligned} \big|\mathcal{I}_2^\theta(t)\big| \leq & A\bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx\bigg)^{\frac{3}{4}} \\ & \times \bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, e^{\frac{|x|}{2}}\, dx\bigg)^{\frac{1}{4}}. \end{aligned}$$

The inequality in Step 2 then follows again from (51).

We next prove a similar estimate for $\mathcal{I}_2^a(t)$.

Step 3. *There exists a positive number A_3, depending only on the constants M_2 in Corollary 3 and K_ϕ in Proposition 11, such that we have*

$$\big|\mathcal{I}_2^a(t)\big| \leq A_3\bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx\bigg)^{\frac{3}{4}},$$

for any $t \in \mathbb{R}$.

The difference with respect to the integral $\mathcal{I}_2^\theta(t)$ lies in the presence of the third derivative $\partial_{xxx}\varepsilon_1^*$ in the expression of $\mathcal{I}_2^a(t)$. In order to bound this term, we integrate it by parts. We can check that

$$\partial_x \mathcal{T}_{c^*}(\varepsilon^*) = c^* \mathcal{L}_{c^*}^+(\varepsilon^*) - \sqrt{2} R_{c^*} \mathcal{L}_{c^*}^-(\varepsilon^*) - \sqrt{2} c^* \langle \varepsilon^*, \partial_x \varepsilon^* \rangle_{\mathbb{C}}, \tag{3.19}$$

so that we obtain

$$\begin{aligned} - \int_{\mathbb{R}} \phi_{c^*} \mathcal{T}_{c^*}(\varepsilon^*) \partial_{xxx}\varepsilon_1^* = & \int_{\mathbb{R}} \phi_{c^*} \partial_{xx}\varepsilon_1^* \big(c^* \mathcal{L}_{c^*}^+(\varepsilon^*) - \sqrt{2} R_{c^*} \mathcal{L}_{c^*}^-(\varepsilon^*) - \sqrt{2} c^* \langle \varepsilon^*, \partial_x \varepsilon^* \rangle_{\mathbb{C}} \big) \\ & + \int_{\mathbb{R}} \partial_x \phi_{c^*} \mathcal{T}_{c^*}(\varepsilon^*) \partial_{xx}\varepsilon_1^*. \end{aligned}$$

Inserting this identity into (3.14), and arguing as in the proof of Step 2, we are led to

$$\big| \mathcal{I}_2^a(t) \big| \leq A \big(1 + \| \varepsilon^* \|_{L^\infty} \big) \bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x, t) \, dx \bigg)^{\frac{3}{4}}.$$

In particular, we have to bound uniformly the perturbation ε^*. Invoking (40) and applying the Sobolev embedding theorem to the function $(1 - |U_0|^2)^{1/2} \varepsilon^*$ provide the inequality

$$(1 - |U_0|^2)^{\frac{1}{2}} \big| \varepsilon^* \big| \leq A \beta_*. \tag{3.20}$$

On the other hand, we deduce from (51) the existence of a positive number R, depending only on M_1, such that

$$\big| \varepsilon^*(\pm x, t) - \varepsilon^*(\pm R, t) \big| \leq \beta_*,$$

for any $x \geq R$ and $t \in \mathbb{R}$. Combining with (3.20), we deduce the existence of a positive number A, depending on M_1 through its dependence on R, such that

$$\| \varepsilon^* \|_{L^\infty} \leq A \beta_*. \tag{3.21}$$

Decreasing possibly the value of β_*, we can assume that $\beta_* \leq 1$, which completes the proof of Step 3.

Applying the Cauchy-Schwarz inequality to (3.15), we similarly prove

Step 4. *There exists a positive number A_4, depending only on the constant K_ϕ in Proposition 11, such that we have*

$$\big| \mathcal{I}_2^c(t) \big| \leq A_4 \bigg(\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x, t) \, dx \bigg)^{\frac{1}{2}},$$

for any $t \in \mathbb{R}$.

Concerning the integral $\mathcal{I}_2^L(t)$, a direct computation provides

Step 5. *There exists a positive number A_5, depending only the constants M_2 in Corollary 3, and K_ϕ in Proposition 11, such that we have*

$$\Big| \mathcal{I}_2^L(t) - \mathcal{G}^*(t) + \mathcal{R}^*(t) \Big| \leq A_5 \beta_* \int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x, t) \, dx,$$

for any $t \in \mathbb{R}$.

This further estimate essentially follows from integrating by parts the expression in (3.16). Invoking (3.19), we indeed obtain

$$\mathcal{I}_2^L(t) = \mathcal{G}^*(t) - \mathcal{R}^*(t) + \sqrt{2}c^*(t) \int_{\mathbb{R}} \Big[\partial_x \phi_{c^*(t)} \langle \varepsilon^*, \partial_x \varepsilon^* \rangle_{\mathbb{C}}\, \mathcal{L}_{c^*(t)}^-(\varepsilon^*) + \phi_{c^*(t)} \Big(\langle \varepsilon^*, \partial_{xx} \varepsilon^* \rangle_{\mathbb{C}}\, \mathcal{L}_{c^*(t)}^-(\varepsilon^*)$$
$$+ |\partial_x \varepsilon^*|^2 \mathcal{L}_{c^*(t)}^-(\varepsilon^*) + c^*(t) \langle \varepsilon^*, \partial_x \varepsilon^* \rangle_{\mathbb{C}}\, \mathcal{L}_{c^*(t)}^+(\varepsilon^*) - \sqrt{2} R_{c^*(t)} \langle \varepsilon^*, \partial_x \varepsilon^* \rangle_{\mathbb{C}}\, \mathcal{T}_{c^*(t)}(\varepsilon^*) \Big) \Big](x,t)\, dx.$$

In view of definitions (53) and (54), assumptions (57), and estimate (40), this provides

$$\begin{aligned} \Big| \mathcal{I}_2^L(t) - \mathcal{G}^*(t) + \mathcal{R}^*(t) \Big| \leq A\beta_* \int_{\mathbb{R}} \big(1 + |x|\big) \Big(|\partial_x \varepsilon^*(x,t)| + |\varepsilon^*(x,t)| \Big) \times \\ \times \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx. \end{aligned} \tag{3.22}$$

In order to bound the right-hand side of this inequality, we face the difficulty that the perturbation ε^* does not necessarily decay exponentially at infinity. As a matter of fact, we can combine (51) with (3.18) to obtain

$$\begin{aligned} \int_{\mathbb{R}} \big(1 + |x|\big) \big| \partial_x \varepsilon^*(x,t) \big| \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx \\ \leq A \int_{\mathbb{R}} \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx, \end{aligned} \tag{3.23}$$

for any $t \in \mathbb{R}$, but we cannot apply directly this argument for the term containing the absolute value $|\varepsilon^*|$. Instead, we deduce from (3.18) that

$$\begin{aligned} \int_{\mathbb{R}} \big(1 + |x|\big) \big| \varepsilon^*(x,t) \big| \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx \\ \leq A \int_{\mathbb{R}} \Big(|\varepsilon^*(x,t)| e^{-\frac{|x|}{8}} \Big) \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, e^{\frac{|x|}{4}}\, dx, \end{aligned}$$

so that, by (51),

$$\begin{aligned} \int_{\mathbb{R}} \big(1 + |x|\big) \big| \varepsilon^*(x,t) \big| \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx \\ \leq A \int_{\mathbb{R}} \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + |\varepsilon^*|^2 e^{-\frac{|\cdot|}{4}} + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx. \end{aligned} \tag{3.24}$$

At this stage, we can use the inequality

$$|\varepsilon^*(x,t)| \leq |\varepsilon^*(0,t)| + |x|^{\frac{1}{2}} \|\partial_x \varepsilon^*(\cdot,t)\|_{L^2},$$

and the Sobolev embedding theorem to check the existence of a positive number A such that

$$\int_{\mathbb{R}} |\varepsilon^*(x,t)|^2\, e^{-\frac{|x|}{4}}\, dx \leq A \int_{\mathbb{R}} \Big[|\partial_x \varepsilon^*|^2 + (1 - |U_{c^*}|^2) |\varepsilon^*|^2 \Big](x,t)\, dx. \tag{3.25}$$

The number A does not depend on c^* due to bound (40). Inserting into (3.24), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \big(1 + |x|\big) \big| \varepsilon^*(x,t) \big| \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx \\ \leq A \int_{\mathbb{R}} \Big[|\partial_{xx} \varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*}|^2) |\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big](x,t)\, dx. \end{aligned} \tag{3.26}$$

Similarly, we deduce from (3.18) and (3.21) that

$$\int_{\mathbb{R}} \big(1+|x|\big)\big(1-|U_{c^*}(x)|^2\big)\big|\varepsilon^*(x,t)\big|^3\,dx \leq A\beta_* \int_{\mathbb{R}} \big|\varepsilon^*(x,t)\big|^2 e^{-\frac{|x|}{4}}\,dx,$$

Here, we have assumed, decreasing the value of β_* if necessary, that $|c^*| \leq 1$. Combining with (3.23) and (3.26), and inserting into (3.22), we obtain the estimate in Step 5 (when $\beta_* \leq 1$, which we can assume without loss of generality).

We finally turn to the integral $\mathcal{I}_2^N(t)$.

Step 6. *There exists a positive number A_6, depending only the constants M_k in Corollary 3, and K_ϕ in Proposition 11, such that we have*

$$\begin{aligned}\Big|\mathcal{I}_2^N(t)\Big| \leq A_6\bigg(\int_{\mathbb{R}} \Big[|\partial_x\varepsilon^*|^2 + (1-|U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\,dx,\bigg)^{\frac{9}{8}} \\ \times\bigg(1+\bigg(\int_{\mathbb{R}} \Big[|\partial_x\varepsilon^*|^2 + (1-|U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\,dx,\bigg)^{\frac{1}{8}}\bigg),\end{aligned}$$

for any $t \in \mathbb{R}$.

In order to prove this inequality, we first check that

$$\big|\partial_x\eta_{\varepsilon^*}\big| \leq A\Big(|\partial_x\varepsilon^*| + (1-|U_{c^*}|^2)|\varepsilon^*|\Big), \tag{3.27}$$

while

$$\big|\partial_{xx}\eta_{\varepsilon^*}\big| \leq A\Big(|\partial_{xx}\varepsilon^*| + |\partial_x\varepsilon^*| + (1-|U_{c^*}|^2)|\varepsilon^*|\Big),$$

where A refers to a positive number depending only on the constant M_1 in Corollary 3. For $\beta_* \leq 1$, this follows from the bound for c^* in (40), and estimates (51) and (3.21).

Invoking (57) and (3.18), we next estimate the right-hand side of (3.17) as

$$\begin{aligned}\Big|\mathcal{I}_2^N(t)\Big| \leq A\int_{\mathbb{R}} \Big[\big(|\partial_{xx}\varepsilon^*| + |\partial_x\varepsilon^*| + |\varepsilon^*|\big)\times \\ \times\big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1-|U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\big)\Big](x,t)\,e^{\frac{|x|}{8}}\,dx.\end{aligned}$$

We then modify slightly the arguments in the proof of Step 5. We first invoke the exponential decay of ε^* in (51), and the exponential decay of $1-|U_{c^*}|^2$ to write

$$\Big|\mathcal{I}_2^N(t)\Big| \leq A\int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^{\frac{5}{2}} + |\partial_x\varepsilon^*|^{\frac{5}{2}} + e^{-\frac{5|\cdot|}{16}}|\varepsilon^*|^{\frac{5}{2}} + \eta_{\varepsilon^*}^{\frac{5}{2}}\Big](x,t)\,dx.$$

Observe that the positive number A does not depend on c^* due to bound (40). Applying the Sobolev embedding theorem, and invoking (3.27), we are led to

$$\Big|\mathcal{I}_2^N(t)\Big| \leq A\bigg(\int_{\mathbb{R}} \Big[|\partial_{xxx}\varepsilon^*|^2 + |\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + e^{-\frac{|\cdot|}{4}}|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\,dx\bigg)^{\frac{5}{4}}. \tag{3.28}$$

At this stage, recall that

$$\int_{\mathbb{R}} (\partial_x^p f)^2 = (-1)^p\int_{\mathbb{R}} f(\partial_x^{2p} f) \leq \bigg(\int_{\mathbb{R}} f^2\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}} (\partial_x^{2p} f)^2\bigg)^{\frac{1}{2}},$$

for any integer p and any function $f \in H^{2p}(\mathbb{R})$. In view of (51), we can apply inductively these inequalities to the function $\partial_x \varepsilon^*$ in order to obtain

$$\int_{\mathbb{R}} |\partial_{xxx}\varepsilon^*(x,t)|^2\, dx \leq A\Big(\int_{\mathbb{R}} |\partial_x \varepsilon^*(x,t)|^2\, dx\Big)^{\frac{9}{10}}.$$

The proof of Step 6 then reduces to introduce this estimate into (3.28) and apply again (3.25).

In order to conclude the proof of Proposition 11, we now gather the estimates in the six previous steps. In view of (40) and (41), they provide the inequality

$$\begin{aligned}&\Big|\frac{d}{dt}\big(\mathcal{M}^*(t)\big) - \mathcal{G}^*(t) + \mathcal{R}^*(t)\Big|\\ &\quad \leq A\Big(\beta_* + \big(\beta_*\big)^{\frac{1}{2}} + \big(\beta_*\big)^{\frac{1}{4}}\Big) \int_{\mathbb{R}} \Big[|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + (1 - |U_{c^*(t)}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big](x,t)\, dx,\end{aligned}$$

where the number A depends only on the constant K_ϕ in Proposition 11, and the numbers M_k in Corollary 3. When $\beta_* \leq 1$, this amounts to (58), which ends the proof of Proposition 11. □

3.3.2 Proof of Proposition 12

For $|c| < \sqrt{2}$, we set

$$\begin{aligned}\mathfrak{G}_c^1(\varepsilon) = \frac{1}{2}\int_{\mathbb{R}} \big(1 - |U_c|^2\big)\Big(&3\big(\partial_{xx}\varepsilon_1 + c\partial_x\varepsilon_2 + (1 - |U_c|^2 + 2R_c^2)\varepsilon_1 - \sqrt{2}cR_c\varepsilon_2\big)^2\\ &+ \Big(\partial_{xx}\varepsilon_2 + \sqrt{2}R_c\partial_x\varepsilon_2 - 2c\big(\partial_x\varepsilon_1 + c\varepsilon_2 + \sqrt{2}R_c\varepsilon_1\big)\Big)^2\Big).\end{aligned}$$

The quadratic form $\mathfrak{G}_c^1$ is well-defined for any function $\varepsilon \in \mathcal{H}(\mathbb{R})$, with $\partial_{xx}\varepsilon \in L^2(\mathbb{R})$. Moreover, it appears as the main component of the quantity $\mathcal{G}_1^*$. As a matter of fact, since $\eta_{\varepsilon^*} = 2\langle U_{c^*}, \varepsilon^*\rangle_{\mathbb{C}} + |\varepsilon^*|^2$, we have

$$\Big|\mathcal{G}_1^* - \mathfrak{G}_{c^*}^1(\varepsilon^*)\Big| \leq K\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\big|\varepsilon^*\big|^2\Big(\big|\partial_{xx}\varepsilon^*\big| + \big|\partial_x\varepsilon^*\big| + \big|\varepsilon^*\big| + \big|\varepsilon^*\big|^2\Big),$$

where K refers, here as in the sequel, to a universal constant. In view of (3.21), this provides

$$\Big|\mathcal{G}_1^* - \mathfrak{G}_{c^*}^1(\varepsilon^*)\Big| \leq K\beta_*\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\Big(\big|\partial_{xx}\varepsilon^*\big|^2 + \big|\partial_x\varepsilon^*\big|^2 + \big|\varepsilon^*\big|^2\Big), \tag{3.29}$$

when $\beta_* \leq 1$. As a consequence, decreasing if necessary the value of β_*, the proof of (59) for the quantity $\mathcal{G}_1^*$ reduces to establish the same estimate for the quadratic form $\mathfrak{G}_{c^*}^1(\varepsilon^*)$.

In order to do so, we rely on a perturbative argument. Indeed, under the orthogonality conditions in (16), the coercivity of the functional $\mathfrak{G}_c^1$ for c small enough results from the coercivity of $\mathfrak{G}_0^1$. More precisely, let us introduce the change of variables

$$v = (v_1, v_2) := \big((1 - |U_c|^2)^{\frac{1}{2}}\varepsilon_1, (1 - |U_c|^2)^{\frac{1}{2}}\varepsilon_2\big). \tag{3.30}$$

Using the identity $R_c' = (1 - |U_c|^2)/\sqrt{2}$, we compute

$$(1 - |U_c|^2)^{\frac{1}{2}}\partial_x\varepsilon = \partial_x v + \frac{R_c}{\sqrt{2}}v, \quad \text{and} \quad (1 - |U_c|^2)^{\frac{1}{2}}\partial_{xx}\varepsilon = \partial_{xx}v + \sqrt{2}R_c\partial_x v + \frac{1}{2}\big(1 - |U_c|^2 + R_c^2\big)v.$$

As a consequence, the quadratic form $\mathfrak{G}_c^1(\varepsilon)$ writes in terms of the variables v_1 and v_2 as

$$\mathfrak{G}_c^1(\varepsilon) = \frac{1}{2} J_c(v) := \frac{1}{2} \int_{\mathbb{R}} \Big(3\Big(- \partial_{xx} v_1 - \sqrt{2} R_c \partial_x v_1 - c \partial_x v_2 - \frac{3}{2}(1 - |U_c|^2 - R_c^2) v_1 + \frac{c}{\sqrt{2}} R_c v_2 \Big)^2 + \Big(- \partial_{xx} v_2 + 2c \partial_x v_1 - 2\sqrt{2} R_c \partial_x v_2 + 3\sqrt{2} c R_c v_1 - \frac{1}{2}(1 - |U_c|^2 + 3R_c^2 - 4c^2) v_2 \Big)^2 \Big). \tag{3.31}$$

In view of (4), the quadratic form J_c is an analytic perturbation of J_0 for c small enough. As a first step, we establish the coercivity of $J_0(v)$ under three orthogonality conditions for v, which correspond to the conditions for ε in (16) (when $c = 0$).

Step 1. *There exists a positive number κ such that*

$$\begin{aligned} J_0(v) =& 3 \int_{\mathbb{R}} \Big(\partial_{xx} v_1 + \sqrt{2} U_0 \partial_x v_1 + \Big(\frac{3}{2} - 3U_0^2 \Big) v_1 \Big)^2 + \int_{\mathbb{R}} \Big(\partial_{xx} v_2 + 2\sqrt{2} U_0 \partial_x v_2 + \Big(\frac{1}{2} + U_0^2 \Big) v_2 \Big)^2 \\ \geq & \kappa \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big), \end{aligned} \tag{3.32}$$

for any function $v \in H^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v_1 (U_0')^{\frac{3}{2}} = \int_{\mathbb{R}} v_2 (U_0')^{\frac{1}{2}} = \int_{\mathbb{R}} v_2 U_0 (U_0')^{\frac{1}{2}} = 0. \tag{3.33}$$

Indeed, we may write the quadratic form J_0 as $J_0(v) := \langle \mathcal{J}_0^1(v_1), v_1 \rangle_{L^2} + \langle \mathcal{J}_0^2(v_2), v_2 \rangle_{L^2}$, with

$$\mathcal{J}_0^1(v_1) = \partial_{xxxx} v_1 - \partial_x \big((9U_0^2 - 4) \partial_x v_1 \big) + \Big(\frac{27}{2} U_0^2 - \frac{9}{4} - 9U_0^4 \Big) v_1,$$

and

$$\mathcal{J}_0^2(v_2) = \partial_{xxxx} v_2 - \partial_x ((8U_0^2 - 3) \partial_x v_2) + (10U_0^4 - 8U_0^2 + 1/4) v_2.$$

The operators $\mathcal{J}_0^1$ and $\mathcal{J}_0^2$ are self-adjoint and non-negative on $L^2(\mathbb{R})$, with domain $H^4(\mathbb{R})$. In view of (3.32), their kernels are spanned by the function $(U_0')^{3/2}$, respectively $(U_0')^{1/2}$ and $U_0 (U_0')^{1/2}$. The Weyl criterion shows that their essential spectrum is equal to the interval $[9/4, +\infty)$. As a consequence, there exists a positive number κ such that

$$J_0(v) \geq \kappa \int_{\mathbb{R}} |v|^2, \tag{3.34}$$

when v satisfies the orthogonality conditions in (3.33).

Using the inequalities $|U_0| \leq 1$ and

$$\| \partial_x v \|_{L^2} \leq \| v \|_{L^2}^{\frac{1}{2}} \| \partial_{xx} v \|_{L^2}^{\frac{1}{2}}, \tag{3.35}$$

we observe that

$$J_0(v) \geq \frac{1}{2} \int_{\mathbb{R}} |\partial_{xx} v|^2 - \frac{41}{4} \int_{\mathbb{R}} v_1^2 - \frac{49}{4} \int_{\mathbb{R}} v_2^2,$$

for any $v \in H^2(\mathbb{R})$. Given a parameter $0 < \tau < 1$, we deduce from (3.34) that

$$J_0(v) \geq \frac{\tau}{2} \int_{\mathbb{R}} |\partial_{xx} v|^2 + \Big(\kappa (1 - \tau) - \frac{49}{4} \tau \Big) \int_{\mathbb{R}} |v|^2.$$

Choosing τ small enough, and invoking (3.35), we obtain (3.32) for a further choice of the positive number κ.

We now extend the conclusions of Step 1 to the quadratic form J_c for c small enough.

Step 2. *There exists a number $\sigma \in (0, \sqrt{2})$ such that, when $|c| \leq \sigma$, we have*

$$J_c(v) \geq \frac{\kappa}{2} \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big), \tag{3.36}$$

for any function $v \in H^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v_1 (U_c')^{\frac{3}{2}} = \int_{\mathbb{R}} v_2 (U_c')^{\frac{1}{2}} = \int_{\mathbb{R}} v_2 R_c (U_c')^{\frac{1}{2}} = 0. \tag{3.37}$$

Assume first that v satisfies the orthogonality conditions in (3.33). We then infer from (3.32) that

$$J_c(v) \geq \kappa \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big) - \big| J_c(v) - J_0(v) \big|. \tag{3.38}$$

In view of (3.31), we have

$$\big| J_c(v) - J_0(v) \big| \leq A \big(c + \| R_c - U_0 \|_{L^\infty} \big) \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big). \tag{3.39}$$

where, here as in the sequel, A is a positive number. On the other hand, given any positive numbers μ and λ with $\lambda > \mu$, we compute

$$\begin{aligned} \Big| \tanh\Big(\frac{x}{\sqrt{2}}\Big) - \tanh\Big(\frac{\sqrt{2-c^2}}{2} x\Big) \Big| = & \Big| x \int_{\frac{\sqrt{2-c^2}}{2}}^{\frac{1}{\sqrt{2}}} (1 - \tanh(sx)^2) \, ds \Big| \\ \leq & \big(\sqrt{2} - \sqrt{2-c^2} \big) |x| \Big| 1 - \tanh\Big(\frac{\sqrt{2-c^2}}{2} x \Big)^2 \Big|, \end{aligned} \tag{3.40}$$

for any $x \in \mathbb{R}$. This gives

$$\| R_c - U_0 \|_{L^\infty} \leq A c^2,$$

when $|c| < 1$. In view of (3.38) and (3.39), we derive the existence of a number $\sigma \in (0, \sqrt{2})$ such that

$$J_c(v) \geq \frac{7\kappa}{8} \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big), \tag{3.41}$$

when $|c| \leq \sigma$, and when v satisfies the orthogonality conditions in (3.33).

In order to modify these orthogonality conditions, we argue as in the proof of Corollary 1. Given a function v which satisfies the orthogonality conditions in (3.37), we decompose it as

$$v_1 = w_1 + \mu_1 (U_0')^{\frac{3}{2}}, \quad \text{and} \quad v_2 = w_2 + \mu_2 (U_0')^{\frac{3}{2}} + \nu_2 U_0 (1 - |U_0|^2)^{\frac{1}{2}},$$

where w satisfies (3.33). Coming back to the definition of J_c in (3.31), we check the existence of a further positive number A such that

$$J_c(v) \geq \frac{6}{7} J_c(w) - A \Big(\mu_1^2 + \mu_2^2 + \nu_2^2 \Big).$$

Similarly, we have

$$\int_{\mathbb{R}} \Big(|\partial_{xx} w|^2 + |\partial_x w|^2 + |w|^2 \Big) \geq \frac{5}{6} \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big) - A \Big(\mu_1^2 + \mu_2^2 + \nu_2^2 \Big).$$

Hence, we can apply (3.41) to obtain

$$J_c(v) \geq \frac{5\kappa}{8} \int_{\mathbb{R}} \Big(|\partial_{xx} v|^2 + |\partial_x v|^2 + |v|^2 \Big) - A \Big(\mu_1^2 + \mu_2^2 + \nu_2^2 \Big). \tag{3.42}$$

At this stage, we recall that

$$\mu_1 = \frac{\langle v_1, (U_0')^{\frac{3}{2}} \rangle_{L^2}}{\|U_0'\|_{L^3}^3}, \qquad \mu_2 = \frac{\langle v_2, (U_0')^{\frac{3}{2}} \rangle_{L^2}}{\|U_0'\|_{L^3}^3} \quad \text{and} \quad \nu_2 = \frac{\langle v_2, U_0 (1 - |U_0|^2)^{\frac{1}{2}} \rangle_{L^2}}{\|U_0^2 (1 - |U_0|^2)\|_{L^1}}.$$

Using (3.37), we observe that

$$\big\langle v_1, (U_0')^{\frac{3}{2}} \big\rangle_{L^2} = \big\langle v_1, (U_0')^{\frac{3}{2}} - (U_c')^{\frac{3}{2}} \big\rangle_{L^2}.$$

Given a positive number α, we argue as in (3.40) to derive the existence of a positive number A_α, depending only on α, such that

$$\big| (U_c'(x))^\alpha - (U_0(x)')^\alpha \big| \leq A_\alpha c^2 (U_c'(x))^{\frac{\alpha}{2}},$$

for any $x \in \mathbb{R}$ and any $|c| \leq 1$. This yields the inequality

$$|\mu_1| \leq A c^2 \|v_1\|_{L^2},$$

and similarly, we obtain

$$|\mu_2| + |\nu_2| \leq A c^2 \|v_2\|_{L^2}.$$

Combining with (3.42), and decreasing, if necessary, the value of the number σ, we obtain (3.36).

We now rewrite this inequality, as well as the orthogonality conditions in (3.37), in terms of the function ε.

Step 3. *There exists a further positive number κ such that, when $|c| \leq \sigma$, we have*

$$\mathfrak{G}_c^1(\varepsilon) \geq \kappa \int_{\mathbb{R}} \big(1 - |U_c|^2\big) \Big(|\partial_{xx}\varepsilon|^2 + |\partial_x \varepsilon|^2 + |\varepsilon|^2 \Big), \tag{3.43}$$

for any function $\varepsilon \in \mathcal{H}(\mathbb{R})$, with $\partial_{xx}\varepsilon \in L^2(\mathbb{R})$, which satisfies the orthogonality conditions in (16).

In view of (3.30) and (3.31), we can rewrite (3.38) as

$$\mathfrak{G}_c^1(\varepsilon) \geq \frac{\kappa}{2} \int_{\mathbb{R}} (1 - |U_c|^2) \Big(\Big| \partial_{xx}\varepsilon - \sqrt{2} R_c \partial_x \varepsilon - \frac{1}{2}(1 - R_c^2 - |U_c|^2)\varepsilon \Big|^2 + \Big| \partial_x \varepsilon - \frac{R_c}{\sqrt{2}} \varepsilon \Big|^2 + |\varepsilon|^2 \Big),$$

and this inequality holds when ε satisfies the three orthogonality conditions

$$\int_{\mathbb{R}} \varepsilon_1 (U_c')^2 = \int_{\mathbb{R}} \varepsilon_2 U_c' = \int_{\mathbb{R}} \varepsilon_2 R_c (1 - |U_c|^2) = 0. \tag{3.44}$$

We then recall that $|R_c| \leq 1$, $|U_c| \leq 1$ and $|c| < \sqrt{2}$, and we apply the inequality

$$\|a - b\|_H^2 \geq \tau \|a\|_H^2 - \frac{\tau}{1 - \tau} \|b\|_H^2,$$

which holds for any number $0 < \tau < 1$, and any vectors a and b in an Hilbert space H, in order to obtain (3.43) for a further choice of the number κ.

Comparing the orthogonality conditions in (16) with the ones in (3.44), we observe that it remains to extend (3.43) in the situation where the function ε_1 satisfies the alternative orthogonality condition $\langle \varepsilon_1, U_c' \rangle_{L^2} = 0$. In this case, since $\langle U_c', (U_c')^2 \rangle_{L^2} > 0$, we can decompose ε as

$\varepsilon = \nu U_c' + u$, with $\nu \in \mathbb{R}$ and $\langle u, (U_c')^2\rangle_{L^2} = 0$. The derivative U_c' belongs to the kernel of the quadratic form $\mathfrak{G}_c^1$. Hence, we infer from (3.43) that

$$\mathfrak{G}_c^1(\varepsilon) = \mathfrak{G}_c^1(u) \geq \kappa \int_{\mathbb{R}} \big(1 - |U_c|^2\big)\Big(|\partial_{xx} u|^2 + |\partial_x u|^2 + |u|^2\Big). \tag{3.45}$$

On the other hand, the number ν is equal to $\nu = -\langle u, U_c'\rangle_{L^2}/\|U_c'\|_{L^2}^2$. Since $|c| \leq \sigma$, there exists a positive number A such that

$$\nu^2 \leq A \int_{\mathbb{R}} (1 - |U_c|^2)|u|^2.$$

In view of (3.45), we first conclude that

$$\mathfrak{G}_c^1(u) \geq \frac{\kappa}{A}\nu^2,$$

and then, that (3.43) remains available under the orthogonality condition $\langle \varepsilon_1, U_c'\rangle_{L^2} = 0$.

We are now in position to provide the

Step 4. *End of the proof of Proposition 12.*

In view of (40), we can decrease the value of β_* so that $|c^*(t)| \leq \sigma$ for any $t \in \mathbb{R}$. In this case, we can apply Step 3, since the function $\varepsilon^*(\cdot, t)$ satisfies the orthogonality conditions in (16) by (39). As a consequence of (3.29) and (3.43), we deduce that

$$\mathcal{G}_1^*(t) \geq \big(\kappa - K\beta_*\big) \int_{\mathbb{R}} \big(1 - |U_{c^*(t)}|^2\big)\Big(\big|\partial_{xx}\varepsilon^*(\cdot, t)\big|^2 + \big|\partial_x \varepsilon^*(\cdot, t)\big|^2 + \big|\varepsilon^*(\cdot, t)\big|^2\Big),$$

for any $t \in \mathbb{R}$. It remains to decrease again, if necessary, the value of β_*, to conclude the proof of Proposition 12. □

3.3.3 Proof of Lemma 1

In view of (4), the numbers $\phi_c(x)$ depend analytically on $x \in \mathbb{R}$ and $c \in (-\sqrt{2}, \sqrt{2})$, so that the functions ϕ_c are smooth on $\mathbb{R}$, and depend smoothly on $c \in (-\sqrt{2}, \sqrt{2})$. When $|c| < 1$, we can find a positive number A such that

$$\Big|\phi_c(x) - \sqrt{\frac{2}{2 - c^2}}|x|\Big| + \Big|\partial_x\phi_c(x) - \frac{2}{2 - c^2}R_c(x)\Big| + \big|\partial_{xx}\phi_c(x)\big| \leq A\big(1 + |x|\big)\big(1 - |U_c(x)|^2\big), \tag{3.46}$$

for any $x \in \mathbb{R}$. Similarly, we have

$$\partial_c\phi_c(x) = -\frac{x\partial_c R_c(x)}{R_c(x)^2} = \frac{c}{2 - c^2}\Big(\frac{x}{R_c(x)} + \frac{x^2}{\sqrt{2}R_c(x)^2}\big(1 - |U_c(x)|^2\big)\Big),$$

so that

$$\Big|\partial_c\phi_c(x)\Big| \leq A\big(1 + |x|\big).$$

Combining with (3.46), we obtain (57). This completes the proof of Lemma 1. □

3.3.4 Proof of Proposition 13

For the choice $\phi_c(x) = x/R_c(x)$, it follows from (55) and (56) that

$$\begin{aligned} & \mathcal{G}_2^*(t) - \mathcal{R}_2^*(t) \\ & = \int_{\mathbb{R}} \Big[\Big(\frac{1}{\sqrt{2}} + (1 - |U_{c^*}|^2)\phi_{c^*} \Big) \mathcal{L}_{c^*}^+(\varepsilon^*)^2 + \Big(\frac{1}{\sqrt{2}} + \sqrt{2}(\partial_x \phi_{c^*}) R_{c^*} \Big) \mathcal{L}_{c^*}^-(\varepsilon^*)^2 + \frac{1}{\sqrt{2}} \mathcal{T}_{c^*}(\varepsilon^*)^2 \\ & \quad - c^*(\partial_x \phi_{c^*}) \mathcal{L}_{c^*}^+(\varepsilon^*) \big(\mathcal{L}_{c^*}^-(\varepsilon^*) + \mathcal{T}_{c^*}(\varepsilon^*) \big) - \big(\partial_{xx}\phi_{c^*} + \phi_{c^*}(1 - |U_{c^*}|^2) \big) \mathcal{T}_{c^*}(\varepsilon^*) \mathcal{L}_{c^*}^-(\varepsilon^*) \Big](x,t)\, dx, \end{aligned}$$

for any $t \in \mathbb{R}$. Applying the inequality $|ab| \leq (a^2 + b^2)/2$, we obtain

$$\begin{aligned} \mathcal{G}_2^*(t) - \mathcal{R}_2^*(t) \geq & \frac{1}{2} \int_{\mathbb{R}} \Big[\big(\sqrt{2} + 2(1 - |U_{c^*}|^2)\phi_{c^*} - 2|c^*||\partial_x \phi_{c^*}| \big) \mathcal{L}_{c^*}^+(\varepsilon^*)^2 \Big](x,t)\, dx \\ & + \frac{1}{2} \int_{\mathbb{R}} \Big[\big(\sqrt{2} + 2\sqrt{2}(\partial_x \phi_{c^*}) R_{c^*} - |c^*||\partial_x \phi_{c^*}| - |\partial_{xx}\phi_{c^*}| - |\phi_{c^*}|(1 - |U_{c^*}|^2) \big) \mathcal{L}_{c^*}^-(\varepsilon^*)^2 \Big](x,t)\, dx \\ & + \frac{1}{2} \int_{\mathbb{R}} \Big[\big(\sqrt{2} - |c^*||\partial_x \phi_{c^*}| - |\partial_{xx}\phi_{c^*}| - |\phi_{c^*}|(1 - |U_{c^*}|^2) \big) \mathcal{T}_{c^*}(\varepsilon^*)^2 \Big](x,t)\, dx. \end{aligned} \tag{3.47}$$

In order to estimate this quantity, we now decrease, if necessary, the value of β_* so that we can deduce from (40) that $|c^*(t)| < 1$ for any $t \in \mathbb{R}$. In this case, we can apply (3.46) and decrease again the value of β_* in order to obtain the inequality

$$2|c^*(t)| \|\partial_x \phi_{c^*(t)}\|_{L^\infty} \leq \sqrt{2} - 1,$$

for any $t \in \mathbb{R}$. Invoking again (3.46), we can find a positive number A such that we have the three inequalities

$$1 + 2(1 - |U_{c^*(t)}|^2)\phi_{c^*(t)} \geq \frac{1}{2} - 2A(1 - |U_{c^*(t)}|^2),$$

$$1 + 2\sqrt{2}\partial_x \phi_{c^*(t)} R_{c^*(t)} - |\partial_{xx}\phi_{c^*(t)}| - |\phi_{c^*(t)}|(1 - |U_{c^*(t)}|^2) \geq \frac{1}{2} - 2A(1 - |U_{c^*(t)}|^2),$$

and

$$1 - |\partial_{xx}\phi_{c^*(t)}| - |\phi_{c^*(t)}|(1 - |U_{c^*(t)}|^2) \geq \frac{1}{2} - 2A(1 - |U_{c^*(t)}|^2).$$

On the other hand, we infer from (53) and (54) that

$$\mathcal{L}_{c^*}^+(\varepsilon^*)^2 + \mathcal{L}_{c^*}^-(\varepsilon^*)^2 + \mathcal{T}_{c^*}(\varepsilon^*)^2 \leq A\Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + (1 - |U_{c^*}|^2)|\varepsilon^*|^2 + \eta_{\varepsilon^*}^2 \Big),$$

so that, by (3.21),

$$\big(1 - |U_{c^*(t)}|^2\big)\Big(\mathcal{L}_{c^*}^+(\varepsilon^*)^2 + \mathcal{L}_{c^*}^-(\varepsilon^*)^2 + \mathcal{T}_{c^*}(\varepsilon^*)^2 \Big) \leq A\big(1 - |U_{c^*(t)}|^2\big)\Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + |\varepsilon^*|^2 \Big).$$

Gathering all these inequalities and applying them to (3.47), we are left with the estimate

$$\begin{aligned} \mathcal{G}_2^*(t) - \mathcal{R}_2^*(t) \geq & \frac{1}{4} \int_{\mathbb{R}} \Big[\mathcal{L}_{c^*}^+(\varepsilon^*)^2 + \mathcal{L}_{c^*}^-(\varepsilon^*)^2 + \mathcal{T}_{c^*}(\varepsilon^*)^2 \Big](x,t)\, dx \\ & - A \int_{\mathbb{R}} \Big[\big(1 - |U_{c^*}|^2\big)\big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + |\varepsilon^*|^2 \big) \Big](x,t)\, dx. \end{aligned} \tag{3.48}$$

It now remains to bound the first integral in the right-hand side of (3.48) according to (60). We first address the integral of the quantity $\mathcal{L}_{c^*}^+(\varepsilon^*)^2$. Coming back to (53), we check that

$$\int_{\mathbb{R}} \mathcal{L}_c^+(\varepsilon^*)^2 \geq \int_{\mathbb{R}} \Big(\partial_{xx}\varepsilon_1^* + c^* \partial_x \varepsilon_2^* - R_{c^*}\eta_{\varepsilon^*} \Big)^2 - A \int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big) \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x \varepsilon^*|^2 + |\varepsilon^*|^2 \Big). \tag{3.49}$$

Here, we have used the inequality

$$\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\eta_{\varepsilon^*}^2 \leq A \int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)|\varepsilon^*|^2,$$

which is a consequence of (3.21).

At this stage, we expand and integrate by parts the first integral in the right-hand side of (3.49) in order to get

$$\begin{aligned}\int_{\mathbb{R}} \Big(\partial_{xx}\varepsilon_1^* + c^*\partial_x\varepsilon_2^* - R_{c^*}\eta_{\varepsilon^*}\Big)^2 = \int_{\mathbb{R}} \bigg(&(\partial_{xx}\varepsilon_1^*)^2 + (c^*)^2(\partial_x\varepsilon_2^*)^2 + 2R_{c^*}(\partial_x\varepsilon_1^*)(\partial_x\eta_\varepsilon^*) + R_{c^*}^2\eta_{\varepsilon^*}^2 \\ &+ 2c^*(\partial_x\varepsilon_2^*)\Big(\partial_{xx}\varepsilon_1^* - R_{c^*}\eta_{\varepsilon^*}\Big) + 2(\partial_x R_{c^*})(\partial_x\varepsilon_1^*)\eta_{\varepsilon^*}\bigg).\end{aligned}$$

We next recall that $R_c^2 = (2 - c^2)/2 - (1 - |U_c|^2)$ and $\partial_x R_c = (1 - |U_c|^2)/\sqrt{2}$. Moreover, we observe that

$$\begin{aligned}\int_{\mathbb{R}} R_{c^*}(\partial_x\varepsilon_1^*)(\partial_x\eta_\varepsilon^*) = \int_{\mathbb{R}} \Big(&2R_{c^*}^2(\partial_x\varepsilon_1^*)^2 + \sqrt{2}c^*R_{c^*}(\partial_x\varepsilon_1^*)(\partial_x\varepsilon_2^*) \\ &+ 2R_{c^*}(\partial_x\varepsilon_1^*)\big((\partial_x R_{c^*})\varepsilon_1^* + \langle\varepsilon^*, \partial_x\varepsilon^*\rangle_{\mathbb{C}}\big)\Big).\end{aligned}$$

Introducing these identities into (3.49), we are led to

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}_{c^*}^+(\varepsilon^*)^2 \geq &\int_{\mathbb{R}} \Big((\partial_{xx}\varepsilon_1^*)^2 + 4(\partial_x\varepsilon_1^*)^2 + \eta_{\varepsilon^*}^2\Big) - A\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + |\varepsilon^*|^2\Big) \\ &- A\big(|c^*| + \|\varepsilon^*\|_{L^\infty}\big)\int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big).\end{aligned} \tag{3.50}$$

Arguing similarly for the quantities $\mathcal{L}_{c^*}^-(\varepsilon^*)^2$ and $\mathcal{T}_{c^*}(\varepsilon^*)^2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}_{c^*}^-(\varepsilon^*)^2 \geq &\int_{\mathbb{R}} \big(\partial_{xx}\varepsilon_2^*\big)^2 - A|c^*|\int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big) \\ &- A\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + |\varepsilon^*|^2\Big),\end{aligned} \tag{3.51}$$

and

$$\int_{\mathbb{R}} \mathcal{T}_{c^*}(\varepsilon^*)^2 \geq 2\int_{\mathbb{R}} \big(\partial_x\varepsilon_2^*\big)^2 - A|c^*|\int_{\mathbb{R}} \Big(|\partial_x\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big) - A\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\Big(|\partial_x\varepsilon^*|^2 + |\varepsilon^*|^2\Big). \tag{3.52}$$

Inserting (3.50), (3.51) and (3.52) into (3.48), we conclude that

$$\begin{aligned}\mathcal{G}_2^*(t) - \mathcal{R}_2^*(t) \geq &\frac{1}{4}\int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + 2|\partial_x\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big) - A\int_{\mathbb{R}} \big(1 - |U_{c^*}|^2\big)\Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + |\varepsilon^*|^2\Big) \\ &- A\big(|c^*| + \|\varepsilon^*\|_{L^\infty}\big)\int_{\mathbb{R}} \Big(|\partial_{xx}\varepsilon^*|^2 + |\partial_x\varepsilon^*|^2 + \eta_{\varepsilon^*}^2\Big).\end{aligned} \tag{3.53}$$

For β_* small enough, we can invoke (40) and (3.21) in order to bound the third integral in the right-hand side of (3.53) by one half of the first one. This provides (60), which completes the proof of Proposition 13. □

A Definition and properties of the momentum

This appendix is devoted to the proofs of Proposition 4, which links the quantities $\mathcal{P}$ and $[P]$, and of Proposition 5, which gives a first-order expansion of $\mathcal{P}$ for a perturbation of the soliton U_c.

A.1 Proof of Proposition 4

Let $\Psi \in \mathcal{V}_0(\alpha)$, with $\alpha < \alpha_1$. In view of (18), there exists a positive number A such that

$$d_0(\Psi_{\rm mod}, U_0) \leq A\alpha.$$

Set $\varepsilon_0 := \Psi_{\rm mod} - U_0$ and $\eta_0 := |\Psi_{\rm mod}|^2 - U_0^2$. Invoking the Sobolev embedding theorem, we have

$$\|\eta_0\|_{L^\infty}^2 \leq 4\|\eta_0\|_{L^2}\big\|\langle U_0', \varepsilon_0\rangle_{\mathbb{C}} + \langle U_0, \varepsilon_0'\rangle_{\mathbb{C}} + \langle \varepsilon_0, \varepsilon_0'\rangle_{\mathbb{C}}\big\|_{L^2}.$$

Since $|U_0| \leq 1$, $U_0' = (1 - U_0^2)/\sqrt{2}$ and $|\varepsilon_0| \leq 3 + |\eta_0|$, we deduce that

$$\|\eta_0\|_{L^\infty}^2 \leq A\big(1 + \|\eta_0\|_{L^\infty}\big)\big\|\varepsilon_0\big\|_{\mathcal{H}_0}\|\eta_0\|_{L^2} \leq A\big(1 + \|\eta_0\|_{L^\infty}\big)\alpha^2. \tag{A.1}$$

Since $U_0(1) \geq 5\sqrt{2}/12$, this gives

$$|\Psi_{\rm mod}(x)|^2 \geq |U_0(x)|^2 - \|\eta_0\|_{L^\infty} \geq \frac{25}{72} - A\alpha,$$

for any $|x| \geq 1$. As a consequence, we can choose α small enough such that $|\Psi_{\rm mod}| \geq 1/2$ outside $(-1, 1)$. In particular, we are allowed to write $\Psi_{\rm mod} := \varrho_{\rm mod} \exp i\varphi_{\rm mod}$. Moreover, we can apply the Sobolev embedding theorem to obtain

$$\big\|(1 - U_0^2)^{1/2}(\Psi_{\rm mod} - U_0)\big\|_{L^\infty} \leq A\big\|\Psi_{\rm mod} - U_0\big\|_{\mathcal{H}_0} \leq A\alpha. \tag{A.2}$$

Therefore, we can fix α_2 so that, when $\alpha \leq \alpha_2$, the choice of $\varphi_{\rm mod}$ is uniquely given by (26). In this case, invoking (25), as well as the identity

$$\langle i\Psi_{\rm mod}, \Psi_{\rm mod}'\rangle_{\mathbb{C}} = \varrho_{\rm mod}^2\varphi_{\rm mod}', \tag{A.3}$$

which holds outside $(-1, 1)$, we have

$$\begin{aligned}\Big|\langle i\Psi_{\rm mod}, \Psi_{\rm mod}'\rangle_{\mathbb{C}} - \big((1-\chi)\varphi_{\rm mod}\big)'\Big| = \Big|(1 - \varrho_{\rm mod}^2)\varphi_{\rm mod}'\Big| \leq & 2\big|1 - \varrho_{\rm mod}^2\big|\big|\varrho_{\rm mod}\varphi_{\rm mod}'\big| \\ \leq & (1 - |\Psi_{\rm mod}|^2)^2 + |\Psi_{\rm mod}'|^2,\end{aligned} \tag{A.4}$$

outside $(-2, 2)$. Therefore, since $\Psi \in \mathcal{E}(\mathbb{R})$, the quantity $\mathcal{P}(\Psi)$ is well-defined.

On the other hand, the renormalized momentum $[P]$ is by definition invariant under translation and multiplication by a constant of modulus one. Hence, we have

$$[P](\Psi) = [P](\Psi_{\rm mod}) = \lim_{S \to +\infty} P_S(\Psi_{\rm mod}) \mod \pi,$$

where

$$P_S(\Psi_{\rm mod}) := \frac{1}{2}\int_{-S}^S \langle i\Psi_{\rm mod}, \Psi_{\rm mod}'\rangle_{\mathbb{C}} - \frac{1}{2}\big(\varphi_{\rm mod}(S) - \varphi_{\rm mod}(-S)\big).$$

It remains to check that

$$P_S(\Psi_{\rm mod}) \to \mathcal{P}(\Psi), \tag{A.5}$$

as $S \to +\infty$.

For $S \geq 1$, we set $\chi_S = \chi(\cdot/S)$. The function $\chi_S - \chi$ is then compactly supported outside $(-1, 1)$, so that

$$\int_{\mathbb{R}} \big((\chi_S - \chi)\varphi_{\rm mod}\big)' = 0.$$

As a consequence of (A.3), we are led to

$$P_S(\Psi_{\rm mod}) - \mathcal{P}(\Psi) = \frac{1}{2} \int_{\mathbb{R}\setminus(-S,S)} \big(1 - \varrho_{\rm mod}^2\big)\varphi_{\rm mod}'.$$

The convergence in (A.5) then follows from the integrability of the function $(1 - \varrho_{\rm mod}^2)\varphi_{\rm mod}'$ at infinity, which is a consequence of (A.4). □

A.2 Proof of Proposition 5

Recall that, when Ψ lies in $\mathcal{V}_0(\alpha_2)$, the map $\Psi_{\rm mod}$ may be written as $\Psi_{\rm mod} = |\Psi_{\rm mod}| \exp i\varphi_{\rm mod}$, with $|\Psi_{\rm mod}| \geq 1/2$, outside the interval $(-1, 1)$. Coming back to decomposition (17), and definition (27), we can expand the quantity $\mathcal{P}(\Psi)$ with respect to $\varepsilon := \Psi_{\rm mod} - U_c$ (where $c = \boldsymbol{c}(\Psi)$) as

$$\mathcal{P}(\Psi) - \mathcal{P}(U_c) + \int_{\mathbb{R}} \langle iU_c', \varepsilon\rangle_{\mathbb{C}} = R_c(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}} \Big(\big[\langle iU_c, \varepsilon\rangle_{\mathbb{C}} - (1 - \chi)(\varphi_{\rm mod} - \varphi_c)\big]' + \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}\Big).$$

Since χ has compact support, we may rewrite this as

$$R_c(\varepsilon) = \frac{1}{2} \int_{\mathbb{R}} \Big(\Big[(1 - \chi)\big(\langle iU_c, \varepsilon\rangle_{\mathbb{C}} - \varphi_{\rm mod} + \varphi_c\big)\Big]' + \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}\Big).$$

Recall next that

$$|\varepsilon(x)| \leq |\varepsilon(0)| + |x|^{\frac{1}{2}} \|\varepsilon'\|_{L^2}, \tag{A.6}$$

when $\varepsilon \in \mathcal{H}(\mathbb{R})$. As a consequence, we obtain

$$\int_{\mathbb{R}} \Big[\frac{1 - \chi}{|U_c|^2}(1 - |U_c|^2)\langle iU_c, \varepsilon\rangle_{\mathbb{C}}\Big]' = 0,$$

so that we finally get

$$\begin{aligned} R_c(\varepsilon) =& \frac{1}{2} \int_{\mathbb{R}} (1 - \chi)\Big(\Big[\frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big]' - \varphi_{\rm mod}' + \varphi_c' + \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}\Big) + \frac{1}{2} \int_{\mathbb{R}} \chi\langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}} \\ & + \frac{1}{2} \int_{\mathbb{R}} \chi'\Big(\varphi_{\rm mod} - \varphi_c - \frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big). \end{aligned} \tag{A.7}$$

In order to complete the proof of Proposition 5, it remains to estimate the integrals in the right-hand side of (A.7) according to (29).

Concerning the first integral, we can write

$$\varphi_{\rm mod}' - \varphi_c' = \frac{\langle i(U_c + \varepsilon), U_c' + \varepsilon'\rangle_{\mathbb{C}}}{|U_c|^2 + \eta_\varepsilon} - \frac{\langle iU_c, U_c'\rangle_{\mathbb{C}}}{|U_c|^2},$$

outside the interval $(-1, 1)$. Expanding this expression with respect to ε and η_ε yields

$$-\varphi_{\rm mod}' + \varphi_c' + \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}} = 2\frac{\langle U_c, \varepsilon\rangle_{\mathbb{C}}\langle iU_c, U_c'\rangle_{\mathbb{C}}}{|U_c|^4} - \frac{\langle iU_c, \varepsilon'\rangle_{\mathbb{C}}}{|U_c|^2} + \frac{\langle iU_c', \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2} + \Phi_{c,\varepsilon}, \tag{A.8}$$

where we denote

$$\begin{aligned}\Phi_{c,\varepsilon} :=& \frac{|\varepsilon|^2 \langle iU_c, U_c'\rangle_{\mathbb{C}}}{|U_c|^4} + \frac{\eta_\varepsilon \langle iU_c, \varepsilon'\rangle_{\mathbb{C}}}{|U_c|^2 |\Psi_{\rm mod}|^2} - \frac{\eta_\varepsilon \langle iU_c', \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2 |\Psi_{\rm mod}|^2} - \frac{\eta_\varepsilon^2 \langle iU_c, U_c'\rangle_{\mathbb{C}}}{|U_c|^4 |\Psi_{\rm mod}|^2} \\ & - \frac{(1-|U_c|^2)\langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}}{|U_c|^2} + \frac{\eta_\varepsilon \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}}{|U_c|^2 |\Psi_{\rm mod}|^2}.\end{aligned} \tag{A.9}$$

On the other hand, the identity $|U_c|^2\, U_c' = \langle U_c, U_c'\rangle_{\mathbb{C}}\, U_c + \langle iU_c, U_c'\rangle_{\mathbb{C}}\, iU_c$ provides

$$\Big[\frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big]' = \frac{\langle iU_c, \varepsilon'\rangle_{\mathbb{C}}}{|U_c|^2} - \frac{\langle iU_c', \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2} - 2\frac{\langle U_c, \varepsilon\rangle_{\mathbb{C}} \langle iU_c, U_c'\rangle_{\mathbb{C}}}{|U_c|^4},$$

outside $(-1,1)$. Inserting into (A.8), we obtain

$$\Big| \int_{\mathbb{R}} (1-\chi)\Big(\Big[\frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big]' - \varphi_{\rm mod}' + \varphi_c' + \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}\Big)\Big| \leq \int_{\mathbb{R}} |1-\chi| |\Phi_{c,\varepsilon}|. \tag{A.10}$$

In order to estimate $\Phi_{c,\varepsilon}$, we first recall that $|\Psi_{\rm mod}| \geq 1/2$ outside $(-1,1)$. Concerning the function U_c, we invoke (18) and decrease, if necessary, the value of α_2 so that

$$|U_c| \geq 1/4, \tag{A.11}$$

outside $(-1,1)$. Similarly, we can argue as in the proof of (A.1) in order to prove that $\|\eta_\varepsilon\|_{L^\infty} \leq 1$ and $\|\varepsilon\|_{L^\infty} \leq 4$ for a possible further value of α_2. Finally, we can invoke (A.6) and the Sobolev embedding theorem to establish the existence of a positive number A such that

$$\big\|(1-|U_c|^2)^{\frac{1}{2}} \varepsilon\big\|_{L^2} \leq A \|\varepsilon\|_{\mathcal{H}_0}.$$

Coming back to (A.9) and (A.10), we conclude that

$$\Big| \int_{\mathbb{R}} (1-\chi)\Big(\Big[\frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big]' - \varphi_{\rm mod}' + \varphi_c' + \langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}\Big)\Big| \leq A\Big(\|\varepsilon\|_{\mathcal{H}_0}^2 + \|\eta_\varepsilon\|_{L^2}^2\Big), \tag{A.12}$$

for a further positive number A.

Concerning the second integral, there exists a positive number A such that $\chi \leq A(1-U_0^2)^{1/2}$ on $\mathbb{R}$. As a consequence, we obtain

$$\int_{\mathbb{R}} |\chi| \big|\langle i\varepsilon, \varepsilon'\rangle_{\mathbb{C}}\big| \leq A \|\varepsilon\|_{\mathcal{H}_0}^2. \tag{A.13}$$

At this stage, the proof reduces to bound the third integral in the right-hand side of (A.7). We come back to the definition of the phase function $\varphi_{\rm mod}$. Arguing as in (A.2), we have

$$\|\varepsilon\|_{L^\infty(-2,2)} \leq A \|\varepsilon\|_{\mathcal{H}_0} \leq A\alpha_2. \tag{A.14}$$

In view of (A.11), we deduce (possibly for a further choice of α_2) that the complex number $\Psi_{\rm mod}(x)$ remains in the open disk with centre $U_c(x)$ and radius $|U_c(x)|$ for any $x \in (-2,-1) \cup (1,2)$. As a consequence, the maps $\Psi_{\rm mod}$ and U_c restricted to $(1,2)$, respectively $(-2,-1)$, lie in a common domain of holomorphy for the complex logarithmic function. In other words, we can write

$$\varphi_{\rm mod} - \varphi_c = i \log\Big(\frac{U_c}{|U_c|}\Big) - i \log\Big(\frac{U_c + \varepsilon}{|U_c + \varepsilon|}\Big),$$

where log refers to an analytic determination of the logarithm. Expanding this expression with respect to ε, we are led to the estimate

$$\Big|\varphi_{\rm mod} - \varphi_c - \frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big| \leq A|\varepsilon|^2.$$

In view of (A.14), this gives

$$\int_{\mathbb{R}} |\chi'|\Big|\varphi_{\rm mod} - \varphi_c - \frac{\langle iU_c, \varepsilon\rangle_{\mathbb{C}}}{|U_c|^2}\Big| \leq A\|\varepsilon\|_{\mathcal{H}_0}^2. \tag{A.15}$$

Estimate (29) then results from (A.12), (A.13) and (A.15). This concludes the proof of Proposition 5. □

Acknowledgments. The authors are grateful to F. Béthuel, R. Côte and Y. Martel for interesting and helpful discussions. They also acknowledge support from the project "Schrödinger equations and applications" (ANR-12-JS01-0005-01) of the Agence Nationale de la Recherche.

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