

On the stability of the Ginzburg-Landau vortex

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Abstract

We introduce a functional framework tailored to investigate the minimality and stability properties of the Ginzburg-Landau vortex of degree one on the whole plane. We prove that a renormalized Ginzburg-Landau energy is well-defined in that framework and that the vortex is its unique global minimizer up to the invariances by translation and phase shift. Our main result is a nonlinear coercivity estimate for the renormalized energy around the vortex, from which we can deduce its orbital stability as a solution to the Gross-Pitaevskii equation, the natural Hamiltonian evolution equation associated to the Ginzburg-Landau energy.

1 Introduction

We are interested in the complex Ginzburg-Landau equation in the plane

$$\Delta\Psi + (1 - |\Psi|^2)\Psi = 0. \quad (1)$$

For each $d \in \mathbb{Z}^*$, this equation possesses a well-known solution called the vortex of degree d at infinity. It has the equivariant form

$$V_d(x) = \rho_d(r)e^{id\theta}, \quad (2)$$

for $x = (r \cos(\theta), r \sin(\theta))$. The profile ρ_d is real-valued, increasing, smooth, and satisfies $\rho_d(0) = 0$ and $\rho_d(r) \rightarrow 1$ as $r \rightarrow +\infty$. Equation (1) is invariant by translations and by constant phase shifts, so that functions of the form $e^{i\varphi}V_d(\cdot - a)$ for arbitrary $a \in \mathbb{R}^2$ and $\varphi \in \mathbb{R}$ are also solutions. It is also invariant by complex conjugacy. This is reflected in the equality $V_d = \bar{V}_{-d}$. For this reason, we restrict in the sequel our attention to the case $d \geq 1$.

Associated to the Ginzburg-Landau equation is the Ginzburg-Landau energy

$$\mathcal{E}_{\text{GL}}(\Psi) \equiv \int_{\mathbb{R}^2} e_{\text{GL}}(\Psi) := \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla\Psi|^2 + \frac{1}{4} (1 - |\Psi|^2)^2 \right).$$

The vortices V_d do *not* have finite energy. This has long been a source of difficulty for their analysis, leading to various strategies based on suitable forms of *renormalization*. The overall picture is that V_1 possesses some minimizing and stability

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properties, while V_d is unstable for any $d \geq 2$. Over the years, this has been shown in different frameworks, some of which will be recalled below, depending on what is precisely meant by minimality and/or stability.

Regarding the stability of V_1 , a very natural question is that of its nonlinear dynamical stability as a stationary solution to the corresponding Hamiltonian evolution equation, the Gross-Pitaevskii equation

$$i\partial_t\Psi + \Delta\Psi + (1 - |\Psi|^2)\Psi = 0. \quad (3)$$

Our goal in this work is two-fold. First, improving on some of the existing variational stability estimates for V_1 . This will involve nonlinear coercivity estimates for (a renormalized version of) \mathcal{E}_{GL} around V_1 . Second, proving the orbital stability of V_1 as a solution to (3) in a natural energy space. This was an open problem even for smooth and compactly supported initial perturbations.

In the remaining part of this introduction, we present the functional framework and state our main results. The strategy leading to these results and how they relate to earlier works in the literature is the object of the next section.

We introduce the complex Hilbert space

$$H := \{\Psi \in L^2_{loc}(\mathbb{R}^2, \mathbb{C}) \text{ s.t. } \|\Psi\|_H < +\infty\},$$

corresponding to the norm

$$\|\Psi\|_H^2 := \int_{\mathbb{R}^2} (|\nabla(\Psi\bar{V}_1)|^2 + (1 - |V_1|^2)|\nabla\Psi|^2),$$

where we recall that $1 - |V_1|^2 = 1 - \rho_1^2 > 0$. We define the energy space E as

$$E := \{\Psi \in H \text{ s.t. } 1 - |\Psi|^2 \in L^2(\mathbb{R}^2)\}.$$

It is a complete metric space for the distance

$$d_E(\Psi_1, \Psi_2) := \|\Psi_1 - \Psi_2\|_H + \| |\Psi_1|^2 - |\Psi_2|^2 \|_{L^2}.$$

The vortex V_1 belongs to E . An important feature of this space is that the infinitesimal generators of the invariance groups, i.e. $\partial_{x_1}V_1$ and $\partial_{x_2}V_1$ for the translations, and iV_1 for the phase shifts, all belong to H . Actually, although the norm $\|\cdot\|_H$ itself is not translation invariant, the spaces H and E are invariant by translations and by constant phase shifts.

Our first result shows that the space E is a natural framework for the renormalization of the Ginzburg-Landau energy with respect to V_1 .

Proposition 1. *The renormalized Ginzburg-Landau energy*

$$\mathcal{E}(\Psi) := \lim_{r \rightarrow +\infty} \int_{B_r} (e_{GL}(\Psi) - e_{GL}(V_1)) \quad (4)$$

is well-defined on E . Besides, it is invariant by translations and constant phase shifts.

It turns out that the condition $\Psi \in E$ actually *encodes* the fact that Ψ has a “degree one” at infinity, even when the zero set of Ψ might be unbounded and the notion of topological degree in the classical meaning would make no direct sense. The existence of the limit defining the renormalized energy \mathcal{E} in Proposition 1 is also a consequence of the fact that $\Psi \in E$, and it could not be used as a standalone “definition” of an appropriate functional framework ¹.

Regarding the minimality of V_1 , we show

¹The limit might exist for oscillating pure phases, e.g., although such fields would have a well-defined topological degree being zero, not one.

Proposition 2. *The vortex V_1 is the unique global minimizer of the energy \mathcal{E} on E , up to translations and constant phase shifts.*

Our next result, and the core of this work, is a coercivity estimate for the renormalized energy in E . We denote the orbit under the invariance groups of a function $\Psi \in E$ by

$$\text{Orb}(\Psi) := \{e^{-i\varphi}\Psi(\cdot + a), \varphi \in \mathbb{R} \text{ and } a \in \mathbb{R}^2\}.$$

Notice that, by construction, $\mathcal{E}(V_1) = 0$.

Theorem 1. *There exist $\kappa > 0$ and $\rho > 0$ such that*

$$\mathcal{E}(\Psi) \geq \kappa d_E(V_1, \text{Orb}(\Psi))^2,$$

whenever $\Psi \in E$ satisfies $d_E(V_1, \text{Orb}(\Psi)) < \rho$.

The distance $d_E(V_1, \text{Orb}(\Psi))$ vanishes if and only if $\Psi \in \text{Orb}(V_1)$. Therefore, Theorem 1 is a (nonlinear) coercivity estimate in terms of the distance d_E to the orbit of V_1 .

In this statement, it might have appeared more familiar to use the quantity $d_E(\Psi, \text{Orb}(V_1))$ instead of $d_E(V_1, \text{Orb}(\Psi))$, but the former is not invariant under translations of Ψ , and the norm $\|\cdot\|_H$, which is part of the definition of d_E , is tailored to measure perturbations around V_1 . Instead, the inequality in Theorem 1 is invariant by translation, since both $\mathcal{E}(\Psi)$ and $\text{Orb}(\Psi)$ are.

Concerning the Gross-Pitaevskii equation (3), the affine space $\Psi_0 + H^1(\mathbb{R}^2, \mathbb{C})$ is contained in E for arbitrary $\Psi_0 \in E$, and we can prove

Proposition 3. *For any $\Psi_0 \in E$, the Gross-Pitaevskii equation (3) possesses a unique global solution $t \mapsto \Psi_t \in C^0(\mathbb{R}, \Psi_0 + H^1(\mathbb{R}^2, \mathbb{C}))$ with initial datum Ψ_0 . Moreover, the renormalized energy \mathcal{E} is conserved along the flow.*

It is conceivable that the Cauchy problem for (3) is actually globally well-posed on E , and not just on fibers of the form $\Psi_0 + H^1(\mathbb{R}^2, \mathbb{C})$. We have not settled that question, but it is not necessary to prove our stability result. Combining Theorem 1 and Proposition 3, we indeed deduce

Theorem 2. *There exist $\delta > 0$ and $C > 0$ such that, if $\Psi_0 \in E$ and $d_E(V_1, \Psi_0) \leq \delta$, then the solution Ψ_t with initial datum Ψ_0 of Proposition 3 satisfies*

$$d_E(V_1, \text{Orb}(\Psi_t)) \leq C d_E(V_1, \Psi_0),$$

for any $t \in \mathbb{R}$. In particular, the vortex V_1 is orbitally stable.

The statement in Theorem 2 does not provide any information on the location of the solution with respect to the orbit. In the course of the proof, we actually construct positions $a(t) \in \mathbb{R}^2$ and phase shifts $\varphi(t) \in \mathbb{R}$ such that

$$d_E(V_1, e^{-i\varphi(t)}\Psi_t(\cdot + a(t))) \leq C d_E(V_1, \Psi_0),$$

for any $t \in \mathbb{R}$. The following proposition provides a first control on the evolution of these parameters.

Proposition 4. *There exist $\tau > 0$ and $C > 0$ such that, if $\Psi_0 \in E$ and $d_E(V_1, \Psi_0) \leq \tau$, then there exist two functions $a \in C^1(\mathbb{R}, \mathbb{R}^2)$ and $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ such that*

$$d_E(V_1, e^{-i\varphi(t)}\Psi_t(\cdot + a(t))) \leq C d_E(V_1, \Psi_0),$$

and

$$|a'(t)| + |\varphi'(t)| \leq C d_E(V_1, \Psi_0), \tag{5}$$

for any $t \in \mathbb{R}$.

The question whether V_1 is stable as a stationary solution to (3), and not only orbitally stable, is still open. There is no immediate obstruction to that stronger form of stability since, although there exist travelling waves of (3) with arbitrarily small speed (see [4, 8]), they are not small perturbations of the vortex but instead perturbations of a vortex-antivortex pair, and have finite Ginzburg-Landau energy.

Asymptotic stability of V_1 (or maybe only of its orbit) is also an open question. Although (3) is an Hamiltonian equation, asymptotic stability could hold for a topology in which the renormalized energy is not continuous (e.g. through dispersion at infinity). In one space dimension, it was proved in [12] that the black soliton, the 1d equivalent of V_1 , is asymptotically stable in an orbital sense.

Finally, we mention that the dynamics of (3) for sequences of initial data that converge to V_1 or more generally to suitable combinations of well separated vortices of degrees ± 1 has already been studied on finite time intervals (see e.g. [9, 16, 3, 15] and the references therein). In particular, it is known that a related notion of modulation parameters asymptotically obey a limit point-vortex ODE, but only on finite time intervals. These results are based on different kind of rigidity estimates for V_1 , where the closeness is measured through the Jacobian (see [14] for the sharpest known statement in that direction). It is tempting to investigate whether the results of the present paper could be used to extend the description of this dynamics to longer time scales.

In the sequel, we use all along the following notation. We set $x^\perp = (-x_2, x_1)$ for any $x \in \mathbb{R}^2$. We also use the notation B_r for the open ball of \mathbb{R}^2 with center $(0, 0)$ and radius $r > 0$. Finally, we define the scalar product of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ as $\langle z_1, z_2 \rangle_{\mathbb{C}} = \operatorname{Re}(z_1 \overline{z_2}) = a_1 a_2 + b_1 b_2$.

2 Strategy for the proofs

In this section, we present the key ingredients that are needed in the proofs of the results stated in the introduction. This is also the occasion to discuss how our arguments relate to earlier works in the literature. The detailed proofs or their completions is postponed to the subsequent sections.

2.1 Concerning renormalization and Proposition 1

We start by motivating the introduction of the Hilbert space H . At the same time, we show how its definition implies the existence of the limit in (4) providing the renormalized Ginzburg-Landau energy \mathcal{E} .

To do so, we first recall the well-known fact that the divergence of the Ginzburg-Landau energy of V_1 is *only* due to a too slow decay of the gradient of its phase at infinity, because of its non-trivial topological degree. Taking the gradient of (2), we obtain

$$\nabla V_1(x) = e^{i\theta} \left(\rho_1'(|x|) \frac{x}{|x|} + i\rho_1(|x|) \frac{x^\perp}{|x|^2} \right), \quad (6)$$

and therefore

$$\frac{1}{2} \int_{B_r} |\nabla V_1|^2 = \pi \int_0^r \left(\rho_1'(r)^2 + \frac{\rho_1(r)^2}{r^2} \right) r dr.$$

From the asymptotic properties of ρ_1 , which we have recalled in Lemma A.1 in Appendix A, it follows that

$$\pi \int_0^r \frac{\rho_1(r)^2}{r^2} r dr = \pi \log(r) + \mathcal{O}(1),$$

as $r \rightarrow +\infty$, while

$$\int_0^{+\infty} \left(\frac{\rho_1'(r)^2}{2} + \frac{(1 - \rho_1(r)^2)^2}{4} \right) r \, dr < +\infty. \quad (7)$$

The definition of the space H through the norm

$$\|\Psi\|_H^2 := \int_{\mathbb{R}^2} (|\nabla(\Psi\bar{V}_1)|^2 + (1 - |V_1|^2)|\nabla\Psi|^2),$$

is then related to our requirement that the infinitesimal generators $\partial_{x_1}V_1$, $\partial_{x_2}V_1$ and iV_1 of the invariance groups should all belong to H in order to eventually derive *optimal* coercivity estimates. It follows from the properties listed in Lemma A.1 that the derivatives $\partial_{x_1}V_1$ and $\partial_{x_2}V_1$ have gradients in $L^2(\mathbb{R}^2)$. As we have just recalled, the same does *not* hold for the function iV_1 . On the other hand, if $\Psi = iV_1$ then $\Psi\bar{V}_1 = i|V_1|^2$, and the latter has a gradient in $L^2(\mathbb{R}^2)$, in view of (7) and the boundedness of $|V_1|$.

From the Leibniz rule, we compute

$$|\nabla(\Psi\bar{V}_1)|^2 = |\nabla\Psi|^2|V_1|^2 + |\Psi|^2|\nabla V_1|^2 + 2\langle\nabla\Psi\bar{V}_1, \Psi\nabla\bar{V}_1\rangle_{\mathbb{C}}, \quad (8)$$

where, here as in the sequel, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ the scalar product of $\mathbb{C} \simeq \mathbb{R}^2$. With the definition (4) of the renormalized Ginzburg-Landau energy in mind, we write

$$|\nabla\Psi|^2 - |\nabla V_1|^2 = |\nabla(\Psi\bar{V}_1)|^2 + (1 - |V_1|^2)|\nabla\Psi|^2 - (1 - |\Psi|^2)|\nabla V_1|^2 - 2\langle\nabla(\Psi\bar{V}_1), \Psi\nabla\bar{V}_1\rangle_{\mathbb{C}}.$$

Note that the first two terms in the right-hand side of the previous pointwise identity are precisely those entering in the definition of the norm in H , and hence by construction they are integrable over \mathbb{R}^2 when $\Psi \in H$. The third term is also integrable when in addition $\Psi \in E$. This follows from the Cauchy-Schwarz inequality since $1 - |\Psi|^2 \in L^2(\mathbb{R}^2)$ for $\Psi \in E$, and on the other hand $\nabla V_1 \in L^4(\mathbb{R}^2)$. In order to prove the existence of the limit in (4), it therefore only remains to prove the existence of the limit

$$\lim_{r \rightarrow +\infty} \int_{B_r} \langle\nabla(\Psi\bar{V}_1), \Psi\nabla\bar{V}_1\rangle_{\mathbb{C}}, \quad (9)$$

whenever $\Psi \in E$. This requires the identification of some cancellation phenomenon. For this and also later purposes, we split the integral in (9) into a local part and a part at infinity. More precisely, here and throughout the paper, we fix a smooth, radial, radially non-increasing cut-off function $0 \leq \chi \leq 1$ such that $\chi \equiv 1$ in B_1 and χ is supported in B_2 . For arbitrary $R > 0$, we set $\chi_R(x) := \chi(x/R)$. Having in mind (6), we decompose $\langle\nabla(\Psi\bar{V}_1), \Psi\nabla\bar{V}_1\rangle_{\mathbb{C}}$ as

$$\left\langle \nabla(\Psi\bar{V}_1), \left(\nabla\bar{V}_1 + i(1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \bar{V}_1 \right) \Psi \right\rangle_{\mathbb{C}} - \frac{x^\perp}{|x|^2} (1 - \chi_R)^2 \langle i\Psi\bar{V}_1, \nabla(\Psi\bar{V}_1) \rangle_{\mathbb{C}}.$$

The first term in the previous decomposition is integrable on \mathbb{R}^2 . As a matter of fact, it follows from (6) that

$$\left(\nabla\bar{V}_1 + i(1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \bar{V}_1 \right) \Psi = e^{i\theta} \left(\rho_1' \frac{x}{|x|} + i\chi_R(2 - \chi_R)\rho_1 \frac{x^\perp}{|x|^2} \right) \Psi,$$

and the latter quantity belongs to $L^2(\mathbb{R}^2)$. Indeed, we first infer from Lemma A.1 again that $\rho_1'(r) \sim 1/r^3$, as $r \rightarrow +\infty$. Second, the function χ_R has compact support, while the function $\rho_1 x^\perp / |x|^2$ is bounded. The previous claim finally results from the fact that Ψ is in the space H , which continuously embeds into the weighted space

$L^2_{-s}(\mathbb{R}^2)$ for arbitrary $s > 1$ (as shown in Lemma B.2). Here, we only need the case $s = 3$.

At this stage, we have reduced the existence of the renormalized energy to the next key lemma, where we additionally gain some smallness estimate in the vicinity of V_1 . Note that this gain will be important for our later perturbation analysis.

Lemma 1. *Let $\Psi \in E$. The quantity*

$$P_R(\Psi) := \lim_{r \rightarrow +\infty} 2 \int_{B_r} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle i\Psi\bar{V}_1, \nabla(\Psi\bar{V}_1) \rangle_{\mathbb{C}} \quad (10)$$

is well-defined for any $R > 0$. Moreover, there exist universal constants $\delta > 0$, $\Lambda > 0$ and $K \geq 1$ such that

$$|P_R(\Psi)| \leq \frac{K}{R} d_E(\Psi, V_1)^2, \quad (11)$$

provided that $d_E(\Psi, V_1) \leq \delta$ and $R \geq \Lambda$.

The proof of the first statement in Lemma 1 is concise, and we present it here next. For $\Psi \in E$, it is shown in Lemma B.5 that $\Psi\bar{V}_1$ has finite Ginzburg-Landau energy. Therefore, it follows from a result of P. Gérard [11] that $\Psi\bar{V}_1$ may be decomposed as

$$\Psi\bar{V}_1 = e^{i\varphi} + w, \quad (12)$$

for some real-valued function φ such that $\nabla\varphi \in L^2(\mathbb{R}^2)$ and some complex-valued function $w \in H^1(\mathbb{R}^2)$. We then split the scalar product in (10) as

$$\langle i\Psi\bar{V}_1, \nabla(\Psi\bar{V}_1) \rangle_{\mathbb{C}} = \langle 1, \nabla\varphi \rangle_{\mathbb{C}} + \langle iw, \nabla w \rangle_{\mathbb{C}} + \langle w, \nabla\varphi e^{i\varphi} \rangle_{\mathbb{C}} + \langle ie^{i\varphi}, \nabla w \rangle_{\mathbb{C}},$$

and we treat separately each of the four corresponding terms. Given any $r > 0$, we first write

$$\int_{B_r} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle 1, \nabla\varphi \rangle_{\mathbb{C}} = \int_{B_r} \operatorname{div} \left((1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \langle 1, \varphi \rangle_{\mathbb{C}} \right) = 0.$$

Here, we have used the fact that $\nabla\chi_R(x) \cdot x^\perp = 0$ pointwise since χ_R is radial, and also that the flux of x^\perp through ∂B_r vanishes pointwise. Next, the second and third terms are vector fields, which are integrable over \mathbb{R}^2 , since $w \in H^1(\mathbb{R}^2)$ and $\nabla\varphi \in L^2(\mathbb{R}^2)$. Therefore, their integral against the bounded vector field $(1 - \chi_R)^2 x^\perp / |x|^2$ is well-defined on \mathbb{R}^2 . Finally, we use as above that

$$\int_{B_r} \operatorname{div} \left((1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle ie^{i\varphi}, w \rangle_{\mathbb{C}} \right) = 0,$$

in order to obtain the identity for the fourth term

$$\int_{B_r} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle ie^{i\varphi}, \nabla w \rangle_{\mathbb{C}} = \int_{B_r} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle \nabla\varphi e^{i\varphi}, w \rangle_{\mathbb{C}}.$$

The last integral now has a well-defined limit when $r \rightarrow +\infty$, since both w and $\nabla\varphi$ belong to $L^2(\mathbb{R}^2)$. This completes the proof of the existence of $P_R(\Psi)$ in Lemma 1, and therefore also of $\mathcal{E}(\Psi)$ in Proposition 1. We refer to Section 3 below for the proof of the invariance with respect to translations and phase shifts of this latter quantity. Note also that, with the help of decomposition (12), we have obtained the formula

$$P_R(\Psi) = 2 \int_{\mathbb{R}^2} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle iw, \nabla w + 2i\nabla\varphi e^{i\varphi} \rangle_{\mathbb{C}}. \quad (13)$$

The proof of (11) follows similar lines, the main difference being that decomposition (12) needs to be adapted to a perturbative setting. This is done in Lemma 4 of Section 3. Note that the integrand in (10) identically vanishes for $\Psi = V_1$, since $\langle i|V_1|^2, \nabla(|V_1|^2) \rangle_{\mathbb{C}} = 0$ on \mathbb{R}^2 due to the real-valued nature of the function $|V_1|$. This may serve as an intuition to why (11) actually holds.

For later reference, we also make here explicit the decomposition of $\mathcal{E}(\Psi)$ which we have obtained so far, namely

$$\begin{aligned} \mathcal{E}(\Psi) &= \frac{1}{2} \|\Psi\|_H^2 - \int_{\mathbb{R}^2} |\nabla V_1|^2 (1 - |\Psi|^2) \\ &\quad - \int_{\mathbb{R}^2} \left\langle \nabla(\Psi \bar{V}_1), \left(\nabla \bar{V}_1 + i(1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \bar{V}_1 \right) \Psi \right\rangle_{\mathbb{C}} + \frac{1}{2} P_R(\Psi) \\ &\quad + \int_{\mathbb{R}^2} \frac{1}{4} ((1 - |\Psi|^2)^2 - (1 - |V_1|^2)^2). \end{aligned} \quad (14)$$

2.2 Concerning minimality and Proposition 2

A solution Ψ to (1) is called a locally minimizing solution if

$$\mathcal{E}_{\text{GL}}(\Psi + \varepsilon, B_R) \geq \mathcal{E}_{\text{GL}}(\Psi, B_R),$$

for any $R > 0$ and any $\varepsilon \in H_0^1(B_R, \mathbb{C})$. Here, we have set

$$\mathcal{E}_{\text{GL}}(\Psi, B_R) := \int_{B_R} e_{\text{GL}}(\Psi).$$

The next characterization was obtained by P. Mironescu [18]².

Theorem A ([18]). *The vortex solution V_1 is a locally minimizing solution to (1). Moreover, it is the only non-constant locally minimizing solution, up to translations and constant phase shifts.*

Since any minimizer of \mathcal{E} in E is necessarily also a locally minimizing solution to (1), and since $\mathcal{E}(V_1) = 0$ by construction, the proof of Proposition 2 reduces to show that \mathcal{E} is non-negative. For that purpose, we shall appeal, after suitable rescalings, to results (for example [18, Corollaire 2]) regarding the asymptotics of Ginzburg-Landau minimizers on a fixed bounded domain with fixed boundary data. The reduction to the latter case from our framework requires some elementary surgery on the boundary of large balls, the necessary details of which are presented in Section 3.

2.3 Concerning coercivity and Theorem 1

A quantitative stability estimate with respect to compactly supported perturbations of V_1 was also obtained by P. Mironescu in [17]. For that purpose, he decomposed

$$\mathcal{E}_{\text{GL}}(V_1 + \varepsilon, B_R) = \mathcal{E}_{\text{GL}}(V_1, B_R) + \frac{1}{2} B(\varepsilon) + \mathcal{O}(\|\varepsilon\|_{H_0^1(B_R)}^3), \quad (15)$$

for any function $\varepsilon \in H_0^1(B_R)$. Here, B is the real quadratic form given by

$$B(\varepsilon) := \int_{\mathbb{R}^2} (|\nabla \varepsilon|^2 - (1 - |V_1|^2)|\varepsilon|^2 + 2\langle V_1, \varepsilon \rangle_{\mathbb{C}}).$$

²The fact that the vortex solution V_1 is locally minimizing is not explicitly stated in [18], but it follows from properties listed in there, in particular Corollaire 2 and the remark following it.

Theorem B ([17]). *For any $R > 0$, there exists $\kappa_R > 0$ such that*

$$B(\varepsilon) \geq \kappa_R \|\varepsilon\|_{H_0^1(B_R)}^2,$$

for any $\varepsilon \in H_0^1(B_R)$.

The fact that the invariance by translation and by phase shift is not reflected in the previous coercivity estimate is due to the restriction $\varepsilon \in H_0^1(B_R)$, which prevents those groups to act. In turn, this can be used to show that necessarily $\kappa_R \rightarrow 0$ as $R \rightarrow +\infty$.

In order to derive a stability estimate without restricting to compactly supported perturbations, M. del Pino, P. Felmer and M. Kowalczyk [10] considered an Hilbert space H_B naturally associated to the decomposition (15) (see also [19] for previous approach in the space $L^2(\mathbb{R}^2)$). This space was defined from the norm ³

$$\|\varepsilon\|_{H_B}^2 := \int_{\mathbb{R}^2} (|\nabla\varepsilon|^2 + (1 - |V_1|^2)|\varepsilon|^2 + 2\langle V_1, \varepsilon \rangle_{\mathbb{C}}).$$

One can note that H_B is a strict subspace of H . With the definition of this space at hand, they obtained

Theorem C ([10]). *The real quadratic form B is positive semi-definite on H_B , and its kernel coincides with the real vector space spanned by $\partial_{x_1}V_1$ and $\partial_{x_2}V_1$.*

The invariance by translation of the Ginzburg-Landau energy is reflected in Theorem C in the fact that $\partial_{x_1}V_1$ and $\partial_{x_2}V_1$ belong to the kernel of B . The invariance by phase shifts cannot be accounted for by working in the space H_B though, since $iV_1 \notin H_B$. This is the reason why the latter is not present in the kernel of B in H_B .

It is possible to extend Theorem C with some quantitative coercivity estimates for B under suitable orthogonality conditions of ε with respect to $\partial_{x_1}V_1$, $\partial_{x_2}V_1$ and iV_1 (see e.g. [7, Proposition 1.3] and also the Fredholm alternative in [10, Theorem 2]). These estimates however do not allow to control the nonlinear terms arising in the expansion of the renormalized Ginzburg-Landau energy, and it does not seem possible to derive nonlinear stability of V_1 based (exclusively) on the linear analysis of B .

At this stage, it is worth comparing the quadratic form B with our previous decomposition (14) of \mathcal{E} . For that purpose, we first write $\Psi = V_1 + \varepsilon$ in (14). Using the fact that V_1 is a solution to (1), we obtain

Lemma 2. *For $\Psi = V_1 + \varepsilon \in E$, we have*

$$\mathcal{E}(V_1 + \varepsilon) = \frac{1}{2}\mathcal{Q}_R(\varepsilon) + \frac{1}{2}\mathcal{P}_R(\varepsilon) + \frac{1}{4} \int_{\mathbb{R}^2} \eta_\varepsilon^2. \quad (16)$$

In this identity, $\mathcal{Q}_R(\varepsilon)$ is the quadratic form on H given by

$$\begin{aligned} \mathcal{Q}_R(\varepsilon) := & \|\varepsilon\|_H^2 - \int_{\mathbb{R}^2} (1 - |V_1|^2 - |\nabla V_1|^2)|\varepsilon|^2 \\ & - 2 \int_{\mathbb{R}^2} \left\langle \nabla(\varepsilon \bar{V}_1), \left(\nabla \bar{V}_1 + i \frac{x^\perp}{|x|^2} (1 - \chi_R)^2 \bar{V}_1 \right) \varepsilon \right\rangle_{\mathbb{C}}, \end{aligned} \quad (17)$$

$\mathcal{P}_R(\varepsilon) := P_R(V_1 + \varepsilon)$, where P_R is defined in Lemma 1, and

$$\eta_\varepsilon := (1 - |V_1 + \varepsilon|^2) - (1 - |V_1|^2) = -2\langle \varepsilon, V_1 \rangle_{\mathbb{C}} - |\varepsilon|^2. \quad (18)$$

³Note the sign change in the middle term with respect to the quadratic form B .

An important feature concerning the decomposition in (16) is that η_ε^2 being a square, it is (pointwise) non-negative. If we develop η_ε^2 according to definition (18), we obtain the identity

$$B(\varepsilon) = \mathcal{Q}_R(\varepsilon) + \mathcal{P}_R(\varepsilon) + 2 \int_{\mathbb{R}^2} \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2,$$

but the latter only makes sense provided that $\varepsilon \in H_B$, because of the third term. Besides, as we have already mentioned, the coercivity properties of B are insufficient to derive the nonlinear stability of V_1 .

We modify the previous strategy in two ways. First, we only develop the square of η_ε according to (18) locally in space. More precisely, we write

$$\frac{1}{4} \int_{\mathbb{R}^2} \eta_\varepsilon^2 = \frac{1}{4} \int_{\mathbb{R}^2} (1 - \chi_R^2) \eta_\varepsilon^2 + \frac{1}{4} \int_{\mathbb{R}^2} \chi_R^2 \eta_\varepsilon^2 = N_R(\varepsilon) + \mathcal{I}_R(\varepsilon).$$

Here, the term

$$\mathcal{I}_R(\varepsilon) := \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2$$

is now a real quadratic form well-defined on H , and the nonlinear quantity

$$N_R(\varepsilon) := \frac{1}{4} \int_{\mathbb{R}^2} (1 - \chi_R^2) \eta_\varepsilon^2 + \frac{1}{4} \int_{\mathbb{R}^2} \chi_R^2 (|\varepsilon|^4 + 4 \langle \varepsilon, V_1 \rangle_{\mathbb{C}} |\varepsilon|^2)$$

contains only non-negative terms except possibly the cubic one in ε , which however is localized in B_{2R} .

Second, although the quantity $\mathcal{P}_R(\varepsilon)$ is quadratic in ε , we keep it out of our linear analysis. More precisely, we finally write

$$\mathcal{E}(V_1 + \varepsilon) = \frac{1}{2} \mathcal{Q}_R(\varepsilon) + N_R(\varepsilon) + \frac{1}{2} \mathcal{P}_R(\varepsilon), \quad (19)$$

where $\mathcal{Q}_R := \mathcal{Q}_R + 2\mathcal{I}_R$ is the real quadratic form on H for which we shall prove a coercivity estimate. The quadratic form \mathcal{P}_R will eventually be controlled using both \mathcal{Q}_R and the nonlinear term N_R . A careful analysis shows that it cannot be included in \mathcal{Q}_R , since it would otherwise induce an infinite number of negative directions.

The coercivity of \mathcal{Q}_R is given by

Proposition 5. *There exist universal constants $\kappa_0 > 0$ and $N_0 > 0$ such that, given any $\varepsilon \in H$ verifying the orthogonality conditions*

$$\int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_x V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_y V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle \varepsilon, iV_1 \rangle_{\mathbb{C}} = 0, \quad (20)$$

and any $R_0 \geq 1$, there exists $R_0 \leq R \leq 2^{N_0} R_0$ such that

$$\mathcal{Q}_R(\varepsilon) \geq \kappa_0 \left(\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 \right).$$

The proof of Proposition 5 is presented in Section 6. In a few words, the idea is to decompose the perturbation ε into a local (compactly supported) part, for which the analysis is very similar to the one in [10], and a second part at infinity, for which the exact form of our decomposition, and in particular the fact that \mathcal{P}_R has been left aside, plays a crucial role. The necessary glue between the two analysis explains the fact that the cut-off location R is perturbation dependent (in a range with universal extent, though).

Observe that the local orthogonality conditions in (20) reflect the geometric invariances by translation and phase shift of the Ginzburg-Landau energy. In the functional framework of the Hilbert space H , the L^2 -scalar products between an arbitrary perturbation ε and the functions $\partial_x V_1$, $\partial_y V_1$, respectively iV_1 , do not necessarily make sense. This explains the introduction of the smooth radial cut-off function χ in the three integrals of (20). Note that the choice of this special function is somewhat arbitrary. One can check that it can be replaced by any smooth, non-negative and non-increasing function, identically equal to 1 on $[0, 1]$, and compactly supported.

Concerning the nonlinear term N_R , it is straightforward to derive

Lemma 3. *For any $R > 0$, $0 < \kappa < 1$, and $\varepsilon \in H$, we have*

$$N_R(\varepsilon) + \kappa \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 \geq \frac{\kappa}{4} \int_{\mathbb{R}^2} \eta_\varepsilon^2 - \|\varepsilon\|_{L^3(B_{2R})}^3.$$

Recalling that $\eta_\varepsilon = -2\langle \varepsilon, V_1 \rangle_{\mathbb{C}} - |\varepsilon|^2$, we indeed compute that

$$\begin{aligned} \frac{1}{4} \eta_\varepsilon^2 - (1 - \kappa) \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 &\geq \frac{\kappa}{4} \eta_\varepsilon^2 + (1 - \kappa) \chi_R^2 \left(\frac{1}{4} \eta_\varepsilon^2 - \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 \right) \\ &\geq \frac{\kappa}{4} \eta_\varepsilon^2 + (1 - \kappa) \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}} |\varepsilon|^2, \end{aligned}$$

and the conclusion follows after integration on \mathbb{R}^2 , using that $|V_1| \leq 1$ and $|1 - \kappa| \leq 1$.

We are now in position to present the detail leading to a nonlinear coercivity estimate for \mathcal{E} around V_1 by combining the results in Proposition 5 for Q_R , Lemma 3 for N_R , and the second statement in Lemma 1 for \mathcal{P}_R .

Recall the decomposition in (19). In Proposition 5, we fix the value of R_0 as

$$R_0 = \max \left\{ \Lambda, \frac{16K}{\kappa_0} \right\},$$

where $\Lambda \geq 1$ and $K > 0$ are the universal constants provided by Lemma 1, and $\kappa_0 > 0$ is the universal constant provided by Proposition 5. Under the orthogonality conditions in (20), we can find $R \in [R_0, 2^{N_0} R_0]$ such that

$$\frac{1}{2} Q_R(\varepsilon) \geq \frac{\kappa_0}{2} \left(\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 \right). \quad (21)$$

By Lemma 1 and our choice of R_0 , we also obtain

$$\left| \frac{1}{2} \mathcal{P}_R(\varepsilon) \right| \leq \frac{K}{2R} d_E(V_1 + \varepsilon, V_1)^2 \leq \frac{\kappa_0}{16} \left(\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2 \right).$$

Finally, it follows from Lemma 3 with $\kappa = \frac{\kappa_0}{2}$ that

$$N_R(\varepsilon) + \frac{\kappa_0}{2} \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 \geq \frac{\kappa_0}{8} \|\eta_\varepsilon\|_{L^2}^2 - \|\varepsilon\|_{L^3(B_{2R})}^3.$$

After summation, the previous three inequalities and (19) yield

$$\mathcal{E}(V_1 + \varepsilon) \geq \frac{\kappa_0}{16} \left(\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2 - \|\varepsilon\|_{L^3(B_{2R})}^3 \right).$$

Now, remark that $2R \leq R_1 := 2^{N_0+1} R_0$, which is a universal constant. Therefore, we derive, in particular from Lemma B.2, that

$$\|\varepsilon\|_{L^3(B_{2R})}^3 \leq \|\varepsilon\|_{L^3(B_{R_1})}^3 \leq K_1 \|\varepsilon\|_H^3,$$

for some further universal constant $K_1 > 0$. If $\|\varepsilon\|_H$ satisfies the smallness condition $K_1 \|\varepsilon\|_H \leq \kappa_0/32$, we may absorb the remaining cubic term in the quadratic one. Therefore, we have proved

Proposition 6. *There exist $\kappa > 0$ and $\delta > 0$ such that, given any function $\Psi = V_1 + \varepsilon \in E$ such that*

$$\|\varepsilon\|_H + \|\eta_\varepsilon\|_{L^2(\mathbb{R}^2)} \leq \delta,$$

and ε satisfies the three orthogonality conditions in (20), we have

$$\mathcal{E}(V_1 + \varepsilon) \geq \kappa(\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2).$$

Deriving Theorem 1 from Proposition 6 is then mostly a matter of fixing appropriately the orthogonality conditions in (20). The strategy to achieve this is classical (see e.g. [20, 21]). It relies on the introduction of modulation parameters corresponding to the geometric invariances. Consider the neighbourhoods in H of the orbit of V_1 defined as

$$\mathcal{V}(\alpha) := \left\{ \Psi \in H \text{ s.t. } \inf_{(a, \varphi) \in \mathbb{R}^2 \times \mathbb{R}} \|e^{-i\varphi} \Psi(\cdot + a) - V_1\|_H < \alpha \right\},$$

for any $\alpha > 0$. Given a function $\Psi \in \mathcal{V}(\alpha)$, we decompose it as $e^{-i\varphi} \Psi(\cdot + a) = V_1 + \varepsilon$ for $a \in \mathbb{R}^2$ and $\varphi \in \mathbb{R}$. When α is small enough, we can choose the modulation parameters a and φ so as to satisfy the orthogonality conditions in (20). More precisely, we shall prove in Section 7 below

Proposition 7. *There exist $\alpha > 0$ and $A > 0$, and two functions $a \in \mathcal{C}^1(\mathcal{V}(\alpha), \mathbb{R}^2)$ and $\varphi \in \mathcal{C}^1(\mathcal{V}(\alpha), \mathbb{R}/2\pi\mathbb{Z})$, such that any function $\Psi \in \mathcal{V}(\alpha)$ can be written as*

$$\Psi = e^{i\varphi(\Psi)} \left(V_1(\cdot - a(\Psi)) + \varepsilon(\cdot - a(\Psi)) \right), \quad (22)$$

where ε satisfies the orthogonality conditions in (20). Moreover, given parameters $(b, \vartheta) \in \mathbb{R}^2 \times \mathbb{R}$ such that

$$\|e^{-i\vartheta} \Psi(\cdot + b) - V_1\|_H < \alpha,$$

we have the estimate

$$\|\varepsilon\|_H + |a(\Psi) - b| + |e^{i\varphi(\Psi)} - e^{i\vartheta}| \leq A \|e^{-i\vartheta} \Psi(\cdot + b) - V_1\|_H. \quad (23)$$

We may now complete the

Proof of Theorem 1. Let $\Psi \in E$ be such that $d := d_E(V_1, \text{Orb}(\Psi)) < \rho/2$, where $0 < \rho \leq 1$ is a universal constant, which will be fixed in the course of the proof. By definition of the distance to the orbit, there exist $\vartheta \in \mathbb{R}$ and $b \in \mathbb{R}^2$ such that ⁴

$$d_E(V_1, e^{-i\vartheta} \Psi(\cdot + b)) \leq 2d. \quad (24)$$

In particular, we have $\|V_1 - e^{-i\vartheta} \Psi(\cdot + b)\|_H \leq 2d < \rho$. We require that $\rho \leq \alpha$, where $\alpha > 0$ is the constant provided by Proposition 7. Applying this proposition, we derive the existence of $a \in \mathbb{R}^2$ and $\varphi \in \mathbb{R}$ such that $\varepsilon := e^{-i\varphi} \Psi(\cdot + a) - V_1$ satisfies the orthogonality conditions in (20), and besides,

$$\|\varepsilon\|_H + |a - b| \leq 2Ad.$$

⁴We could show that the distance to the orbit is actually achieved but we will not need and spare that argument.

We next estimate

$$\begin{aligned}\|\eta_\varepsilon\|_{L^2} &= \left\| |V_1|^2 - |\Psi(\cdot + a)|^2 \right\|_{L^2} = \left\| |V_1(\cdot + b - a)|^2 - |\Psi(\cdot + b)|^2 \right\|_{L^2} \\ &\leq \left\| |V_1|^2 - |\Psi(\cdot + b)|^2 \right\|_{L^2} + \left\| |V_1(\cdot + b - a)|^2 - |V_1|^2 \right\|_{L^2} \\ &\leq d_E(V_1, e^{-i\vartheta}\Psi(\cdot + b)) + C(2A)|a - b| \leq 2(1 + AC(2A))d,\end{aligned}$$

where the continuous function $C(\cdot)$ is provided by Lemma B.7. After summation, this yields $\|\varepsilon\|_H + \|\eta_\varepsilon\|_{L^2} \leq 2(1 + A(C(2A) + 1))d$. We require that $(1 + A(C(2A) + 1))\rho \leq \delta$, where δ is given by Proposition 6. This proposition then yields the lower bound

$$\mathcal{E}(e^{-i\varphi}\Psi(\cdot + a)) \geq \kappa(\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2).$$

To conclude, we finally observe that

$$\mathcal{E}(e^{-i\varphi}\Psi(\cdot + a)) = \mathcal{E}(\Psi),$$

by Proposition 1, and that

$$\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_{L^2}^2 \geq \frac{1}{2}d_E(V_1, e^{-i\varphi}\Psi(\cdot + a))^2 \geq \frac{1}{2}d_E(V_1, \text{Orb}(\Psi))^2,$$

by definition of the latter. This completes the proof of Theorem 1, with the choice $\rho = \min\{1, \alpha, \delta/(1 + A(C(2A) + 1))\}$, and a value of κ being half of the corresponding value in Proposition 6. \square

2.4 Concerning orbital stability and Theorem 2

The proof of Theorem 2 assuming Theorem 1 and Proposition 3 is standard.

Proof of Theorem 2. Let $\Psi_0 \in E$ be such that

$$d := d_E(V_1, \Psi_0) < \delta,$$

where $0 < \delta \leq 1$ is a universal constant that will be fixed in the course of the proof. For $d = 0$, the conclusion in Theorem 2 holds since V_1 is a stationary solution. Therefore, we assume in the sequel that $d > 0$.

First we show that, if δ is taken smaller than the corresponding value in Lemma 1, then we have the estimate

$$\mathcal{E}(\Psi_0) \leq Kd^2, \tag{25}$$

for some universal constant $K > 0$. Indeed, take $R = \Lambda$, where Λ is the universal constant provided by the statement of Lemma 1, and decompose $\mathcal{E}(\Psi_0)$ according to (19), i.e.

$$\mathcal{E}(\Psi_0) = \frac{1}{2}Q_\Lambda(\varepsilon_0) + N_\Lambda(\varepsilon_0) + \frac{1}{2}\mathcal{P}_\Lambda(\varepsilon_0),$$

where $\varepsilon_0 := \Psi_0 - V_1$. By Lemma 1, we obtain that

$$|\mathcal{P}_\Lambda(\varepsilon_0)| \leq K\|\varepsilon_0\|_H^2 \leq Kd^2,$$

for some universal $K > 0$. A similar estimate holds for $Q_\Lambda(\varepsilon_0)$, since Q_Λ is a continuous quadratic form on H due to Lemma B.2. Finally, we check that

$$|N_\Lambda(\varepsilon_0)| \leq \frac{1}{4}\|\eta_{\varepsilon_0}\|_{L^2}^2 + \mathcal{I}_\Lambda(\varepsilon_0) \leq K(\|\eta_{\varepsilon_0}\|_{L^2}^2 + \|\varepsilon_0\|_H^2) = Kd^2,$$

where $\eta_{\varepsilon_0} := |V_1|^2 - |\Psi_0|^2$, as before.

Define next the constant $C := \max\{2, (2K/\kappa)^{1/2}\}$, where $K > 0$ is the constant in (25), and $\kappa > 0$ is the constant provided by Theorem 1. Define then the constant $\delta := \rho/(2C)$, where $\rho > 0$ is also provided by Theorem 1.

Let Ψ_t be the solution of the Gross-Pitaevskii equation with initial data Ψ_0 . We claim that

$$d_E(V_1, \text{Orb}(\Psi_t)) < C d_E(V_1, \Psi_0), \quad (26)$$

for any $t \in \mathbb{R}$. Since the map $t \mapsto \Psi_t$ is continuous with values into E , and since (26) holds for $t = 0$ because $C \geq 2$, it suffices to show that the equality

$$d_E(V_1, \text{Orb}(\Psi_t)) = C d_E(V_1, \Psi_0) \quad (27)$$

for some $t \in \mathbb{R}$ leads to a contradiction. Since $C d_E(V_1, \Psi_0) \leq C\delta = \frac{\rho}{2}$, we may apply Theorem 1 to Ψ_t when (27) holds, and conclude that

$$\mathcal{E}(\Psi_t) \geq \kappa d_E(V_1, \text{Orb}(\Psi_t))^2 = \kappa C^2 d_E(V_1, \Psi_0) = 2Kd^2. \quad (28)$$

Since $\mathcal{E}(\Psi_t) = \mathcal{E}(\Psi_0)$, we deduce from (25) and (28) that $Kd^2 \geq 2Kd^2$. This is a contradiction since $K > 0$ and $d > 0$, and so (26) holds for any $t \in \mathbb{R}$. \square

3 Renormalization of the energy - Proofs of Lemma 1 and Proposition 1 completed

We first complete the

Proof of Proposition 1. We have already shown in the introduction that

$$\mathcal{E}(\Psi) = \lim_{r \rightarrow +\infty} \int_{B_r} (e_{\text{GL}}(\Psi) - e_{\text{GL}}(V_1))$$

is a well-defined quantity when $\Psi \in E$. It remains to prove its invariance by translation and phase shift. The latter is immediate, but the former requires a short argument.

First, it follows from Lemma A.1 that

$$\lim_{r \rightarrow +\infty} \int_{B_{R+r} \setminus B_r} e_{\text{GL}}(V_1) = 0,$$

for any $R \geq 0$. Let then $\Psi \in E$ and $a \in \mathbb{R}^2$. Since the density e_{GL} is non-negative, we have

$$\int_{B_r} e_{\text{GL}}(\Psi(\cdot + a)) \leq \int_{B_{r+|a|}} e_{\text{GL}}(\Psi),$$

and therefore,

$$\int_{B_r} (e_{\text{GL}}(\Psi(\cdot + a)) - e_{\text{GL}}(V_1)) \leq \int_{B_{r+|a|}} (e_{\text{GL}}(\Psi) - e_{\text{GL}}(V_1)) + \int_{B_{r+|a|} \setminus B_r} e_{\text{GL}}(V_1).$$

Taking the limit as $r \rightarrow +\infty$, we obtain that $\mathcal{E}(\Psi(\cdot + a)) \leq \mathcal{E}(\Psi)$. It suffices then to interchange the roles of Ψ and $\Psi(\cdot + a)$ to obtain the reverse inequality. This completes the proof of Proposition 1. \square

We next turn to the end of the

Proof of Lemma 1. Recall that the existence of the quantity

$$P_R(\Psi) = \lim_{r \rightarrow +\infty} 2 \int_{B_r} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \cdot \langle i\Psi \bar{V}_1, \nabla(\Psi \bar{V}_1) \rangle_{\mathbb{C}}$$

was already established in the introduction for any function $\Psi \in E$. It remains to prove that the existence of universal constants $\delta > 0$, $\Lambda > 0$ and $K \geq 1$ such that

$$|P_R(\Psi)| \leq \frac{K}{R} d_E(\Psi, V_1)^2, \quad (29)$$

provided that $d_E(\Psi, V_1) \leq \delta$ and $R \geq \Lambda$. For that purpose, we replace the decomposition $\Psi \bar{V}_1 = e^{i\varphi} + w$ in (12) by a slight variant, but only available on B_Λ^c , namely

$$\Psi \bar{V}_1 = |V_1|^2 (e^{i\varphi_\varepsilon} + w_\varepsilon), \quad (30)$$

where $w_\varepsilon \in H^1(\mathbb{R}^2, \mathbb{C})$ and $\varphi_\varepsilon \in \dot{H}^1(\mathbb{R}^2, \mathbb{R})$ will satisfy suitable smallness estimates. Using the property that $R \geq \Lambda$, we can modify (13) into

$$P_R(\Psi) = 2 \int_{\mathbb{R}^2} (1 - \chi_R)^2 |V_1|^4 \frac{x^\perp}{|x|^2} \cdot \langle iw_\varepsilon, \nabla w_\varepsilon + 2i\nabla\varphi_\varepsilon e^{i\varphi_\varepsilon} \rangle_{\mathbb{C}}.$$

Estimate (29) then follows from the next lemma and the Cauchy-Schwarz inequality, using that

$$\left\| (1 - \chi_R)^2 |V_1|^4 \frac{x^\perp}{|x|^2} \right\|_{L^\infty} \leq \frac{K}{R}.$$

□

The decomposition in (30) is provided by the following lemma.

Lemma 4. *There exist constants $K > 0$, $\Lambda > 0$ and $\delta > 0$ such that, given any $\Psi = V_1 + \varepsilon \in E$ with*

$$\|\eta_\varepsilon\|_{L^2} + \|\nabla(\varepsilon \bar{V}_1)\|_{L^2} \leq \delta,$$

there exist $\varphi_\varepsilon \in \dot{H}^1(\mathbb{R}^2, \mathbb{R})$ and $w_\varepsilon \in H^1(\mathbb{R}^2, \mathbb{C})$ such that

$$\Psi \bar{V}_1 = |V_1|^2 (e^{i\varphi_\varepsilon} + w_\varepsilon)$$

on B_Λ^c . Moreover, the functions φ_ε and w_ε satisfy

$$\|w_\varepsilon\|_{H^1(\mathbb{R}^2)} \leq K (\|\nabla(\varepsilon \bar{V}_1)\|_{L^2} + \|\eta_\varepsilon\|_{L^2}),$$

and

$$\|\nabla\varphi_\varepsilon\|_{L^2(\mathbb{R}^2)} \leq K \|\nabla(\varepsilon \bar{V}_1)\|_{L^2}.$$

Proof. In the whole proof, the notation $A \lesssim B$, for any arbitrary non-negative numbers A and B , means that $A \leq KB$ for some universal constant K , whose exact value is irrelevant for the argument. We denote by $\chi(D)$ the cut-off in Fourier space induced by the cut-off function χ . Recall that χ is radial, with $\chi \equiv 1$ in B_1 and $\chi \equiv 0$ outside B_2 , and that $\chi(D)(f) := \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}(f)(\xi))$, so that $\chi(D)$ only keeps the small frequencies. For $\Lambda > 1$ and $\Psi = V_1 + \varepsilon$, we decompose the function $\varepsilon \bar{V}_1$ on B_Λ^c as

$$\varepsilon \bar{V}_1 = \chi(D)(\varepsilon \bar{V}_1) + \frac{(1 - \chi(D))(\varepsilon \bar{V}_1)}{|V_1|^2} |V_1|^2.$$

We let $w_0 := (1 - \chi(D))(\varepsilon \bar{V}_1)/|V_1|^2$ on B_Λ^c , and check using Lemma A.1 that

$$\|w_0\|_{H^1(B_\Lambda^c)} \lesssim \|\nabla(\varepsilon \bar{V}_1)\|_{L^2}.$$

We now aim to prove that, provided that $\Lambda > 1$ is large enough and $0 < \delta \leq 1$ is small enough, the function $w := \chi(D)(\varepsilon\bar{V}_1) + |V_1|^2$ satisfies $|w| \geq 1/2$ on B_Λ^c . For that purpose, we decompose

$$1 - |w|^2 = 1 - \left| |V_1|^2 + \varepsilon\bar{V}_1 \right|^2 - \left| (1 - \chi(D))(\varepsilon\bar{V}_1) \right|^2 + 2\langle (1 - \chi(D))(\varepsilon\bar{V}_1), |V_1|^2 + \varepsilon\bar{V}_1 \rangle_{\mathbb{C}}, \quad (31)$$

and we start by estimating these three terms in $L^2(B_\Lambda^c)$. For the first one, we rewrite

$$1 - \left| |V_1|^2 + \varepsilon\bar{V}_1 \right|^2 = 1 - |V_1|^4 - 2\langle \varepsilon, V_1 \rangle_{\mathbb{C}} |V_1|^2 - |V_1|^2 |\varepsilon|^2 = 1 - |V_1|^4 + \eta_\varepsilon |V_1|^2,$$

and therefore,

$$\left\| 1 - \left| |V_1|^2 + \varepsilon\bar{V}_1 \right|^2 \right\|_{L^2(B_\Lambda^c)} \leq \left\| 1 - |V_1|^4 \right\|_{L^2(B_\Lambda^c)} + \left\| \eta_\varepsilon \right\|_{L^2(B_\Lambda^c)} \lesssim \Lambda^{-1} + \delta.$$

Here, we have used Lemma A.1 in order to estimate the decay with respect to Λ . For the second term, we infer from the Sobolev embedding theorem that

$$\left\| \left| (1 - \chi(D))(\varepsilon\bar{V}_1) \right|^2 \right\|_{L^2} \lesssim \left\| (1 - \chi(D))(\varepsilon\bar{V}_1) \right\|_{H^1}^2 \lesssim \left\| \nabla(\varepsilon\bar{V}_1) \right\|_{L^2}^2 \leq \delta^2 \leq \delta.$$

Finally, since $\left| |V_1|^2 + \varepsilon\bar{V}_1 \right| \lesssim 1 + |\Psi| \lesssim 1 + |\eta_\varepsilon|^{1/2}$ and $|\eta_\varepsilon|^{1/2} \in L^4(\mathbb{R}^2)$, we deduce again from the Sobolev embedding theorem that the third term is controlled by

$$\begin{aligned} & \left\| \langle (1 - \chi(D))(\varepsilon\bar{V}_1), |V_1|^2 + \varepsilon\bar{V}_1 \rangle_{\mathbb{C}} \right\|_{L^2} \\ & \lesssim \left\| (1 - \chi(D))(\varepsilon\bar{V}_1) \right\|_{L^2} + \left\| (1 - \chi(D))(\varepsilon\bar{V}_1) \right\|_{H^1} \left\| \eta_\varepsilon \right\|_{L^2}^{\frac{1}{2}} \\ & \lesssim \left\| \nabla(\varepsilon\bar{V}_1) \right\|_{L^2} (1 + \left\| \eta_\varepsilon \right\|_{L^2}^{\frac{1}{2}}) \lesssim \delta(1 + \delta^{\frac{1}{2}}) \lesssim \delta. \end{aligned}$$

Combining these estimates in (31), we deduce that

$$\left\| 1 - |w|^2 \right\|_{L^2(B_\Lambda^c)} \lesssim \Lambda^{-1} + \delta.$$

To obtain a uniform control rather than an L^2 -one, we rely on the Gagliardo-Nirenberg type inequality

$$\left\| 1 - |w|^2 \right\|_{L^\infty(B_\Lambda^c)} \lesssim \left\| 1 - |w|^2 \right\|_{L^2(B_\Lambda^c)}^{\frac{1}{2}} \left\| D^2(|w|^2) \right\|_{L^2(B_\Lambda^c)}^{\frac{1}{2}},$$

where the proportionality constant does not depend on $\Lambda \geq 1$, as it can be seen by scaling from B_1^c and invoking a standard Sobolev extension theorem. Then we bound

$$\left\| D^2(|w|^2) \right\|_{L^2(B_\Lambda^c)} \lesssim \left\| D^2 w \right\|_{L^\infty(B_\Lambda^c)} \left\| 1 - |w|^2 \right\|_{L^2(B_\Lambda^c)} + \left\| D^2 w \right\|_{L^2(B_\Lambda^c)} + \left\| Dw \right\|_{L^4(B_\Lambda^c)}^2.$$

From Lemma A.1 and the fact that $\chi(D)(\varepsilon\bar{V}_1)$ only has small frequencies, we also have

$$\left\| D^k w \right\|_{L^2(B_\Lambda^c)} \lesssim \Lambda^{-1} + \delta, \quad (32)$$

with a constant depending only on $k \geq 1$. Combining the previous inequalities and the Sobolev embedding theorem, we deduce

$$\left\| 1 - |w|^2 \right\|_{L^\infty(B_\Lambda^c)} \lesssim \Lambda^{-1} + \delta.$$

Therefore, we obtain $|w| \geq 1/2$ on B_Λ^c as claimed, provided that $\Lambda > 0$ is chosen sufficiently large and $\delta > 0$ sufficiently small. Under these conditions, there exists a function $\varphi_\varepsilon \in \dot{H}^1(B_\Lambda^c)$ such that

$$\chi(D)(\varepsilon\bar{V}_1) + |V_1|^2 = w = \left| \chi(D)(\varepsilon\bar{V}_1) + |V_1|^2 \right| e^{i\varphi_\varepsilon}.$$

We decompose $\chi(D)(\varepsilon\bar{V}_1) = |V_1|^2(e^{i\varphi_\varepsilon} - 1) + |V_1|^2w_1$, with

$$w_1 := \frac{1}{|V_1|^2}(|V_1|^2 + \chi(D)(\varepsilon\bar{V}_1) - |V_1|^2)e^{i\varphi_\varepsilon} = \frac{1}{|V_1|^2}(|w| - |V_1|^2)e^{i\varphi_\varepsilon}.$$

Since $|w| \geq 1/2$ on B_Λ^c , we observe that $||w| - |V_1|^2| \lesssim ||w|^2 - |V_1|^4|$ on B_Λ^c , which implies that

$$\|w_1\|_{L^2(B_\Lambda^c)} \lesssim \||w|^2 - |V_1|^4\|_{L^2(B_\Lambda^c)} \lesssim \|\eta_\varepsilon\|_{L^2} + \|\nabla(\varepsilon\bar{V}_1)\|_{L^2}.$$

We next write

$$\|\nabla w_1\|_{L^2(B_\Lambda^c)} \lesssim \|\nabla(w_1e^{-i\varphi_\varepsilon})\|_{L^2(B_\Lambda^c)} + \|w_1\|_{L^\infty(B_\Lambda^c)}\|\nabla\varphi_\varepsilon\|_{L^2(B_\Lambda^c)}.$$

We first observe that

$$\nabla(w_1e^{-i\varphi_\varepsilon}) = |V_1|^2\nabla\left(\frac{1}{|V_1|^2}\right)w_1e^{-i\varphi_\varepsilon} + \frac{1}{|V_1|^2}\nabla\chi(D)(\varepsilon\bar{V}_1).$$

For $\Lambda > 1$, we therefore obtain

$$\begin{aligned} \|\nabla(w_1e^{-i\varphi_\varepsilon})\|_{L^2(B_\Lambda^c)} &\lesssim \|w_1\|_{L^2(B_\Lambda^c)} + \|\nabla\chi(D)(\varepsilon\bar{V}_1)\|_{L^2(B_\Lambda^c)} \\ &\lesssim \|\nabla(\varepsilon\bar{V}_1)\|_{L^2} + \|\eta_\varepsilon\|_{L^2}. \end{aligned}$$

We also have

$$\|w_1\|_{L^\infty(B_\Lambda^c)} \lesssim \|1 - |w|^2\|_{L^\infty(B_\Lambda^c)} + 1 \lesssim 1.$$

Finally, since $e^{i\varphi_\varepsilon} = w/|w|$, with $|w| \geq 1/2$ on B_Λ^c , we have

$$\begin{aligned} \|\nabla\varphi_\varepsilon\|_{L^2(B_\Lambda^c)} &\lesssim \|\langle iw, \nabla w \rangle_{\mathbb{C}}\|_{L^2(B_\Lambda^c)} \\ &= \left\| \left\langle i(\chi(D)(\varepsilon\bar{V}_1) + |V_1|^2), \nabla(\chi(D)(\varepsilon\bar{V}_1) + |V_1|^2) \right\rangle_{\mathbb{C}} \right\|_{L^2(B_\Lambda^c)} \end{aligned}$$

and the latter expression is controlled by

$$\|\chi(D)(\varepsilon\bar{V}_1)\|_{L^2(B_\Lambda^c)} \left(1 + \|\chi(D)(\varepsilon\bar{V}_1)\|_{L^\infty(B_\Lambda^c)}\right) \lesssim \|\nabla(\varepsilon\bar{V}_1)\|_{L^2},$$

since $\delta \leq 1$. This leads to

$$\|\nabla w_1\|_{L^2(B_\Lambda^c)} \lesssim \|\nabla(\varepsilon\bar{V}_1)\|_{L^2} + \|\eta_\varepsilon\|_{L^2}.$$

We define $w_\varepsilon := w_0 + w_1$, which satisfies

$$\|w_\varepsilon\|_{H^1(B_\Lambda^c)} \lesssim \|\nabla(\varepsilon\bar{V}_1)\|_{L^2} + \|\eta_\varepsilon\|_{L^2}.$$

We check that, by construction, $\Psi\bar{V}_1 = |V_1|^2(e^{i\varphi_\varepsilon} + w_\varepsilon)$ on B_Λ^c . It suffices then to extend w_ε and φ_ε to \mathbb{R}^2 using the Sobolev extension theorem in order to complete the proof. \square

4 Minimality - Proof of Proposition 2 completed

In the introduction, we have already mentioned that in order to prove Proposition 2, it suffices to show that the renormalized Ginzburg-Landau energy is non-negative. For that purpose, we invoke results from [18] regarding the asymptotics

of the Ginzburg-Landau energy for functions on a fixed bounded domain with fixed boundary data.

Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $R_n \rightarrow +\infty$. Define $\varepsilon_n := R_n^{-1}$ and let $u_n(x) := \Psi(x/\varepsilon_n)$ on B_1 . By scaling, we have

$$\int_{B_{R_n}} e_{\text{GL}}(\Psi) = \int_{B_1} e_{\text{GL}\varepsilon_n}(u_n) := \int_{B_1} \left(\frac{|\nabla u_n|^2}{2} + \frac{(1 - |u_n|^2)^2}{4\varepsilon_n^2} \right),$$

and therefore we only need to show that

$$\liminf_{n \rightarrow +\infty} \int_{B_1} \left(e_{\text{GL}\varepsilon_n}(u_n) - e_{\text{GL}\varepsilon_n} \left(V_1 \left(\frac{\cdot}{\varepsilon_n} \right) \right) \right) \geq 0.$$

Since the boundary data of u_n on ∂B_1 is not fixed, we cannot invoke the results of [18] directly. Instead, we choose the sequence $(R_n)_{n \in \mathbb{N}}$ so that u_n has almost minimal energy on ∂B_1 , and then perform an extension to a slightly larger domain with a fixed boundary data. More precisely, since $\nabla(\Psi \bar{V}_1)$ and $1 - |\Psi|^2$ both belong to $L^2(\mathbb{R}^2)$, we may find a sequence $(R_n)_{n \in \mathbb{N}}$ such that $R_n \rightarrow +\infty$ and

$$\int_{\partial B_{R_n}} \left(|\nabla(\Psi \bar{V}_1)|^2 + (1 - |\Psi|^2)^2 \right) = o\left(\frac{1}{R_n}\right), \quad (33)$$

as $n \rightarrow +\infty$. For n sufficiently large, it follows that we may write

$$u_n(e^{i\theta}) = |u_n(e^{i\theta})| e^{i(\theta + \phi_n(\theta))} = \rho_n(\theta) e^{i(\theta + \phi_n(\theta))}$$

on ∂B_1 , where

$$\int_0^{2\pi} \left(\frac{|\partial_\theta \rho_n|^2}{2} + \rho_n^2 \frac{|\partial_\theta \phi_n|^2}{2} + \frac{(1 - \rho_n^2)^2}{4\varepsilon_n^2} \right) d\theta = o(1), \quad (34)$$

as $n \rightarrow +\infty$. Let $\bar{\phi}_n$ denote the mean of ϕ_n on ∂B_1 . Fix $\delta > 0$ and consider the extension

$$u_n(re^{i\theta}) = \left(\lambda(r)\rho_n(\theta) + 1 - \lambda(r) \right) e^{i(\theta + \lambda(r)\phi_n(\theta) + (1 - \lambda(r))\bar{\phi}_n)}$$

of u_n for $1 \leq r \leq 1 + \delta$, where $\lambda(r) := (1 + \delta - r)/\delta$. We deduce from (34) and elementary computations that the function $v_n := e^{-i\phi_n} u_n$, which is defined on $B_{1+\delta}$ and with fixed boundary data $v_n((1 + \delta)e^{i\theta}) = e^{i\theta}$ on $\partial B_{1+\delta}$, satisfies

$$\int_{B_{1+\delta}} e_{\text{GL}\varepsilon_n}(v_n) \leq \int_{B_1} e_{\text{GL}\varepsilon_n}(u_n) + \pi \log(1 + \delta) + o(1),$$

as $n \rightarrow +\infty$. By [18, Corollaire 2] and the remark following it, we have

$$\liminf_{n \rightarrow +\infty} \int_{B_{1+\delta}} \left(e_{\text{GL}\varepsilon_n}(v_n) - e_{\text{GL}\varepsilon_n} \left(V_1 \left(\frac{\cdot}{\varepsilon_n} \right) \right) \right) \geq 0,$$

and therefore

$$\liminf_{n \rightarrow +\infty} \int_{B_1} \left(e_{\text{GL}\varepsilon_n}(u_n) - e_{\text{GL}\varepsilon_n} \left(V_1 \left(\frac{\cdot}{\varepsilon_n} \right) \right) \right) \geq -\pi \log(1 + \delta).$$

The conclusion follows letting $\delta \rightarrow 0$. □

5 Second order expansion - Proof of Lemma 2

First recall the decomposition in (14) of the energy

$$\begin{aligned}\mathcal{E}(\Psi) &= \frac{1}{2}\|\Psi\|_H^2 - \int_{\mathbb{R}^2} |\nabla V_1|^2(1 - |\Psi|^2) \\ &\quad - \int_{\mathbb{R}^2} \left\langle \nabla(\Psi \bar{V}_1), \left(\nabla \bar{V}_1 + i(1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \bar{V}_1 \right) \Psi \right\rangle_{\mathbb{C}} + \frac{1}{2}P_R(\Psi) \\ &\quad + \int_{\mathbb{R}^2} \frac{1}{4}((1 - |\Psi|^2)^2 - (1 - |V_1|^2)^2).\end{aligned}$$

We write $\Psi = V_1 + \varepsilon \in E$ and then expand the various terms in the previous decomposition. Note that no use is made of the equation (1) satisfied by V_1 until the very end of the argument. The qualitative properties of V_1 in Lemma A.1 associated to the embedding given by Lemma B.2 for $\varepsilon \in H$ are sufficient to justify all computations.

After elementary algebra and integration by parts, we obtain

$$\frac{1}{2}\|\Psi\|_H^2 = \frac{1}{2}\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} (T_{0a} + \langle T_{1a}, \varepsilon \rangle_{\mathbb{C}}),$$

where

$$T_{0a} := \frac{1}{2}|\nabla(|V_1|^2)|^2 + \frac{1}{2}|\nabla V_1|^2(1 - |V_1|^2),$$

and

$$T_{1a} := -\Delta(|V_1|^2)V_1 - \operatorname{div}((1 - |V_1|^2)\nabla V_1).$$

Using that $|V_1|$ and χ_R are radial functions, we also obtain

$$\begin{aligned}& - \int_{\mathbb{R}^2} \left\langle \nabla(\Psi \bar{V}_1), \left(\nabla \bar{V}_1 + i(1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \bar{V}_1 \right) \Psi \right\rangle_{\mathbb{C}} \\ &= - \int_{\mathbb{R}^2} \left\langle \nabla(\varepsilon \bar{V}_1), \left(\nabla \bar{V}_1 + i(1 - \chi_R)^2 \frac{x^\perp}{|x|^2} \bar{V}_1 \right) \varepsilon \right\rangle_{\mathbb{C}} + \int_{\mathbb{R}^2} (T_{0b} + \langle T_{1b}, \varepsilon \rangle_{\mathbb{C}}),\end{aligned}$$

where

$$T_{0b} := -\frac{1}{2}|\nabla(|V_1|^2)|^2 \quad \text{and} \quad T_{1b} := \Delta(|V_1|^2)V_1 - \operatorname{div}(|V_1|^2\nabla V_1) - V_1|\nabla V_1|^2.$$

Finally, recalling the definition $\eta_\varepsilon = -2\langle \varepsilon, V_1 \rangle_{\mathbb{C}} - |\varepsilon|^2$ in (18), we have

$$-\frac{1}{2}|\nabla V_1|^2(1 - |\psi|^2) = T_{0c} + \langle T_{1c}, \varepsilon \rangle_{\mathbb{C}} + \frac{1}{2}|\nabla V_1|^2|\varepsilon|^2,$$

where

$$T_{0c} := -\frac{1}{2}|\nabla V_1|^2(1 - |V_1|^2) \quad \text{and} \quad T_{1c} := |\nabla V_1|^2V_1.$$

Similarly, we compute

$$\frac{1}{4}((1 - |\psi|^2)^2 - (1 - |V_1|^2)^2) = \langle T_{1d}, \varepsilon \rangle_{\mathbb{C}} - \frac{1}{2}(1 - |V_1|^2)|\varepsilon|^2 + \frac{1}{4}\eta_\varepsilon^2,$$

where

$$T_{1d} := -(1 - |V_1|^2)V_1.$$

It is immediate that $T_{0a} + T_{0b} + T_{0c} = 0$, and we check that

$$T_{1a} + T_{1b} + T_{1c} + T_{1d} = -\Delta V_1 - (1 - |V_1|^2)V_1 = 0,$$

since V_1 is a solution to (1). Lemma 2 then follows after summing the previous identities. \square

6 Quadratic form - Proof of Proposition 5

In this section, we establish the coercivity properties of the quadratic form Q_R as stated in Proposition 5. In particular, we assume throughout this section that $R \geq 1$. For $\varepsilon \in H$, recall that

$$Q_R(\varepsilon) := \mathcal{Q}_R(\varepsilon) + 2\mathcal{I}_R(\varepsilon)$$

where

$$\begin{aligned} \mathcal{Q}_R(\varepsilon) = & \|\varepsilon\|_H^2 - \int_{\mathbb{R}^2} (1 - |V_1|^2 - |\nabla V_1|^2) |\varepsilon|^2 \\ & - 2 \int_{\mathbb{R}^2} \left\langle \nabla(\varepsilon \bar{V}_1), \left(\nabla \bar{V}_1 + i \frac{x^\perp}{|x|^2} (1 - \chi_R)^2 \bar{V}_1 \right) \varepsilon \right\rangle_{\mathbb{C}}, \end{aligned}$$

and

$$\mathcal{I}_R(\varepsilon) = \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2.$$

Combining the decay and regularity properties of V_1 stated in Lemma A.1 with the embeddings for the Hilbert space H in Lemma B.2, we can check that the quadratic form Q_R is well-defined and continuous on H . Moreover, its associated self-adjoint operator is a compact perturbation of the identity on H .

In the course of the proof, we shall use multiple times the following elementary consequence of the Fredholm theory.

Lemma 5. *Let $(X, \|\cdot\|_X)$ be a real Hilbert space and Q be a real continuous quadratic form on X . Assume that, for an inner product on X whose norm is equivalent to the original one, Q is associated to a compact perturbation of the identity on X . Assume also that $Q(x) > 0$ for all $x \in X \setminus \{0\}$. Then there exists $\kappa_0 > 0$ such that*

$$Q(x) \geq \kappa_0 \|x\|_X^2,$$

for any $x \in X$.

The Hilbert space H is naturally decomposed into orthogonal Fourier sectors through the formula ⁵

$$\varepsilon(r, \theta) = \sum_{j \in \mathbb{Z}} \varepsilon_j(r) e^{i(j+1)\theta}.$$

Elementary computations show that

$$\|\varepsilon_j e^{i(j+1)\theta}\|_H \simeq \|\varepsilon_j\|_{H_j},$$

with universal similarity constants independent of $j \in \mathbb{Z}$. Here, the Hilbert space H_j is defined through the norm

$$\|e\|_{H_j}^2 := \int_0^{+\infty} \left(|e'(r)|^2 + \left(\frac{j^2}{1+r^2} + \frac{(j+1)^2}{r^2(1+r^2)} \right) |e(r)|^2 \right) r dr. \quad (35)$$

In this framework, the quadratic forms \mathcal{Q}_R and \mathcal{I}_R may be decomposed as

$$\mathcal{Q}_R(\varepsilon) = \sum_{j \in \mathbb{Z}} \left(Q_{R,j}(a_j) + Q_{R,j}(b_j) \right), \quad (36)$$

and

$$\mathcal{I}_R(\varepsilon) = I_R(a_0) + \frac{1}{2} \sum_{j=1}^{+\infty} \left(I_R(a_j + a_{-j}) + I_R(b_j - b_{-j}) \right), \quad (37)$$

⁵The shift in the exponent is convenient for later symmetry purposes.

where we have set $\varepsilon_j =: a_j + ib_j$, with a_j and b_j being real-valued functions. In the previous identities, the quadratic forms $Q_{R,j}$ and I_R are given by

$$Q_{R,j}(e) := \int_0^{+\infty} \left(|e'|^2 + \left(\frac{(j+1)^2}{r^2} - 2j \frac{(1-\chi_R)^2}{r^2} \rho_1^2 - (1-\rho_1^2) \right) |e|^2 \right) r dr,$$

and

$$I_R(e) := \int_0^{+\infty} \rho_1^2 \chi_R^2 |e|^2 r dr.$$

We now describe the coercivity properties of the quadratic forms $Q_{R,j}$ and I_R , as well as of suitable combinations of them, beginning by $Q_{R,0}$.

Lemma 6. *The quadratic form $Q_{R,0}$ is independent of the number R and it satisfies*

$$Q_0(e) := Q_{R,0}(e) = \int_0^{+\infty} \rho_1^2 \left| \left(\frac{e}{\rho_1} \right)' \right|^2 r dr \geq 0. \quad (38)$$

for any $e \in H_0$.

Proof. By definition, the quadratic form $Q_{R,0}$ does not depend on the number R . Moreover, it follows from the Leibniz rule that

$$\int_0^{+\infty} \rho_1^2 \left| \left(\frac{e}{\rho_1} \right)' \right|^2 r dr = \int_0^{+\infty} \left(|e'|^2 + \frac{(\rho_1')^2}{\rho_1^2} |e|^2 - (|e|^2)' \frac{\rho_1'}{\rho_1} \right) r dr.$$

Integrating by parts, we obtain

$$- \int_0^{+\infty} (|e|^2)' \frac{\rho_1'}{\rho_1} r dr = \int_0^{+\infty} \left(\frac{\rho_1''}{\rho_1} + \frac{\rho_1'}{r \rho_1} \right) |e|^2 r dr.$$

Indeed, the boundary terms vanish. This follows from Lemma A.1, as well as the fact that a function $e \in H_0$ satisfies $e(0) = 0$ and has an at most logarithmic growth at infinity. In particular, this latter property guarantees that $|e(r)|^2/r^2 = o(1)$ as $r \rightarrow +\infty$. Identity (38) then follows from using the equation (77) for the function ρ_1 . \square

We next compare the quadratic forms $Q_{R,j}$ with Q_0 .

Lemma 7. *Let $j \in \mathbb{Z} \setminus \{-1\}$ and $e \in H_j \subset H_0$. We have*

$$Q_{R,j}(e) \geq Q_0(e) \geq 0.$$

More precisely, for $j \neq -2$, we have

$$Q_{R,j}(e) - Q_0(e) \geq \frac{1}{3} \int_0^{+\infty} \frac{j^2}{r^2} |e|^2 r dr, \quad (39)$$

while for $j = -2$,

$$Q_{R,-2}(e) - Q_0(e) = 4 \int_0^{+\infty} \rho_1^2 (1-\chi_R)^2 \frac{|e|^2}{r^2} r dr. \quad (40)$$

Proof. We compute

$$Q_{R,j}(e) - Q_0(e) = \int_0^{+\infty} \left(j^2 + 2j(1 - (1-\chi_R)^2) \rho_1^2 \right) \frac{|e|^2}{r^2} r dr,$$

which gives (40) for $j = -2$. For $j \geq 0$, the conclusion follows from the inequality $j^2 + 2j(1 - (1-\chi_R)^2) \rho_1^2 \geq j^2$. For $j \leq -3$, we instead write $j^2 + 2j(1 - (1-\chi_R)^2) \rho_1^2 \geq j^2 + 2j$, and then $j^2 + 2j \geq j^2/3$. \square

As a consequence of the formula (38) for Q_0 , we obtain the following coercivity result for this quadratic form.

Corollary 1. *There exists $\kappa_0 > 0$ such that*

$$Q_0(e) + 2I_R(e) \geq \kappa_0 \left(\|e\|_{H_0}^2 + I_R(e) \right),$$

for any $e \in H_0$. Moreover, if e satisfies the orthogonality condition

$$\int_0^{+\infty} \chi(r)e(r)\rho_1(r) r dr = 0,$$

then we have

$$Q_0(e) \geq \kappa_0 \|e\|_{H_0}^2.$$

Proof. We deduce from Lemma 6 that the quadratic form $e \mapsto Q_0(e) + I_1(e)$ is positive definite. Moreover, it is as Q_0 associated to a compact perturbation of the identity in H_0 . Therefore, there exists some constant $\kappa > 0$, independent of R , such that

$$Q_0(e) + I_R(e) \geq Q_0(e) + I_1(e) \geq \kappa \|e\|_{H_0}^2,$$

so that

$$Q_0(e) + 2I_R(e) \geq \kappa \|e\|_{H_0}^2 + I_R(e).$$

Similarly, the quadratic form $e \mapsto Q_0(e) + \left(\int_0^{+\infty} \chi(r)e(r)\rho_1(r) r dr \right)^2$ is positive definite and associated to a compact perturbation of the identity. The conclusion follows for some suitable $\kappa_0 \leq \min\{\kappa, 1\}$. \square

Similarly, we derive from (38) and (39) the following coercivity result for the quadratic forms $Q_{R,j}$, with $j \neq -2, -1, 0$.

Corollary 2. *There exists $\kappa_0 > 0$, independent of R , such that*

$$Q_{R,j}(e) \geq \kappa_0 \|e\|_{H_j}^2,$$

for any $j \in \mathbb{Z} \setminus \{-2, -1, 0\}$ and any $e \in H_j$.

Proof. As for the proof of Corollary 1, the quadratic form $e \mapsto Q_0(e) + \frac{1}{4} \int_0^1 \rho_1^2 |e|^2 r dr$ is positive definite. Moreover, it is as Q_0 associated to a compact perturbation of the identity in H_0 . Therefore, we infer the existence of $\kappa_0 > 0$ such that

$$Q_0(e) \geq \kappa_0 \|e\|_{H_0}^2 - \frac{1}{4} \int_0^1 \rho_1^2 |e|^2 r dr, \quad (41)$$

for any $e \in H_0$. Since $H_j \subset H_0$ for $j \neq -1$, the summation of (39) and (41) yields the conclusion for $j \neq -2$, and for some possibly smaller value of κ_0 , but that can be chosen independently of j . \square

We next consider the more tedious case $j = -2$.

Corollary 3. *There exists $\kappa_0 > 0$, independent of R , such that we have*

$$Q_{R,2}(e) + Q_{R,-2}(f) + I_R(e \pm f) \geq \kappa_0 \left(\|e\|_{H_2}^2 + \|f\|_{H_{-2}}^2 + I_R(e \pm f) \right),$$

for any $e \in H_2$ and $f \in H_{-2}$.

Proof. We derive from Lemma 7 that

$$Q_{R,2}(e) + Q_{R,-2}(f) \geq Q_0(e) + Q_0(f) + \int_0^{+\infty} \frac{|e|^2}{r^2} r dr + \int_0^{+\infty} \rho_1^2 (1 - \chi_R)^2 \frac{|f|^2}{r^2} r dr. \quad (42)$$

Since $0 \leq \rho_1 \leq 1$, we can split

$$\int_0^{+\infty} \frac{|e|^2}{r^2} r dr \geq \int_0^{+\infty} \rho_1^2 \chi_R^2 \frac{|e|^2}{r^2} r dr + \int_0^{+\infty} \rho_1^2 (1 - \chi_R)^2 \frac{|e|^2}{r^2} r dr.$$

After summation of (42) with the inequality

$$I_R(e \pm f) = \int_0^{+\infty} \rho_1^2 \chi_R^2 |e \pm f|^2 r dr \geq \int_1^\infty \rho_1^2 \chi_R^2 \frac{|e \pm f|^2}{r^2} r dr,$$

we obtain the estimate

$$Q_{R,2}(e) + Q_{R,-2}(f) + I_R(e \pm f) \geq Q_0(e) + Q_0(f) + \frac{1}{6} \int_1^{+\infty} \rho_1^2 \left(\frac{|e|^2}{r^2} + \frac{|f|^2}{r^2} \right) r dr, \quad (43)$$

using that $e^2 + (e \pm f)^2 \geq e^2/2 + f^2/3$ and $\chi_R^2 + (1 - \chi_R)^2 \geq 1/2$. The quadratic form in the right-hand side of (43) is positive definite by Lemma 6 and independent of R . It is associated to compact perturbations of the identity for norms that are equivalent to the ones in H_2 and H_{-2} . It follows that

$$Q_{R,2}(e) + Q_{R,-2}(f) + I_R(e \pm f) \geq \kappa \left(\|e\|_{H_2}^2 + \|f\|_{H_{-2}}^2 \right),$$

for some $\kappa > 0$. We conclude by observing that

$$Q_{R,2}(e) + Q_{R,-2}(f) + I_R(e \pm f) \geq \frac{1}{2} \left(Q_{R,2}(e) + Q_{R,-2}(f) + I_R(e \pm f) \right) + \frac{1}{2} I_R(e \pm f),$$

by Lemma 7, and therefore choosing $\kappa_0 = \min\{\kappa, 1\}/2$. \square

We finally establish some coercivity for the quadratic forms $Q_{R,\pm 1}$ under suitable orthogonality conditions.

Proposition 8. *There exist $\kappa_0 > 0$, $R_0 \geq 2$ and $C > 0$ such that, given any $R \geq R_0$ and any real-valued functions $e \in H_1$ and $f \in H_{-1}$ satisfying the orthogonality conditions*

$$\int_0^{+\infty} \chi \left((e \pm f) \rho_1' - (e \mp f) \frac{\rho_1}{r} \right) r dr = 0, \quad (44)$$

we have

$$\begin{aligned} Q_{R,1}(e) + Q_{R,-1}(f) + I_R(e \pm f) \\ \geq \kappa_0 \left(\|e\|_{H_1}^2 + \|f\|_{H_{-1}}^2 + I_R(e \pm f) \right) - C \int_R^{2R} \frac{|e|^2 + |f|^2}{r^2} r dr. \end{aligned}$$

Proof. We split the quantity $Q_{R,1}(e) + Q_{R,-1}(f) + I_R(e \pm f)$ as

$$Q_{\text{loc}}^\pm(\chi_R e, \chi_R f) + Q_\infty((1 - \chi_R)e, (1 - \chi_R)f) + \mathcal{R}_R(e, f),$$

where

$$Q_{\text{loc}}^\pm(u, v) = \int_0^{+\infty} \left(|u'|^2 + |v'|^2 + \frac{4}{r^2} |u|^2 - (1 - \rho_1^2)(|u|^2 + |v|^2) + \rho_1^2 |u \pm v|^2 \right) r dr,$$

$$Q_\infty(u, v) := \int_0^{+\infty} \left(|u'|^2 + |v'|^2 + \left(\frac{4 - 2\rho_1^2}{r^2} - (1 - \rho_1^2) \right) |u|^2 + \left(\frac{2\rho_1^2}{r^2} - (1 - \rho_1^2) \right) |v|^2 \right) r dr,$$

and

$$\begin{aligned} \mathcal{R}_R(u, v) &:= \int_0^{+\infty} 2 \left((u\chi_R)'(u(1 - \chi_R))' + (v\chi_R)'(v(1 - \chi_R))' \right) r dr \\ &\quad + \int_0^{+\infty} 2\chi_R(1 - \chi_R) \left(\frac{4}{r^2} |u|^2 - (1 - \rho_1^2) (|u|^2 + |v|^2) \right) r dr. \end{aligned}$$

We now control each of the previous quantity separately.

Step 1. If u and v are supported outside of the interval $[0, R]$, and R_0 is larger than some universal constant, then we deduce from the decay properties of $1 - \rho_1^2$ in Lemma A.1 that

$$Q_\infty(u, v) \geq \int_0^{+\infty} \left(|u'|^2 + |v'|^2 + \frac{1}{2r^2} (|u|^2 + |v|^2) \right) r dr \geq \kappa_0 (\|u\|_{H_1}^2 + \|v\|_{H_{-1}}^2). \quad (45)$$

Step 2. We next claim that there exists a universal constant $C > 0$ such that

$$\mathcal{R}_R(u, v) \geq -C \int_R^{2R} \frac{|u|^2 + |v|^2}{r^2} r dr. \quad (46)$$

for any $u \in H_1$ and any $v \in H_{-1}$. Concerning the second integral in the definition of $\mathcal{R}_R(u, v)$, this estimate follows from the fact that $\chi_R(1 - \chi_R)$ is supported in $[R, 2R]$ and from the decay properties of $1 - \rho_1^2$ in Lemma A.1. For the first one, an integration by parts provides

$$\begin{aligned} \int_0^{+\infty} 2(u\chi_R)'(u(1 - \chi_R))' r dr &= 2 \int_R^{2R} \chi_R(1 - \chi_R) |u'|^2 r dr \\ &\quad - \int_R^{2R} \left(2(\chi_R')^2 + (\chi_R - \chi_R^2)'' + \frac{(\chi_R - \chi_R^2)'}{r} \right) |u|^2 r dr, \end{aligned}$$

where the first term is non-negative and

$$\left| 2(\chi_R')^2 + (\chi_R - \chi_R^2)'' + \frac{(\chi_R - \chi_R^2)'}{r} \right| \leq \frac{C}{R^2},$$

pointwise on $[R, 2R]$. The analogous inequality holds for v , and inequality (46) therefore follows.

Step 3. We finally claim that, if $u \in H_1$ and $v \in H_{-1}$ are real-valued, compactly supported, and satisfy the orthogonality condition

$$\int_0^{+\infty} \chi \left((u \pm v) \rho_1' - (u \mp v) \frac{\rho_1}{r} \right) r dr = 0, \quad (47)$$

then

$$Q_{\text{loc}}^\pm(u, v) \geq \kappa_0 \left(\|u\|_{H_1}^2 + \|v\|_{H_{-1}}^2 + \int_0^{+\infty} \rho_1^2 |u \pm v|^2 r dr \right). \quad (48)$$

In order to prove this claim, we first recall that it was proved in [10] that the quantities $Q_{\text{loc}}^\pm(u, v)$ are non-negative and vanish if and only if $u \pm v = c\rho_1'$ and $u \mp v = -c\rho_1/r$ for some constant $c \in \mathbb{R}$. As a consequence, the quadratic forms

$$\mathcal{Q}_{\text{loc}}^\pm(u, v) := Q_{\text{loc}}^\pm(u, v) + \left(\int_0^{+\infty} \chi \left((u \pm v) \rho_1' - (u \mp v) \frac{\rho_1}{r} \right) r dr \right)^2,$$

are positive definite on the Hilbert spaces G^\pm associated to the norm

$$\|(u, v)\|_{G^\pm}^2 := \|u\|_{H_1}^2 + \|v\|_{H_{-1}}^2 + \int_0^{+\infty} \rho_1^2 |u \pm v|^2 r dr.$$

The conclusion will follow from Lemma 5 if we can check that the quadratic forms $\mathcal{Q}_{\text{loc}}^\pm$ are associated to a compact perturbation of the identity in G^\pm for some equivalent norm.

We choose the norm given by

$$|||(u, v)|||_{G^\pm}^2 := \int_0^{+\infty} \left(|u'|^2 + |v'|^2 + \frac{4}{r^2} |u|^2 + |u \pm v|^2 \right) r dr - \int_4^\infty \frac{|u|^2 + |v|^2}{r^2} r dr.$$

We can readily check that $|||(u, v)|||_{G^\pm} \leq C \|(u, v)\|_{G^\pm}$ for some universal constant $C > 0$. The converse inequality requires some explanation, actually even the fact that $|||\cdot|||_{G^\pm}$ defines a norm. For that purpose, computing the discriminant gives the pointwise inequality

$$\left(\frac{2}{r^2} + \frac{1}{2} \right) |u|^2 \pm uv + \left(\frac{1}{2} - \frac{3}{2r^2} \right) |v|^2 > 0,$$

for $(u, v) \neq (0, 0)$ and $r > \sqrt{12}$. Therefore, we obtain

$$\frac{4}{r^2} |u|^2 + |u \pm v|^2 - \frac{1}{r^2} (|u|^2 + |v|^2) \geq \frac{1}{r^2} |u|^2 + \frac{1}{2} \rho_1^2 |u \pm v|^2 + \frac{1}{2r^2} |v|^2,$$

for $r \geq 4$. On the other hand, there exists some universal constant $C > 0$ such that

$$\frac{4}{r^2} |u|^2 + |u \pm v|^2 \geq \frac{1}{C} \left(\frac{4}{r^2} |u|^2 + |v|^2 + \rho_1^2 |u \pm v|^2 \right),$$

when $0 \leq r \leq 4$. Combining the previous estimates is enough to guarantee that $|||\cdot|||_{G^\pm}$ is a norm and is equivalent to $\|\cdot\|_{G^\pm}$. Next, we write

$$\begin{aligned} \mathcal{Q}_{\text{loc}}^\pm(u, v) - |||(u, v)|||_{G^\pm}^2 &= \int_4^{+\infty} \left(\frac{1}{r^2} - (1 - \rho_1^2) \right) (|u|^2 + |v|^2) r dr \\ &\quad - \int_0^4 (1 - \rho_1^2) (u^2 + v^2) r dr - \int_0^{+\infty} (1 - \rho_1^2) |u \pm v|^2 r dr, \end{aligned}$$

and each of the three terms on the right-hand-side is compact in view of the decay properties of ρ_1 . Finally, $\mathcal{Q}_{\text{loc}}^\pm - \mathcal{Q}_{\text{loc}}^\pm$ is also compact as the square of a scalar product.

We are now in position to complete the proof of Proposition 8. We apply Step 3 to $u = \chi_R e$ and $v = \chi_R f$. Since $\chi_R \equiv 1$ on the interval $[0, 2]$, the orthogonality condition (44) implies the orthogonality condition (47). Therefore, we have

$$\mathcal{Q}_{\text{loc}}^\pm(\chi_R e, \chi_R f) \geq \kappa_0 \left(\|\chi_R e\|_{H_1}^2 + \|\chi_R f\|_{H_{-1}}^2 + \int_0^{+\infty} \chi_R^2 \rho_1^2 |e \pm f|^2 r dr \right). \quad (49)$$

Next, we apply Step 1 with $u = (1 - \chi_R)e$ and $v = (1 - \chi_R)f$ so as to obtain

$$\mathcal{Q}_\infty((1 - \chi_R)e, (1 - \chi_R)f) \geq \kappa_0 \left(\|(1 - \chi_R)e\|_{H_1}^2 + \|(1 - \chi_R)f\|_{H_{-1}}^2 \right). \quad (50)$$

The summation of (49) and (50) combined with Step 2 yields the conclusion. \square

We finally conclude the

Proof of Proposition 5. In view of (6), we first note that the orthogonality conditions in (20) translate into

$$\int_{\mathbb{R}^2} \chi \langle \varepsilon, iV_1 \rangle_{\mathbb{C}} = \int_0^{+\infty} \chi b_0 \rho_1 r \, dr = 0,$$

$$\int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_x V_1 \rangle_{\mathbb{C}} = \frac{1}{2} \int_0^{+\infty} \chi \left((a_1 + a_{-1}) \rho_1' - (a_1 - a_{-1}) \frac{\rho_1}{r} \right) r \, dr = 0,$$

and

$$\int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_y V_1 \rangle_{\mathbb{C}} = -\frac{1}{2} \int_0^{+\infty} \chi \left((b_1 - b_{-1}) \rho_1' - (b_1 + b_{-1}) \frac{\rho_1}{r} \right) r \, dr = 0.$$

As a consequence, we can estimate the terms in (36) and (37) using Corollaries 1, 2 and 3, and Proposition 8 in order to obtain the lower bound

$$Q_R(\varepsilon) \geq \kappa_0 \left(\sum_{j \in \mathbb{Z}} \|\varepsilon_j\|_{H_j}^2 + \mathcal{I}_R(\varepsilon) \right) - C \int_R^{2R} \frac{|\varepsilon_1|^2 + |\varepsilon_{-1}|^2}{r^2} r \, dr,$$

for some constant $\kappa_0 > 0$ independent of R . Fix $N_0 \geq 1$. By the pigeon-hole principle, we can find $R_0 \leq R \leq 2^{N_0} R_0$ such that

$$\int_R^{2R} \frac{|\varepsilon_1|^2 + |\varepsilon_{-1}|^2}{r^2} r \, dr \leq \frac{1}{N_0} \int_{R_0}^{2^{N_0} R_0} \frac{|\varepsilon_1|^2 + |\varepsilon_{-1}|^2}{r^2} r \, dr \leq \frac{2}{N_0} \left(\|\varepsilon_1\|_{H_1}^2 + \|\varepsilon_{-1}\|_{H_{-1}}^2 \right).$$

We therefore choose N_0 such that $2C/N_0 \leq \kappa_0/2$ and the conclusion follows with κ_0 replaced by $\kappa_0/2$. \square

7 Modulation parameters - Proof of Proposition 7

The proof of Proposition 7 is classical (see e.g. [2] and the references therein). However, we have to handle with care the norm $\|\cdot\|_H$ since it is not left invariant by translation. This is the reason why we provide the following detail.

The main ingredient is to apply the implicit function theorem to the map

$$\Xi(\Psi, b, \varphi) = \left(\int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_x V_1 \rangle_{\mathbb{C}}, \int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_y V_1 \rangle_{\mathbb{C}}, \int_{\mathbb{R}^2} \chi \langle \varepsilon, iV_1 \rangle_{\mathbb{C}} \right). \quad (51)$$

The function Ψ in this expression belongs to H , the vector b is in \mathbb{R}^2 and the number φ in \mathbb{R} . As above, we have set $\varepsilon = e^{-i\varphi} \Psi(\cdot + b) - V_1$. The map Ξ is well-defined from $H \times \mathbb{R}^2 \times \mathbb{R}$ to \mathbb{R}^3 and it satisfies

$$\Xi(e^{i\varphi} V_1(\cdot - b), b, \varphi) = 0. \quad (52)$$

Applying the implicit function theorem, we can expect to construct parameters $a(\Psi) \in \mathbb{R}^2$ and $\varphi(\Psi) \in \mathbb{R}$ so as to guarantee the orthogonality conditions in (20) for any function Ψ in a neighbourhood of any vortex solution $e^{i\varphi} V_1(\cdot - b)$. We first perform this construction for the original vortex solution V_1 .

Lemma 8. *Set $\mathbf{B}_{V_1}(r) := \{\Psi \in H \text{ s.t. } \|\Psi - V_1\|_H < r\}$ for any $r > 0$. There exist $\rho > 0$ and $\Lambda > 0$ such that the following statements hold. There exist two maps $b := (b_1, b_2) \in \mathcal{C}^1(\mathbf{B}_{V_1}(\rho), \mathbb{R}^2)$ and $\varphi \in \mathcal{C}^1(\mathbf{B}_{V_1}(\rho), \mathbb{R})$ such that, given any function*

$\Psi \in \mathbf{B}_{V_1}(\rho)$, the pair $(b, \varphi) = (b(\Psi), \varphi(\Psi))$ is the unique solution in the product set $(-\Lambda\rho, \Lambda\rho)^3$ of the equations

$$\int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_x V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle \varepsilon, \partial_y V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle \varepsilon, iV_1 \rangle_{\mathbb{C}} = 0,$$

where $\varepsilon = e^{-i\varphi} \Psi(\cdot + b) - V_1$. Moreover, the maps b and φ satisfy

$$|b_1(\Psi_2) - b_1(\Psi_1)| + |b_2(\Psi_2) - b_2(\Psi_1)| + |\varphi(\Psi_2) - \varphi(\Psi_1)| \leq \Lambda \|\Psi_2 - \Psi_1\|_H, \quad (53)$$

for any functions $(\Psi_1, \Psi_2) \in \mathbf{B}_{V_1}(\rho)^2$.

Proof. Since the map Ξ is continuously differentiable from $H \times \mathbb{R}^2 \times \mathbb{R}$ to \mathbb{R}^3 , we can compute

$$\begin{aligned} \nabla_b \Xi(\Psi, b, \varphi) &= \left(\int_{\mathbb{R}^2} \chi \langle e^{-i\varphi} \nabla \Psi(\cdot + b), \partial_x V_1 \rangle_{\mathbb{C}}, \int_{\mathbb{R}^2} \chi \langle e^{-i\varphi} \nabla \Psi(\cdot + b), \partial_y V_1 \rangle_{\mathbb{C}}, \right. \\ &\quad \left. \int_{\mathbb{R}^2} \chi \langle e^{-i\varphi} \nabla \Psi(\cdot + b), iV_1 \rangle_{\mathbb{C}} \right), \end{aligned} \quad (54)$$

and

$$\begin{aligned} \partial_\varphi \Xi(\Psi, b, \varphi) &= - \left(\int_{\mathbb{R}^2} \chi \langle ie^{-i\varphi} \Psi(\cdot + b), \partial_x V_1 \rangle_{\mathbb{C}}, \int_{\mathbb{R}^2} \chi \langle ie^{-i\varphi} \Psi(\cdot + b), \partial_y V_1 \rangle_{\mathbb{C}}, \right. \\ &\quad \left. \int_{\mathbb{R}^2} \chi \langle ie^{-i\varphi} \Psi(\cdot + b), iV_1 \rangle_{\mathbb{C}} \right). \end{aligned} \quad (55)$$

Since

$$\int_{\mathbb{R}^2} \chi \langle \partial_x V_1, \partial_y V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle iV_1, \partial_x V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle iV_1, \partial_y V_1 \rangle_{\mathbb{C}} = 0,$$

we deduce from the previous formulae that the differential

$$d_{b_1, b_2, \varphi} \Xi(V_1, 0, 0) = \begin{pmatrix} \int_{\mathbb{R}^2} \chi |\partial_x V_1|^2 & 0 & 0 \\ 0 & \int_{\mathbb{R}^2} \chi |\partial_y V_1|^2 & 0 \\ 0 & 0 & \int_{\mathbb{R}^2} \chi |V_1|^2 \end{pmatrix}, \quad (56)$$

is a continuous isomorphism from $\mathbb{R}^2 \times \mathbb{R}$ to \mathbb{R}^3 . In view of (52), we infer from the implicit function theorem the existence of some $\rho > 0$, of an open neighbourhood \mathcal{U} of $(V_1, 0, 0)$ in $H \times \mathbb{R}^3$ and of two functions $b \in \mathcal{C}^1(\mathbf{B}_{V_1}(\rho), \mathbb{R}^2)$ and $\varphi \in \mathcal{C}^1(\mathbf{B}_{V_1}(\rho), \mathbb{R})$ such that, for any datum $(\Psi, b, \varphi) \in \mathcal{U}$, the equation $\Xi(\Psi, b, \varphi) = 0$ owns a unique solution given by $(b, \varphi) = (b(\Psi), \varphi(\Psi))$. By continuous differentiability of the map $\gamma := (b, \varphi)$, we can decrease the value of ρ such that the operator norm $\|d\gamma(\Psi)\|$ of the differentials $d\gamma(\Psi)$ is less than $\Lambda := 1 + \|d\gamma(V_1)\|$ on the ball $\mathbf{B}_{V_1}(\rho)$. Inequality (53) then follows from the mean value inequality. In turn, we infer from (53) that the map γ is valued into the ball $B_{\Lambda\rho}$ and we can decrease the value of ρ , if necessary, so as to replace the open subset \mathcal{U} by the product set $\mathbf{B}_{V_1}(\rho) \times B_{\Lambda\rho}$. This ends the proof of Lemma 8. \square

We now extend the previous construction to the neighbourhood of any fixed vortex solution $e^{i\phi} V_1(\cdot - a)$ by using the translation and phase invariances.

Corollary 4. *For $a \in \mathbb{R}^2$ and $\phi \in \mathbb{R}$, consider the balls $\mathbf{B}_{(a, \phi)}(r) := \{\Psi \in H \text{ s.t. } \|e^{-i\phi} \Psi(\cdot + a) - V_1\|_H < r\}$ for any $r > 0$, and set*

$$b_{a, \phi}(\psi) = a + b(e^{-i\phi} \psi(\cdot + a)) \quad \text{and} \quad \varphi_{a, \phi}(\psi) = a + \varphi(e^{-i\phi} \psi(\cdot + a)),$$

for any function $\psi \in \mathbf{B}_{(a,\phi)}(\rho)$. Given any function $\Psi \in \mathbf{B}_{(a,\phi)}(\rho)$, the pair $(b, \varphi) = (b_{a,\phi}(\Psi), \varphi_{a,\phi}(\Psi))$ is the unique solution in the product set $(a_1 - \Lambda\rho, a_1 + \Lambda\rho) \times (a_2 - \Lambda\rho, a_2 + \Lambda\rho) \times (\phi - \Lambda\rho, \phi + \Lambda\rho)$ of the equations

$$\int_{\mathbb{R}^2} \chi\langle \varepsilon, \partial_x V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi\langle \varepsilon, \partial_y V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi\langle \varepsilon, iV_1 \rangle_{\mathbb{C}} = 0,$$

where $\varepsilon = e^{-i\varphi}\Psi(\cdot + b) - V_1$. Moreover, the maps $b_{a,\phi}$ and $\varphi_{a,\phi}$ satisfy

$$\begin{aligned} & |[b_{a,\phi}]_1(\Psi_2) - [b_{a,\phi}]_1(\Psi_1)| + |[b_{a,\phi}]_2(\Psi_2) - [b_{a,\phi}]_2(\Psi_1)| + |\varphi_{a,\phi}(\Psi_2) - \varphi_{a,\phi}(\Psi_1)| \\ & \leq \Lambda \|\Psi_2(\cdot + a) - \Psi_1(\cdot + a)\|_H, \end{aligned}$$

for any functions $(\Psi_1, \Psi_2) \in \mathbf{B}_{(a,\phi)}(\rho)^2$.

Proof. Corollary 4 is a direct consequence of Lemma 8 once we have observed that the map $(\Psi, b, \phi) \mapsto (e^{-i\phi}\Psi(\cdot + a), b - a, \varphi - \phi)$ is a bijection from the product set $\mathbf{B}_{(a,\phi)}(\rho) \times (a_1 - \Lambda\rho, a_1 + \Lambda\rho) \times (a_2 - \Lambda\rho, a_2 + \Lambda\rho) \times (\phi - \Lambda\rho, \phi + \Lambda\rho)$ onto $\mathbf{B}_{V_1}(\rho) \times (-\Lambda\rho, \Lambda\rho)^3$. \square

The next step in the proof of Proposition 7 is to extend the previous construction to a neighbourhood of the orbit of V_1 of the form $\mathcal{V}(\alpha)$. By definition, this neighbourhood is equal to

$$\mathcal{V}(\alpha) = \bigcup_{(a,\phi) \in \mathbb{R}^3} \mathbf{B}_{(a,\phi)}(\alpha).$$

For $\alpha \leq \rho$, the existence of modulation parameters so that the orthogonality conditions in (20) are satisfied results from Corollary 4. In order to complete the proof of Proposition 7, it essentially remains to establish that the choice of these parameters can be made in a continuously differentiable way.

In this direction, the main difficulty is to prevent the possibility that a function $\Psi \in \mathcal{V}(\rho)$ belongs to two balls $\mathbf{B}_{(a_1,\phi_1)}(\rho)$ and $\mathbf{B}_{(a_2,\phi_2)}(\rho)$ for points a_1 and a_2 at a large distance from one another. In this case, the translation parameter a can be chosen either close to a_1 or to a_2 in view of (53). Hence it is not so direct to find a continuously differentiable choice for this parameter.

In order to by-pass this difficulty, we first show the following lemma.

Lemma 9. *There exist $\alpha_0 > 0$ and $R_0 > 0$ such that, if*

$$\mathbf{B}_{(a_1,\phi_1)}(\alpha_0) \cap \mathbf{B}_{(a_2,\phi_2)}(\alpha_0) \neq \emptyset,$$

for points $(a_1, a_2) \in \mathbb{R}^4$ and numbers $(\phi_1, \phi_2) \in \mathbb{R}^2$, then

$$|a_1 - a_2| \leq R_0.$$

Proof. Let $\alpha > 0$ and consider a function $\Psi \in \mathbf{B}_{(a_1,\phi_1)}(\alpha) \cap \mathbf{B}_{(a_2,\phi_2)}(\alpha)$ for two points a_1 and a_2 , and two numbers ϕ_1 and ϕ_2 . By definition, this function first satisfies

$$\int_{\mathbb{R}^2} (1 - |V_1|^2) \left| \nabla(\Psi(\cdot + a_1) - e^{i\phi_1} V_1) \right|^2 < \alpha^2. \quad (57)$$

In view of Lemma A.1, the integral

$$I_1 := \|V_1\|_H^2 = \int_{\mathbb{R}^2} \left(|\nabla|V_1|^2|^2 + (1 - |V_1|^2) |\nabla V_1|^2 \right),$$

is finite and positive. Moreover, we can find $R > 0$ such that

$$\int_{B_R} \left(|\nabla|V_1|^2|^2 + (1 - |V_1|^2)|\nabla V_1|^2 \right) = \frac{31I_1}{32}. \quad (58)$$

Hence we infer from (57) and the inequality $(\alpha - \beta)^2 \geq \alpha^2/2 - \beta^2$ that

$$\int_{B_R} (1 - |V_1|^2)|\nabla\Psi(\cdot + a_1)|^2 \geq \frac{31I_1}{32} - \alpha^2.$$

Since $\rho_1 = |V_1| < 1$, we are led to

$$\int_{B_R(a_1)} |\nabla\Psi|^2 \geq \frac{31I_1}{64} - \alpha^2. \quad (59)$$

Here we have set, as in the sequel, $B_r(a) = \{x \in \mathbb{R}^2 \text{ s.t. } |x - a| < r\}$ for any $r > 0$ and any $a \in \mathbb{R}^2$.

Similarly, we know that

$$\int_{\mathbb{R}^2} \left| \nabla(\bar{V}_1(\Psi(\cdot + a_2) - e^{i\phi_2}V_1)) \right|^2 < \alpha^2, \quad (60)$$

so that, by (58),

$$\int_{B_R(a_2)^c} \left| \nabla(\bar{V}_1(\cdot - a_2)\Psi) \right|^2 \leq 2\left(\frac{I_1}{32} + \alpha^2\right).$$

Assuming that $|a_2 - a_1| \geq 2R$, we obtain

$$\frac{1}{2} \int_{B_R(a_1)} |V_1(\cdot - a_2)|^2 |\nabla\Psi|^2 \leq 2\left(\frac{I_1}{32} + \alpha^2\right) + \int_{B_R(a_1)} |\Psi|^2 |\nabla V_1(\cdot - a_2)|^2.$$

At this stage, we can increase, if necessary, the value of the number R such that $|V_1(x)|^2 \geq 1/2$ for $|x| \geq R$. In this case, we obtain

$$\int_{B_R(a_1)} |\nabla\Psi|^2 \leq 8\left(\frac{I_1}{32} + \alpha^2\right) + 4 \int_{B_R(a_1)} |\Psi|^2 |\nabla V_1(\cdot - a_2)|^2.$$

In view of the existence of a universal constant $C > 0$ such that $|\nabla V_1(x)|^2 \leq C/(1 + |x|^2)$, we next have

$$\int_{B_R(a_1)} |\nabla\Psi|^2 \leq 8\left(\frac{I_1}{32} + \alpha^2\right) + \frac{4C}{1 + (|a_2 - a_1| - R)^2} \int_{B_R(a_1)} |\Psi|^2.$$

Going to the proof of Lemma B.2, we next find $K_R > 0$, depending only on R , such that

$$\int_{B_R(a_1)} |\Psi|^2 \leq K_R \|\Psi\|_{H_{a_1}}^2 \leq 2K_R \left(\|\Psi - e^{i\varphi_1}V_1(\cdot - a_1)\|_{H_{a_1}}^2 + I_1 \right),$$

for any number $\varphi \in \mathbb{R}$. Hence we are led to

$$\int_{B_R(a_1)} |\nabla\Psi|^2 \leq 8\left(\frac{I_1}{32} + \alpha^2\right) + \frac{8CK_R}{1 + (|a_2 - a_1| - R)^2} (\alpha^2 + I_1). \quad (61)$$

Combining (59) and (61) next gives

$$\frac{31I_1}{64} - \alpha^2 \leq 8\left(\frac{I_1}{32} + \alpha^2\right) + \frac{8CK_R}{1 + (|a_2 - a_1| - R)^2} (\alpha^2 + I_1),$$

and we can choose $\alpha = \alpha_0 := \sqrt{I_1/64}$ in order to obtain

$$|a_2 - a_1| \leq R + 10\sqrt{CK_R}.$$

The conclusion follows for $R_0 = \max\{2R, R + 10\sqrt{CK_R}\}$. \square

We next refine the bound in Lemma 9 for α small.

Lemma 10. *Let $\mu > 0$. There exists $\nu > 0$ such that, if*

$$\mathbf{B}_{(a_1, \phi_1)}(\nu) \cap \mathbf{B}_{(a_2, \phi_2)}(\nu) \neq \emptyset,$$

for points $(a_1, a_2) \in \mathbb{R}^4$ and numbers $(\phi_1, \phi_2) \in \mathbb{R}^2$, then

$$|a_2 - a_1| + |e^{i\phi_2} - e^{i\phi_1}| < \mu.$$

Proof. We argue by contradiction assuming that the statement in Lemma 10 is wrong. In this case, we can find $\mu > 0$, as well as sequences $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, $(\phi_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ such that

$$\mathbf{B}_{(a_n, \phi_n)}\left(\frac{1}{2^n}\right) \cap \mathbf{B}_{(b_n, \varphi_n)}\left(\frac{1}{2^n}\right) \neq \emptyset, \quad (62)$$

and

$$|b_n - a_n| + |e^{i\varphi_n} - e^{i\phi_n}| \geq \mu, \quad (63)$$

for any $n \in \mathbb{N}$. Up to a subsequence, there exist two numbers ϕ_1 and ϕ_2 such that

$$e^{i\phi_n} \rightarrow e^{i\phi_\infty} \quad \text{and} \quad e^{i\varphi_n} \rightarrow e^{i\varphi_\infty}, \quad (64)$$

as $n \rightarrow +\infty$. For n large enough, we also infer from (62) that the difference $a_n - b_n$ is bounded by the number R_0 in Lemma 9. Up to a further subsequence, we can assume the existence of a point $d_\infty \in \mathbb{R}^2$ such that

$$b_n - a_n \rightarrow d_\infty, \quad (65)$$

as $n \rightarrow +\infty$. Since the norm $|d_\infty|$ is positive by (63), it follows that

$$|b_n - a_n| \leq 2|d_\infty|, \quad (66)$$

for n large enough.

Consider next a sequence of functions Ψ_n in $\mathbf{B}_{(a_n, \phi_n)}(1/2^n) \cap \mathbf{B}_{(b_n, \varphi_n)}(1/2^n)$. Going back to the proof of Lemma B.2, we can find $C > 0$, depending only on $|d_\infty|$, such that

$$\int_{B_{3|d_\infty|}} \left(|e^{-i\phi_n} \Psi_n(\cdot + a_n) - V_1|^2 + |e^{-i\varphi_n} \Psi_n(\cdot + b_n) - V_1|^2 \right) \leq \frac{C}{2^n},$$

for any $n \in \mathbb{N}$. In view of (66), we observe that

$$\begin{aligned} \int_{B_{|d_\infty|}} |e^{i(\varphi_n - \phi_n)} V_1(\cdot - b_n + a_n) - V_1|^2 &\leq 2 \int_{B_{3|d_\infty|}} |e^{-i\phi_n} \Psi_n(\cdot + a_n) - V_1|^2 \\ &\quad + 2 \int_{B_{3|d_\infty|}(b_n - a_n)} |e^{-i\varphi_n} \Psi_n(\cdot + a_n) - V_1(\cdot - b_n + a_n)|^2, \end{aligned}$$

so that

$$\int_{B_{|d_\infty|}} |e^{i(\varphi_n - \phi_n)} V_1(\cdot - b_n + a_n) - V_1|^2 \leq \frac{4C}{2^n}.$$

Combining with (64) and (65), we are led to

$$\int_{B_{|d_\infty|}} |e^{i(\phi_2 - \phi_1)} V_1(\cdot - d_\infty) - V_1|^2 = 0,$$

in the limit $n \rightarrow +\infty$. We conclude that $e^{i(\phi_2 - \phi_1)} V_1(\cdot - d_\infty) = V_1$ on the ball $B_{|d_\infty|}$. Since the function V_1 only vanishes at the origin, we infer that $d_\infty = 0$, and then $e^{i\phi_2} = e^{i\phi_1}$. This is a contradiction with the inequality $|d_\infty| + |e^{i\phi_2} - e^{i\phi_1}| \geq \mu$, that follows from (63) in the limit $n \rightarrow +\infty$. This completes the proof of Lemma 10. \square

We are now in position to conclude the

Proof of Proposition 7. Without loss of generality, we can assume that the numbers ρ and Λ in Lemma 8 and Corollary 4 satisfy $\rho\Lambda < 1$. We consider the number ν provided by Lemma 10 for $\mu = \rho\Lambda/8$ and we set $\alpha := \min\{\rho/2, \nu\}$.

Given a function $\Psi \in \mathcal{V}(\alpha)$, we can find a point a and a number ϕ such that Ψ belongs to the ball $\mathbf{B}_{(a,\phi)}(\alpha)$. Since $\alpha < \rho$, we infer from Corollary 4 the existence of a point $a(\Psi) = b_{a,\phi}(\Psi) \in \mathbb{R}^2$ and of a number $\varphi(\Psi) = \varphi_{a,\phi}(\Psi) \in \mathbb{R}$ such that the orthogonality conditions in (20) are satisfied. We claim that the value of $a(\Psi)$ does not depend on the choice of a and ϕ . The number $\varphi(\Psi)$ is also independent of this choice, but modulo 2π .

Assume indeed that the function Ψ is in $\mathbf{B}_{(b,\vartheta)}(\alpha)$ for another point $b \in \mathbb{R}^2$ and another number $\vartheta \in \mathbb{R}$. The intersection $\mathbf{B}_{(a,\phi)}(\nu) \cap \mathbf{B}_{(b,\vartheta)}(\nu)$ is then not empty, so that by Lemma 10,

$$|b - a| + |e^{i\vartheta} - e^{i\phi}| < \mu = \frac{\rho\Lambda}{8}. \quad (67)$$

Recall here that $|e^{it} - 1| = 2|\sin(t/2)| \geq 2|t|/\pi$ when $t \in [-\pi, \pi]$. Hence, there exists an integer $k \in \mathbb{Z}$ such that

$$|\vartheta + 2\pi k - \phi| < \frac{\pi\mu}{2} < \frac{\rho\Lambda}{4}. \quad (68)$$

On the other hand, we also infer from Corollary 4 that

$$|a_1(\Psi) - a_1| + |a_2(\Psi) - a_2| + |\varphi(\Psi) - \phi| \leq \Lambda \|\Psi(\cdot + a) - e^{i\varphi} V_1(\cdot + a - a)\|_H < \frac{\Lambda\rho}{2}.$$

Combining with (67) and (68), we obtain

$$|a_1(\Psi) - b_1| + |a_2(\Psi) - b_2| + |\varphi(\Psi) - \vartheta - 2\pi k| < \Lambda\rho.$$

Since $\Xi(\Psi, a(\Psi), \varphi(\Psi) + 2k\pi) = 0$, we deduce from Corollary 4 that $a(\Psi) = b_{b,\vartheta}(\Psi)$ and $\varphi(\Psi) = \varphi_{b,\vartheta}(\Psi) + 2k\pi$. In conclusion, the choice of $a(\Psi)$ and $\varphi(\Psi)$ (modulo 2π) does not depend on the choice of a and ϕ such that $\Psi \in \mathbf{B}_{(a,\phi)}(\alpha)$. Therefore, the maps a and φ are well-defined from $\mathcal{V}(\alpha)$ with values in $\mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}$. They are continuous differentiable on $\mathcal{V}(\alpha)$ due to the continuous differentiability of the maps $b_{a,\phi}$ and $\varphi_{a,\phi}$.

We now turn to the proof of (23). When $\|e^{-i\vartheta}\Psi(\cdot + b) - V_1\|_H < \alpha$, we have $a(\Psi) = b_{b,\vartheta}(\Psi)$ and $\varphi(\Psi) = \varphi_{b,\vartheta}(\Psi)$. As a consequence of Corollary 4, we first obtain

$$|a(\Psi) - b| + |\varphi(\Psi) - \vartheta| \leq \sqrt{2}\Lambda \|e^{-i\vartheta}\Psi(\cdot + b) - V_1\|_H. \quad (69)$$

On the other hand, we know that

$$\|\varepsilon\|_H = \|e^{-i\varphi(\Psi)}\Psi(\cdot + a(\Psi)) - V_1\|_H.$$

Since $|a(\Psi) - b| < \sqrt{2}\Lambda\alpha < 1/\sqrt{2}$ by (69), we can invoke the uniform boundedness of the translation operators in Lemma B.3 in order to find $C > 0$ such that

$$\|\varepsilon\|_H \leq C \|e^{-i\varphi(\Psi)}\Psi(\cdot + b) - V_1(\cdot - a(\Psi) + b)\|_H.$$

In particular, we obtain

$$\|\varepsilon\|_H \leq C \left(\|e^{-i\vartheta}\Psi(\cdot + b) - V_1\|_H + |e^{i(\vartheta - \varphi(\Psi))} - 1| \|V_1\|_H + \|V_1 - V_1(\cdot - a(\Psi) + b)\|_H \right).$$

Again since $|a(\Psi) - b| < 1/\sqrt{2}$ by (69), we can infer from Lemma B.4 the existence of a further $C > 0$ such that

$$\|\varepsilon\|_H \leq C \left(\|e^{-i\vartheta}\Psi(\cdot + b) - V_1\|_H + |\varphi(\Psi) - \vartheta| \|V_1\|_H + |a(\Psi) - b| \right).$$

Estimate (23) then follows from (69). This completes the proof of Proposition 7. \square

8 Evolution of the modulation parameters - Proof of Proposition 4

Let $\tau > 0$ to be fixed later. Under the assumption $d_E(V_1, \Psi_0) \leq \tau$, we can go back to the proofs of Theorems 1 and 2 in order to check that the solution Ψ_t lies in a set $\mathcal{V}(\alpha_\tau)$ for any $t \in \mathbb{R}$. Here, the numbers α_τ tend to 0 when $\tau \rightarrow 0$. In particular, we can apply Proposition 7 for τ small enough. This provides modulation parameters $a(t) := a(\Psi_t) \in \mathbb{R}^2$ and $\varphi(t) := \varphi(\Psi_t) \in \mathbb{R}/2\pi\mathbb{Z}$ that satisfy all the statements in Proposition 7.

Recall here that the solution Ψ lies in $\mathcal{C}(\mathbb{R}, \Psi_0 + H^1(\mathbb{R}^2))$ by Proposition 3. Going back to the proof of Lemma B.1, we check that it remains continuous with values in the Hilbert space H . As a consequence of Proposition 7, the previous maps a and φ are also continuous from \mathbb{R} to \mathbb{R}^2 , respectively $\mathbb{R}/2\pi\mathbb{Z}$. Up to the choice of a constant in $2\pi\mathbb{Z}$, we can therefore reduce the map φ to a continuous real-valued function.

In order to prove the continuous differentiability of a and φ , we rely on the decomposition of any function in E given by Lemma B.8. We first assume that the initial datum Ψ_0 takes the form $U_0 + w_0$, with $U_0 \in \mathcal{U}$ and $w_0 \in H^3(\mathbb{R}^2)$. In this case, the corresponding solution Ψ belongs to $\mathcal{C}^0(\mathbb{R}, U_0 + H^3(\mathbb{R}^2))$ by Proposition C.1. In view of (3), it is also in $\mathcal{C}^1(\mathbb{R}, U_0 + H^3(\mathbb{R}^2))$, so in $\mathcal{C}^1(\mathbb{R}, H)$ by Lemma B.1. In view of Proposition 7, the functions a and φ are then in $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^2)$, respectively $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$. In order to extend this property to all initial data in E , we now compute the time derivatives a' and φ' by differentiating the orthogonality conditions in (20). We will eventually rely on these computations and a standard density argument in order to complete the proof of Proposition 4.

Before going into to the computations of the derivatives a' and φ' , we deduce from the previous smoothness properties of the maps a and φ , as well as of the solution Ψ , and from equations (1) and (3) that the function $\varepsilon(x, t) := e^{-i\varphi(t)}\Psi(x+a(t), t) - V_1(x)$ in (22) satisfies

$$i\partial_t\varepsilon + \Delta\varepsilon + (1 - |V_1|^2)\varepsilon + \eta_\varepsilon(V_1 + \varepsilon) - \varphi'(t)(V_1 + \varepsilon) - ia'(t) \cdot (\nabla V_1 + \nabla\varepsilon) = 0, \quad (70)$$

where $\eta_\varepsilon = 1 - |V_1 + \varepsilon|^2 - (1 - |V_1|^2)$, as before. With this equation at hand, we can differentiate the three orthogonality conditions in (20) in order to obtain the system

$$\mathcal{M}_\varepsilon(t) \begin{pmatrix} a'_1(t) \\ a'_2(t) \\ \varphi'(t) \end{pmatrix} = \mathcal{F}_\varepsilon(t). \quad (71)$$

Since

$$\int_{\mathbb{R}^2} \chi \langle \partial_x V_1, \partial_y V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle iV_1, \partial_x V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi \langle iV_1, \partial_y V_1 \rangle_{\mathbb{C}} = 0,$$

the matrix \mathcal{M}_ε in this formula is given by

$$\mathcal{M}_\varepsilon = \mathcal{M}_0 + \begin{pmatrix} \int_{\mathbb{R}^2} \chi \langle \partial_x V_1, \partial_x \varepsilon \rangle_{\mathbb{C}} & \int_{\mathbb{R}^2} \chi \langle \partial_x V_1, \partial_y \varepsilon \rangle_{\mathbb{C}} & \int_{\mathbb{R}^2} \chi \langle i\partial_x V_1, \varepsilon \rangle_{\mathbb{C}} \\ \int_{\mathbb{R}^2} \chi \langle \partial_y V_1, \partial_x \varepsilon \rangle_{\mathbb{C}} & \int_{\mathbb{R}^2} \chi \langle \partial_y V_1, \partial_y \varepsilon \rangle_{\mathbb{C}} & \int_{\mathbb{R}^2} \chi \langle i\partial_y V_1, \varepsilon \rangle_{\mathbb{C}} \\ \int_{\mathbb{R}^2} \chi \langle V_1, i\partial_x \varepsilon \rangle_{\mathbb{C}} & \int_{\mathbb{R}^2} \chi \langle V_1, i\partial_y \varepsilon \rangle_{\mathbb{C}} & \int_{\mathbb{R}^2} \chi \langle V_1, \varepsilon \rangle_{\mathbb{C}} \end{pmatrix}, \quad (72)$$

with

$$\mathcal{M}_0 = \begin{pmatrix} \int_{\mathbb{R}^2} \chi |\partial_x V_1|^2 & 0 & 0 \\ 0 & \int_{\mathbb{R}^2} \chi |\partial_y V_1|^2 & 0 \\ 0 & 0 & \int_{\mathbb{R}^2} \chi |V_1|^2 \end{pmatrix}. \quad (73)$$

Similarly, the right-hand side \mathcal{F}_ε is equal to

$$\mathcal{F}_\varepsilon = \begin{pmatrix} \int_{\mathbb{R}^2} \left(\chi((1 - |V_1|^2) \langle i\partial_x V_1, \varepsilon \rangle_{\mathbb{C}} + \eta_\varepsilon \langle i\partial_x V_1, V_1 + \varepsilon \rangle_{\mathbb{C}}) - \langle i\nabla(\chi\partial_x V_1), \nabla\varepsilon \rangle_{\mathbb{C}} \right) \\ \int_{\mathbb{R}^2} \left(\chi((1 - |V_1|^2) \langle i\partial_y V_1, \varepsilon \rangle_{\mathbb{C}} + \eta_\varepsilon \langle i\partial_y V_1, V_1 + \varepsilon \rangle_{\mathbb{C}}) - \langle i\nabla(\chi\partial_y V_1), \nabla\varepsilon \rangle_{\mathbb{C}} \right) \\ \int_{\mathbb{R}^2} \left(\chi((1 - |V_1|^2) \langle V_1, \varepsilon \rangle_{\mathbb{C}} + \eta_\varepsilon \langle V_1, V_1 + \varepsilon \rangle_{\mathbb{C}}) - \langle \nabla(\chi V_1), \nabla\varepsilon \rangle_{\mathbb{C}} \right) \end{pmatrix}. \quad (74)$$

At this stage, recall the existence of a constant $A > 0$ such that

$$\|\varepsilon(\cdot, t)\|_H \leq A\alpha_\tau,$$

by (23). Combining the definition of the norm $\|\cdot\|_H$, Lemma B.2 and the fact that the function χ is smooth and compactly supported, we deduce from (72) the existence of $C > 0$ such that

$$\|\mathcal{M}_\varepsilon(t) - \mathcal{M}_0\| \leq C\alpha_\tau. \quad (75)$$

Hence we can choose the value of τ small enough, so that the matrix \mathcal{M}_ε is invertible and its operator norm is less than a number $C > 0$ (depending only on α_τ).

Since $\eta_\varepsilon = -2\langle \varepsilon, V_1 \rangle_{\mathbb{C}} - |\varepsilon|^2$, we can similarly invoke the definition of the norm $\|\cdot\|_H$, Lemma B.2, the fact that the function χ is smooth and compactly supported, as well as the Sobolev embedding theorem, in order to control the right-hand side \mathcal{F}_ε by

$$|\mathcal{F}_\varepsilon(t)| \leq C\|\varepsilon(\cdot, t)\|_H, \quad (76)$$

for a further choice of the constant C . In view of (71) and (75), we obtain

$$|a'(t)| + |\varphi'(t)| \leq C\|\varepsilon(\cdot, t)\|_H.$$

Estimate (5) then follows from the proofs of Theorems 1 and 2.

We are now in position to complete the proof of Proposition 4 by a density argument. Consider an arbitrary initial datum $\Psi^0 \in \mathcal{E}(\mathbb{R})$ and the corresponding decomposition $\Psi_0 = U_0 + w_0$ provided by Lemma B.8. We can find a sequence of functions w_0^n in $H^3(\mathbb{R})$ that tend to w_0 in $H^1(\mathbb{R}^2)$ as $n \rightarrow +\infty$. Moreover, the corresponding maps a_n and φ_n are in $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^2)$, respectively $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$, and their time derivatives are given by (71).

Recall here that the Gross-Pitaevskii flow is globally continuous with respect to the initial datum in $U_0 + H^1(\mathbb{R})$ by Proposition C.1. Going back to the proof of Lemma B.1, we observe that the flow map $\Psi_0 \mapsto \Psi(\cdot, t)$ remains continuous from $U_0 + H^1(\mathbb{R})$ to $\mathcal{C}^0([-T, T], H)$ for any $T > 0$. In view of Proposition 7, we first deduce that the maps a_n and φ_n converge in $\mathcal{C}^0([-T, T], \mathbb{R}^2)$, respectively $\mathcal{C}^0([-T, T], \mathbb{R})$, to the maps a and φ corresponding to the initial datum Ψ_0 . We also deduce that the function ε in (22) also depends continuously in $\mathcal{C}^0([-T, T], H)$ on the initial datum in $U_0 + H^1(\mathbb{R})$. Going back to the proofs of (75) and (76), it follows that the time derivatives a'_n and φ'_n are convergent in $\mathcal{C}^0([-T, T], \mathbb{R}^2)$, respectively $\mathcal{C}^0([-T, T], \mathbb{R})$. This is enough to guarantee the continuous differentiability of the maps a and φ , and that their time derivatives a' and φ' satisfy (71) in the limit $n \rightarrow +\infty$. Reproducing the proofs of (75) and (76), we obtain (5) as before. This completes the proof of Proposition 4. \square

A Properties of the vortex solution

Recall that the vortex solution V_1 takes the special form $V_1(x) = \rho_1(r)e^{i\theta}$ for any point $x = (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2$. Several properties of its profile ρ_1 are useful in

the course of our proofs. For the sake of completeness, we have collected them in the next lemma, as well as their consequences on the algebraic decay rate of the lower order derivatives of V_1 .

Lemma A.1. (i) *There exists a unique solution $\rho_1 : [0, \infty) \rightarrow \mathbb{R}$ to the ordinary differential equation*

$$\rho_1''(r) + \frac{\rho_1'(r)}{r} - \frac{\rho_1(r)}{r^2} + \rho_1(r)(1 - \rho_1(r)^2) = 0, \quad (77)$$

with $\rho_1(0) = 0$, and $\rho_1(r) \rightarrow 1$ as $r \rightarrow +\infty$. The function ρ_1 is smooth, increasing and it satisfies

$$\rho_1(r) = A_1 \left(r - \frac{r^3}{8} + \mathcal{O}(r^5) \right) \text{ as } r \rightarrow 0,$$

with $A_1 = \rho_1'(0) > 0$.

(ii) *Moreover, the function ρ_1 satisfies the asymptotics*

$$\rho_1(r) = 1 - \frac{1}{2r^2} - \frac{9}{8r^4} + \mathcal{O}\left(\frac{1}{r^6}\right),$$

as well as

$$\rho_1'(r) = \frac{1}{r^3} + \frac{9}{2r^5} + \mathcal{O}\left(\frac{1}{r^7}\right), \quad \rho_1''(r) = -\frac{3}{r^4} - \frac{45}{2r^6} + \mathcal{O}\left(\frac{1}{r^8}\right) \quad \text{and} \quad \rho_1'''(r) \sim \frac{12}{r^5},$$

in the limit $r \rightarrow +\infty$. In particular, we have

$$1 - \rho_1(r)^2 = \frac{1}{r^2} + \frac{2}{r^4} + \mathcal{O}\left(\frac{1}{r^6}\right),$$

as $r \rightarrow +\infty$.

(iii) *As a consequence, there exists a universal constant $C > 0$ such that*

$$|\nabla V_1(x)| \leq \frac{C}{1 + |x|}, \quad |d^2 V_1(x)| \leq \frac{C}{1 + |x|^2} \quad \text{and} \quad |d^3 V_1(x)| \leq \frac{C}{1 + |x|^3},$$

for any $x \in \mathbb{R}^2$.

Proof. Statement (i) is proved in [6, 13]. Statement (ii) is given in [6, Theorem 3.4], except the expansion of the function $1 - \rho_1^2$ that is a direct consequence of the one for ρ_1 , and the asymptotics for the third derivative ρ_1''' that is obtained by differentiating (77).

Since the vortex solution V_1 is smooth on \mathbb{R}^2 , the proof of Statement (i) reduces to quantify the algebraic decay of the lower order derivatives of V_1 . Going back to (6), a direct computation provides the existence of a universal number such that

$$|\nabla V_1(x)| \leq C \left(\rho_1'(|x|) + \frac{\rho_1(|x|)}{|x|} \right), \quad |d^2 V_1(x)| \leq C \left(|\rho_1''(|x|)| + \frac{\rho_1'(|x|)}{|x|} + \frac{\rho_1(|x|)}{|x|^2} \right),$$

and

$$|d^3 V_1(x)| \leq C \left(|\rho_1'''(|x|)| + \frac{|\rho_1''(|x|)|}{|x|} + \frac{\rho_1'(|x|)}{|x|^2} + \frac{\rho_1(|x|)}{|x|^3} \right),$$

for any $x \in \mathbb{R}^2$. The bounds in Statement (iii) then follow from Statement (ii). This concludes the proof of Lemma A.1. \square

B Properties of the function spaces

In this appendix, we gather some properties related to the Hilbert space H and the metric space E that are useful in the course of our proofs.

B.1 Properties of the Hilbert space H

Recall that the vector space H is defined as

$$H := \left\{ \psi \in L^2_{loc}(\mathbb{R}^2) \text{ s.t. } \nabla(\psi \bar{V}_1) \in L^2(\mathbb{R}^2) \text{ and } (1 - |V_1|^2)^{\frac{1}{2}} \nabla \psi \in L^2(\mathbb{R}^2) \right\},$$

and that it is naturally endowed with the scalar product

$$\langle \psi_1, \psi_2 \rangle_H := \int_{\mathbb{R}^2} \left(\langle \nabla(\psi_1 \bar{V}_1), \nabla(\psi_2 \bar{V}_1) \rangle_{\mathbb{C}} + (1 - |V_1|^2) \langle \nabla \psi_1, \nabla \psi_2 \rangle_{\mathbb{C}} \right).$$

Concerning its topological properties, we show

Lemma B.1. *The vector space H is a Hilbert space for the scalar product $\langle \cdot, \cdot \rangle_H$. Moreover,*

$$H^1(\mathbb{R}^2) \subset H.$$

Proof. Consider a Cauchy sequence $(\psi_n)_{n \in \mathbb{N}}$ of H . Fix $R > 0$ and denote by

$$m_R(f) = \frac{1}{\pi R^2} \int_{B_R} f,$$

the average mean of a locally integrable function f on the ball B_R . As a consequence of the Poincaré-Wirtinger inequality, the sequences $(\psi_n \bar{V}_1 - m_R(\psi_n \bar{V}_1))_{n \in \mathbb{N}}$ and $(\psi_n - m_R(\psi_n))_{n \in \mathbb{N}}$ are Cauchy sequences, therefore convergent sequences, in $L^2(B_R)$. Hence, the sequence $(m_R(\psi_n) \bar{V}_1 - m_R(\psi_n \bar{V}_1))_{n \in \mathbb{N}}$ is also convergent in this space. Taking the scalar product in $L^2(B_R)$ with the function \bar{V}_1 , we deduce that $(m_R(\psi_n))_{n \in \mathbb{N}}$ is a convergent sequence, so that $(\psi_n)_{n \in \mathbb{N}}$ is convergent in $L^2(B_R)$, and more generally in $L^2_{loc}(\mathbb{R}^2)$.

Let us denote by ψ_∞ its limit. Going back to the definition of a Cauchy sequence in H , we know that the sequences $(\nabla(\psi_n \bar{V}_1))_{n \in \mathbb{N}}$ and $((1 - |V_1|^2)^{1/2} \nabla \psi_n)_{n \in \mathbb{N}}$ are convergent in $L^2(\mathbb{R}^2)$. Testing them in front of smooth compactly supported functions and taking the limit $n \rightarrow +\infty$ in the sense of distributions, we check that their limits in $L^2(\mathbb{R}^2)$ are equal to $\nabla(\psi_\infty \bar{V}_1)$, respectively $(1 - |V_1|^2)^{1/2} \nabla \psi_\infty$. Hence the function ψ_∞ lies in H and it is the limit of the Cauchy sequence $(\psi_n)_{n \in \mathbb{N}}$ in this space.

When $\psi \in H^1(\mathbb{R}^2)$, we also have

$$\nabla(\psi \bar{V}_1) = \bar{V}_1 \nabla \psi + \psi \nabla \bar{V}_1 \in L^2(\mathbb{R}^2),$$

since V_1 and ∇V_1 are bounded functions. For the same reason, the function $(1 - |V_1|^2)^{1/2} \nabla \psi$ is in $L^2(\mathbb{R}^2)$. This proves that ψ belongs to H and concludes the proof of Lemma B.1. \square

A drawback of the previous definition for the Hilbert space H lies in the property that its canonical norm $\|\psi\|_H$ does not provide any direct control on the function ψ under consideration, but only on its gradient. In order to recover such a control, we next establish that the Hilbert space H compactly embeds into suitable weighted Lebesgue spaces. As a consequence of this result, we especially gain a local control on the functions in H that turns out to be very useful in our proofs.

Lemma B.2. *Let $s > 1$. The Hilbert space H continuously embeds into the weighted Lebesgue space*

$$L^2_{-s}(\mathbb{R}^2) := \left\{ \psi \in L^2_{loc}(\mathbb{R}^2) \text{ s.t. } \|\psi\|_{L^2_{-s}}^2 := \int_{\mathbb{R}^2} \frac{|\psi(x)|^2}{(1 + |x|^2)^s} dx < \infty \right\}.$$

Moreover this embedding is compact.

Proof. Let $\psi \in H$. In view of Lemma A.1, there exists a universal constant $K > 0$ such that

$$\int_{B_2} |\psi|^2 \leq K \int_{B_2} |\nabla \bar{V}_1|^2 |\psi|^2.$$

Since $\psi \nabla \bar{V}_1 = \nabla(\bar{V}_1 \psi) - \bar{V}_1 \nabla \psi$, we obtain

$$\int_{B_2} |\psi|^2 \leq 2K \int_{B_2} \left(|\nabla(\bar{V}_1 \psi)|^2 + |\bar{V}_1|^2 |\nabla \psi|^2 \right).$$

Invoking once again Lemma A.1, we can find a further universal constant $K > 0$ such that

$$\int_{B_2} |\psi|^2 \leq K \int_{B_2} \left(|\nabla(\bar{V}_1 \psi)|^2 + (1 - |\bar{V}_1|^2) |\nabla \psi|^2 \right) \leq K \|\psi\|_H^2. \quad (78)$$

In particular, there exists a universal constant $K > 0$ such that

$$\|\nabla(\bar{V}_1 \psi)\|_{L^2(\mathbb{R}^2)} + \|\bar{V}_1 \psi\|_{L^2(B_2)} \leq K_0 \|\psi\|_H. \quad (79)$$

Let us now show the existence of $K_s > 0$, depending only on s , such that

$$\int_{\mathbb{R}^2} \frac{|f(x)|^2}{(1 + |x|^2)^s} dx \leq K_s (\|\nabla f\|_{L^2(\mathbb{R}^2)}^2 + \|f\|_{L^2(B_2)}^2) \quad (80)$$

for any function $f \in \dot{H}^1(\mathbb{R}^2)$. Consider, as previously, a smooth, decreasing cut-off function $\chi : \mathbb{R}_+ \rightarrow [0, 1]$ that satisfies $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ for $r \geq 2$. We first show the existence of $K_s > 0$ such that

$$\int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^{2s}} (1 - \chi(|x|)) dx \leq K_s (\|\nabla f\|_{L^2(\mathbb{R}^2)}^2 + \|f\|_{L^2(B_2)}^2). \quad (81)$$

Indeed, we can combine the use of polar coordinates and an integration by parts in order to compute

$$\int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^{2s}} (1 - \chi(|x|)) dx = -\frac{1}{2(s-1)} \int_0^{2\pi} \int_0^{+\infty} \frac{1}{r^{2s-2}} \partial_r (|f|^2 (1 - \chi(r))) dr d\theta.$$

We deduce that

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^{2s}} (1 - \chi(|x|)) dx \\ & \leq \frac{1}{2(s-1)} \int_{\mathbb{R}^2} \left(\frac{|f(x)|^2}{|x|^{2s-1}} |\chi'(|x|)| + \frac{2|f(x)||\nabla f(x)|}{|x|^{2s-1}} (1 - \chi(|x|)) \right) dx. \end{aligned}$$

Estimate (81) then follows from the Cauchy-Schwarz inequality that provides

$$\int_{\mathbb{R}^2} \frac{|f(x)||\nabla f(x)|}{|x|^{2s-1}} (1 - \chi(|x|)) dx \leq \left(\int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^{2s}} (1 - \chi(|x|)) dx \right)^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^2)},$$

when $4s - 2 > 2s$. In turn, estimate (80) follows from checking that

$$\int_{\mathbb{R}^2} \frac{|f(x)|^2}{(1 + |x|^2)^s} dx \leq K \left(\|f\|_{L^2(B_2)}^2 + \int_{\mathbb{R}^2} \frac{|f|^2}{|x|^{2s}} (1 - \chi(|x|)) dx \right),$$

for a further $K > 0$. In view of (79), we finally deduce from (80) for $f = \bar{V}_1 \psi$ and from (78) that the space H continuously embeds into $L^2_{-s}(\mathbb{R}^2)$. Let us now show that this embedding is compact.

Consider a bounded sequence $(\psi_n)_{n \in \mathbb{N}}$ of H . Given any $R > 0$, it follows from the previous embedding that this sequence is also bounded in $H^1(B_R)$. Invoking the Rellich-Kondrachov theorem and performing a diagonal argument, we can construct a function $\psi \in H_{\text{loc}}^1(\mathbb{R}^2)$ such that, up to a subsequence,

$$\|\psi_n - \psi\|_{L^2(B_R)} \rightarrow 0, \quad (82)$$

when $n \rightarrow +\infty$, for any $R > 0$. By weak convergence, the function ψ also belongs to H . In particular, since $(1+s)/2 > 1$ for $s > 1$, we can invoke the previous continuous embedding in order to find $A > 0$ such that

$$\int_{\mathbb{R}^2} \frac{|\psi_n(x)|^2}{(1+|x|^2)^{\frac{1+s}{2}}} dx + \int_{\mathbb{R}^2} \frac{|\psi(x)|^2}{(1+|x|^2)^{\frac{1+s}{2}}} dx \leq A,$$

for any $n \in \mathbb{N}$. Now, take any $\delta > 0$, and let us conclude the proof of the compactness by showing the existence of a number $N_\delta \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^2} \frac{|\psi_n(x) - \psi(x)|^2}{(1+|x|^2)^s} dx \leq \delta, \quad (83)$$

for any $n \geq N_\delta$. Indeed, we have

$$\int_{\mathbb{R}^2 \setminus B_R} \frac{|\psi_n(x) - \psi(x)|^2}{(1+|x|^2)^s} dx \leq \frac{2}{R^{\frac{s-1}{2}}} \int_{\mathbb{R}^2 \setminus B_R} \frac{|\psi_n(x)|^2 + |\psi(x)|^2}{(1+|x|^2)^{\frac{1+s}{2}}} dx \leq \frac{2A}{R^{\frac{s-1}{2}}},$$

for any $R > 0$. Since $(s-1)/2 > 0$, we can choose R_δ such that $2A/R_\delta^{(s-1)/2} \leq \delta/2$. Furthermore, we compute

$$\int_{B_{R_\delta}} \frac{|\psi_n(x) - \psi(x)|^2}{(1+|x|^2)^s} dx \leq \int_{B_{R_\delta}} |\psi_n - \psi|^2.$$

Hence, we deduce from (82) the existence of an integer N_δ such that

$$\int_{B_{R_\delta}} |\psi_n - \psi|^2 \leq \frac{\delta}{2},$$

for any $n \geq N_\delta$. This concludes the proof of (83) and of Lemma B.2. \square

Another crucial property of the Hilbert space H lies in the fact that it is left invariant by translations and phase shifts. More precisely, we can show the following estimates for these two classes of operations.

Lemma B.3. *Let $d \in \mathbb{R}^2$ and $\vartheta \in \mathbb{R}$. Given any function $\psi \in H$, the functions $e^{-i\vartheta}\psi$ and $\psi(\cdot + d)$ also belong to H . Moreover,*

$$\|e^{-i\vartheta}\psi\|_H = \|\psi\|_H,$$

and there exists $C(|d|) > 0$, depending continuously on the norm $|d|$ in \mathbb{R}_+ , such that

$$\|\psi(\cdot + d)\|_H \leq C(|d|)\|\psi\|_H.$$

Proof. Let $\psi \in H$. By definition, we have

$$\|e^{-i\vartheta}\psi\|_H = \|\psi\|_H,$$

so that the function $e^{-i\theta}\psi$ is also in H . Similarly, we will prove that the function $\psi(\cdot + d)$ belongs to H by bounding the quantity

$$\|\psi(\cdot + d)\|_H^2 = \int_{\mathbb{R}^2} \left(|\nabla(\bar{V}_1(\cdot - d)\psi)|^2 + (1 - |V_1(\cdot - d)|^2)|\nabla\psi|^2 \right) =: I(\psi).$$

In this direction, we will split the integral $I(\psi)$ as $I(\psi) = I_1(\psi) + I_2(\psi) + I_3(\psi)$, where

$$I_1(\psi) := \int_{B_{|d|+1}} |\nabla(\bar{V}_1(\cdot - d)\psi)|^2, \quad I_2(\psi) := \int_{B_{|d|+1}^c} |\nabla(\bar{V}_1(\cdot - d)\psi)|^2,$$

and

$$I_3(\psi) = \int_{\mathbb{R}^2} (1 - |V_1(\cdot - d)|^2)|\nabla\psi|^2.$$

Concerning the integral $I_1(\psi)$, we check that

$$\begin{aligned} I_1(\psi) &\leq 2 \int_{B_{|d|+1}} \left(|\nabla V_1(\cdot - d)|^2 |\psi|^2 + |V_1(\cdot - d)|^2 |\nabla\psi|^2 \right) \\ &\leq 2 \|\nabla V_1\|_{L^\infty}^2 \int_{B_{|d|+1}} |\psi|^2 + \frac{1}{1 - \rho_1(|d| + 1)^2} \int_{B_{|d|+1}} (1 - |V_1|^2) |\nabla\psi|^2. \end{aligned}$$

In view of the proof of Lemma B.2, there exists a positive number $C(|d|)$, depending continuously on $|d|$ in \mathbb{R}_+ , such that

$$\int_{B_{|d|+1}} |\psi|^2 \leq C(|d|) \|\psi\|_H^2,$$

so that

$$I_1(\psi) \leq \left(2C(|d|) \|\nabla V_1\|_{L^\infty}^2 + \frac{1}{1 - \rho_1(|d| + 1)^2} \right) \|\psi\|_H^2. \quad (84)$$

In order to estimate the integral $I_2(\psi)$, we use the property that the function V_1 only vanishes at the origin. We obtain

$$\begin{aligned} I_2(\psi) &= \int_{B_{|d|+1}^c} \left| \nabla \left(\frac{\bar{V}_1(\cdot - d)}{\bar{V}_1} \bar{V}_1 \psi \right) \right|^2 \\ &\leq 2 \int_{B_{|d|+1}^c} \left(\left| \nabla \left(\frac{V_1(\cdot - d)}{V_1} \right) \right|^2 |V_1|^2 |\psi|^2 + \frac{|V_1(\cdot - d)|^2}{|V_1|^2} |\nabla(\bar{V}_1 \psi)|^2 \right). \end{aligned}$$

The second term in the last integral of the previous inequality is bounded by

$$\int_{B_{|d|+1}^c} \frac{|V_1(\cdot - d)|^2}{|V_1|^2} |\nabla(\bar{V}_1 \psi)|^2 \leq \frac{1}{\rho_1(|d| + 1)^2} \|\psi\|_H^2. \quad (85)$$

Concerning the first term, we compute

$$|V_1|^2 \left| \nabla \left(\frac{V_1(\cdot - d)}{V_1} \right) \right|^2 = \frac{1}{|V_1|^2} \left| V_1 \nabla V_1(\cdot - d) - V_1(\cdot - d) \nabla V_1 \right|^2,$$

so that by (6), we obtain

$$\begin{aligned} |V_1(x)|^2 \left| \nabla \left(\frac{V_1(x - d)}{V_1(x)} \right) \right|^2 &= \rho_1(x - d)^2 \frac{|d|^2}{|x|^2 |x - d|^2} + \rho_1'(x - d)^2 + \frac{\rho_1(x - d)^2}{\rho_1(x)^2} \rho_1'(x)^2 \\ &\quad - 2 \frac{\rho_1(x - d) x \cdot (x - d)}{\rho_1(x) |x| |x - d|} \rho_1'(x) \rho_1'(x - d). \end{aligned}$$

We check that

$$\frac{1}{|x-d|} \leq \frac{|d|+1}{|x|},$$

when $|x| \geq |d|+1$. In view of Lemma A.1, we deduce that

$$|V_1(x)|^2 \left| \nabla \left(\frac{V_1(x-d)}{V_1(x)} \right) \right|^2 \leq C \left(\frac{|d|^2(|d|+1)^2}{|x|^4} + \frac{(|d|+1)^6}{|x|^6} + \frac{1}{\rho_1(|d|+1)^2|x|^6} \right), \quad (86)$$

for some $C > 0$, not depending on d . Going back to the proof of Lemma B.2, we infer the existence of a further $C(|d|) > 0$, depending continuously on $|d|$ in \mathbb{R}_+ , such that

$$\int_{B_{|d|+1}^c} \left| \nabla \left(\frac{V_1(\cdot-d)}{V_1} \right) \right|^2 |V_1|^2 |\psi|^2 \leq C(|d|) \|\psi\|_H^2. \quad (87)$$

Similarly, we bound the integral $I_3(\psi)$ by

$$I_3(\psi) \leq \left\| \frac{1 - |V_1(\cdot-d)|^2}{1 - |V_1|^2} \right\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} (1 - |V_1|^2) |\nabla \psi|^2.$$

When $|x| \leq |d|+1$, we observe that

$$\left| \frac{1 - |V_1(x-d)|^2}{1 - |V_1(x)|^2} \right| \leq \frac{1}{1 - \rho_1(|d|+1)^2}.$$

On the other hand, we deduce from Lemma A.1 the existence of $C > 0$, not depending on d , such that

$$\left| \frac{1 - |V_1(x-d)|^2}{1 - |V_1(x)|^2} \right| \leq C \frac{|x|^2}{|x-d|^2} \leq 2C \left(1 + \frac{|d|^2}{|x-d|^2} \right) \leq 2C(1 + |d|^2),$$

for $|x| \geq 1 + |d|$. As a consequence, we obtain

$$I_3(\psi) \leq \max \left\{ \frac{1}{1 - \rho_1(|d|+1)^2}, 2C(1 + |d|^2) \right\} \|\psi\|_H^2.$$

The conclusion then follows from (84), (85) and (87). \square

Concerning translation, we can refine the estimate in Lemma B.3 in the special case of the vortex solution V_1 . In the next lemma, we establish some local Lipschitz continuity of the function $V_1(\cdot+d)$ with respect to the translation parameter d . This property is useful in the previous construction of the modulation parameters.

Lemma B.4. *Let $d \in \mathbb{R}^2$. There exists $C(|d|) > 0$, depending continuously on the norm $|d|$ in \mathbb{R}_+ , such that*

$$\|V_1(\cdot+d) - V_1\|_H \leq C(|d|) |d|.$$

Proof. By definition, we have

$$\|V_1(\cdot+d) - V_1\|_H^2 = \int_{\mathbb{R}^2} \Phi_{V_1}(x, d) dx =: I_{V_1}(d),$$

where

$$\Phi_{V_1}(x, d) := \left| \nabla \left(\bar{V}_1(x) (V_1(x+d) - V_1(x)) \right) \right|^2 + (1 - |V_1(x)|^2) \left| \nabla (V_1(x+d) - V_1(x)) \right|^2,$$

for any pair $(x, d) \in \mathbb{R}^2 \times \mathbb{R}^2$. The proof is then based on the property that the integral I_{V_1} is of class \mathcal{C}^2 on \mathbb{R}^2 , with $I_{V_1}(0) = 0$ and $\nabla I_{V_1}(0) = 0$. In this case, we can apply the Taylor formula in order to obtain

$$\|V_1(\cdot + d) - V_1\|_H^2 \leq M(|d|)|d|^2,$$

with $M(|d|) := \max_{|x| \geq |d|} \|d^2 I_{V_1}(x)\|$ being continuous with respect to $|d|$. It is then enough to take the square root of this inequality in order to complete the proof of Lemma B.4.

Hence, we are reduced to check the second order continuous differentiability of the integral I_{V_1} by applying the dominated convergence theorem. In view of Lemma A.1, the function Φ_{V_1} is smooth on $\mathbb{R}^2 \times \mathbb{R}^2$, with

$$\begin{aligned} \partial_{d_i} \Phi_{V_1}(x, d) := & 2 \langle \nabla(\bar{V}_1(V_1(x+d) - V_1(x))), \nabla(\bar{V}_1(x) \partial_{x_i} V_1(x+d)) \rangle_{\mathbb{C}} \\ & + 2(1 - |V_1(x)|^2) \langle \nabla(V_1(x+d) - V_1(x)), \partial_{x_i} \nabla V_1(x+d) \rangle_{\mathbb{C}}, \end{aligned} \quad (88)$$

and

$$\begin{aligned} \partial_{d_i} \partial_{d_j} \Phi_{V_1}(x, d) := & 2 \langle \nabla(\bar{V}_1 \partial_{x_j} V_1(x+d)), \nabla(\bar{V}_1(x) \partial_{x_i} V_1(x+d)) \rangle_{\mathbb{C}} \\ & + 2 \langle \nabla(\bar{V}_1(V_1(x+d) - V_1(x))), \nabla(\bar{V}_1(x) \partial_{x_i} \partial_{x_j} V_1(x+d)) \rangle_{\mathbb{C}} \\ & + 2(1 - |V_1(x)|^2) \langle \partial_{x_j} \nabla V_1(x+d), \partial_{x_i} \nabla V_1(x+d) \rangle_{\mathbb{C}} \\ & + 2(1 - |V_1(x)|^2) \langle \nabla(V_1(x+d) - V_1(x)), \partial_{x_i} \partial_{x_j} \nabla V_1(x+d) \rangle_{\mathbb{C}}, \end{aligned} \quad (89)$$

for $1 \leq i, j \leq 2$ and $(x, d) \in \mathbb{R}^2 \times \mathbb{R}^2$. Recall that

$$\frac{1}{|x+d|} \leq \frac{|d|+1}{|x|},$$

when $|x| \geq |d|+1$. In view of Lemma A.1, this inequality is enough to find $C(|d|) > 0$, depending continuously on $|d|$ in \mathbb{R}_+ , such that

$$\begin{aligned} |V_1(x+d)| + (1+|x|)|\nabla V_1(x+d)| \\ + (1+|x|^2)|d^2 V_1(x+d)| + (1+|x|^3)|d^3 V_1(x+d)| \leq C(|d|). \end{aligned}$$

We then infer from (88) and (89) the bounds

$$|\partial_{d_i} \Phi_{V_1}(x, d)| + (1+|x|)|\partial_{d_i} \partial_{d_j} \Phi_{V_1}(x, d)| \leq \frac{C(|d|)}{1+|x|^3}.$$

Arguing as for the proof of (86), we also obtain the refined bound

$$|\Phi_{V_1}(x, d)| \leq \frac{C(|d|)}{1+|x|^4}.$$

Applying the dominated convergence theorem with d lying in bounded subsets of \mathbb{R}^2 , we conclude that the integral I_{V_1} is of class \mathcal{C}^2 on \mathbb{R}^2 . The facts that $I_{V_1}(0) = 0$ and $\nabla I_{V_1}(0) = 0$ follows from the identities $\Phi_{V_1}(x, 0) = \partial_{d_1} \Phi_{V_1}(x, 0) = \partial_{d_2} \Phi_{V_1}(x, 0) = 0$ for any $x \in \mathbb{R}^2$. This completes the proof of Lemma B.4. \square

B.2 Properties of the metric space E

We now turn to the energy set

$$E = \{\psi \in H \text{ s.t. } 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\},$$

that we have endowed with the distance

$$d_E(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|_H + \| |\psi_2|^2 - |\psi_1|^2 \|_{L^2}.$$

Recall that the classical Hamiltonian framework for the Gross-Pitaevskii equation is given by the set of functions with finite Ginzburg-Landau energy (see e.g. [1, 11] and the references therein). The introduction of the energy set E is reminiscent from this framework. Roughly speaking, this set is composed of functions with infinite Ginzburg-Landau energy due to a topological degree equal to 1 at infinity, but of finite Ginzburg-Landau energy when this degree is suitably brought back to 0. This interpretation can be made more effective through the following observation.

Lemma B.5. *Let $\psi \in E$. The function $\psi\bar{V}_1$ has finite Ginzburg-Landau energy.*

Proof. By definition of the energy set E , the function $\nabla(\psi\bar{V}_1)$ is square integrable. For the potential term, we write

$$1 - |\psi\bar{V}_1|^2 = |V_1|^2(1 - |\psi|^2) + 1 - |V_1|^2.$$

The right-hand side of this formula is also square integrable since the function $|V_1| = \rho_1$ is bounded by 1 and the function $1 - |V_1|^2$ is square integrable. Hence, the Ginzburg-Landau energy of $\psi\bar{V}_1$ is finite. \square

As a consequence of Lemma B.5, it is natural to rely on earlier results about the functions with finite Ginzburg-Landau energy (in particular in [11]) in order to describe the main properties of the energy set E . Our first result in this direction is

Lemma B.6. *The energy set E is a complete metric space for the distance d_E . Moreover, it satisfies*

$$E \subset L^4(\mathbb{R}^2) + L^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2), \quad (90)$$

as well as

$$E + H^1(\mathbb{R}^2) = E. \quad (91)$$

Proof. Observe first that a Cauchy sequence $(\psi_n)_{n \in \mathbb{N}}$ of E is a Cauchy sequence of the Hilbert space H . As a consequence of Lemmas B.1 and B.2, it is convergent in H and in $L^2_{\text{loc}}(\mathbb{R}^2)$ towards a limit function $\psi_\infty \in H$. Since $(1 - |\psi_n|^2)_{n \in \mathbb{N}}$ is a Cauchy sequence of $L^2(\mathbb{R}^2)$, it is also convergent in $L^2(\mathbb{R}^2)$. Moreover, its limit function is necessarily equal to $1 - |\psi_\infty|^2$ by almost everywhere convergence. Hence, the function ψ_∞ is in E and it is the limit of the sequence $(\psi_n)_{n \in \mathbb{N}}$ in this metric space that is therefore complete.

In order to prove (90), we write a given function $\Psi \in E$ as $\Psi = \Psi \mathbb{1}_{|\Psi| \geq 2} + \Psi \mathbb{1}_{|\Psi| < 2}$. The second function in this decomposition is bounded. The first one is in $L^4(\mathbb{R}^2)$. Indeed, we know that $|\Psi| \leq 2(|\Psi|^2 - 1)^{1/2}$ whenever $|\Psi| \geq 2$, and that $(|\Psi|^2 - 1)^{1/2} \in L^4(\mathbb{R}^2)$ by definition of E . This proves that $E \subset L^4(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$. The other inclusion in (90) then follows from the general property that $L^4(\mathbb{R}^2) + L^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$.

Concerning (91), Lemma B.1 guarantees that $H^1(\mathbb{R}^2) \subset H$. Therefore, we are reduced to establish that $1 - |\psi + u|^2$ is in $L^2(\mathbb{R}^2)$ when $\psi \in E$ and $u \in H^1(\mathbb{R}^2)$. For that purpose, we write

$$|\psi + u|^2 - 1 = |\psi|^2 - 1 + |u|^2 + 2\langle u, \psi \rangle_{\mathbb{C}}.$$

In this formula, the function $1 - |\psi|^2$ is in $L^2(\mathbb{R}^2)$ by definition of the metric space E , so as the function $|u|^2$ due to the Sobolev embedding theorem. Since $u \in L^2(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$ again by the Sobolev embedding theorem, and $\psi \in L^\infty(\mathbb{R}^2) + L^4(\mathbb{R}^2)$ by (90), the function $\langle u, \psi \rangle_{\mathbb{C}}$ also belongs to $L^2(\mathbb{R}^2)$. Hence, the function $|\psi + u|^2 - 1$ is in $L^2(\mathbb{R}^2)$, which completes the proof of Lemma B.6. \square

In the spirit of Lemma B.4, we also need at some point the following Lipschitz estimate for translations on V_1 in E .

Lemma B.7. *Let $d \in \mathbb{R}^2$. There exists $C(|d|) > 0$, depending continuously on the norm $|d|$ in \mathbb{R}_+ , such that*

$$\| |V_1(\cdot + d)|^2 - |V_1|^2 \|_{L^2} \leq C(|d|) |d|.$$

Proof. The proof is very similar to the one of Lemma B.4, and therefore we omit it. \square

In order to tackle the Cauchy problem in the energy set E , we now relate it with the functional framework introduced in [5] to solve this problem for functions with non-zero degree at infinity. Recall that this framework was based on the set

$$\begin{aligned} \mathcal{U}(\mathbb{R}^2) := \{ U \in E \text{ s.t. } U \in L^\infty(\mathbb{R}^2), \nabla|U| \in L^2(\mathbb{R}^2), \\ \text{and } \nabla^k U \in L^2(\mathbb{R}^2) \text{ for all } k \geq 2 \}. \end{aligned} \quad (92)$$

Given a fixed function $U \in \mathcal{U}(\mathbb{R}^2)$, it was proved in [5] that the Cauchy problem for (3) is globally well-posed in $U + H^1(\mathbb{R}^2)$. This result can be applied in the context of the energy set E since any function in this set can be decomposed as a function in $\mathcal{U}(\mathbb{R}^2)$ plus an H^1 -function.

Lemma B.8. *Let $\psi \in E$. There exist two functions $U \in \mathcal{U}(\mathbb{R}^2)$ and $w \in H^1(\mathbb{R}^2)$ such that*

$$\psi = U + w.$$

Proof. We fix a smooth, non-negative, compactly supported mollifier ρ . Given an arbitrary function $\psi \in E$, we decompose it as $\psi = U + w$, where

$$U := V_1 \left(1 + \rho * ((\psi - V_1)\bar{V}_1) \right).$$

We first show that w is in $H^1(\mathbb{R}^2)$. Setting $\varepsilon := \psi - V_1$, we compute

$$w = \varepsilon - (\rho * (\varepsilon\bar{V}_1)) V_1 = (\varepsilon\bar{V}_1 - \rho * (\varepsilon\bar{V}_1)) V_1 + \varepsilon(1 - |V_1|^2).$$

Since $\int_{\mathbb{R}^2} \rho = 1$, we can find $C > 0$, depending only on ρ , such that

$$\| f - \rho * f \|_{L^2} \leq C \| \nabla f \|_{L^2},$$

for any function $f \in \dot{H}^1(\mathbb{R}^2)$. We use this inequality for $f = \varepsilon\bar{V}_1$, which belongs to $\dot{H}^1(\mathbb{R}^2)$ by definition of the vector space H . Combined with the facts that $V_1 \in L^\infty(\mathbb{R}^2)$, $1 - |V_1|^2 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\varepsilon \in L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ by Lemma B.6,

we infer that w is in $L^2(\mathbb{R}^2)$. In order to prove that $\nabla w \in L^2(\mathbb{R}^2)$, we use similar arguments, and the fact that $|\nabla \varepsilon|(1 - |V_1|^2) \leq |\nabla \varepsilon|(1 - |V_1|^2)^{1/2}$, where the latter term is in $L^2(\mathbb{R}^2)$ by definition of H .

We next show that $U \in \mathcal{U}(\mathbb{R}^2)$. Since $|V_1| \leq 1$, we first have

$$|U| \leq 1 + \rho * |\varepsilon|.$$

Since $\varepsilon \in L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ by Lemma B.6, and $\rho \in L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we deduce that $U \in L^\infty(\mathbb{R}^2)$. Similarly, we compute

$$|\nabla|U|| \leq |\nabla|V_1|| (1 + \rho * |\varepsilon|) + |\rho * \nabla(\varepsilon \bar{V}_1)|.$$

By Lemma A.1, $\nabla|V_1|$ belongs to $L^2(\mathbb{R}^2)$, while $\nabla(\varepsilon \bar{V}_1)$ is in $L^2(\mathbb{R}^2)$ by definition of H . Hence, $\nabla|U|$ is also in $L^2(\mathbb{R}^2)$. The fact that $1 - |U|^2 \in L^2(\mathbb{R}^2)$ is then a consequence of the fact that $\psi \in E$, $w \in H^1(\mathbb{R}^2)$ and Lemma B.6. It remains to show that $\nabla^k U \in L^2(\mathbb{R}^2)$ for any $k \geq 2$. In this direction, standard tame estimates yield

$$\|\nabla^k U\|_{L^2} \leq C \left(\|\nabla^k V_1\|_{L^2} \|1 + \rho * (\varepsilon \bar{V}_1)\|_{L^\infty} + \|V_1\|_{L^\infty} \|\nabla^{k-1} \rho * \nabla(\varepsilon \bar{V}_1)\|_{L^2} \right),$$

and this quantity is finite by Lemma A.1 and by definition of H . Finally, the function U is also in E by (91). \square

As a direct consequence of Lemma B.8, we deduce that smooth functions are dense in the energy set E .

Corollary B.1. *Let $\psi \in E$. There exist smooth functions $\psi_n \in E$ such that*

$$d_E(\psi_n, \psi) \rightarrow 0,$$

as $n \rightarrow +\infty$.

Proof. In view of Lemma B.8, we can find two functions $U \in \mathcal{U}(\mathbb{R}^2) \cap E$ and $w \in H^1(\mathbb{R}^2)$ such that $\psi = U + w$. Since smooth, compactly supported functions are dense in $H^1(\mathbb{R}^2)$, there exist functions $w_n \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that $w_n \rightarrow w$ in $H^1(\mathbb{R}^2)$ as $n \rightarrow +\infty$. Set

$$\psi_n = U + w_n,$$

for any $n \in \mathbb{N}$. Since U is smooth by definition of $\mathcal{U}(\mathbb{R}^2)$, the functions ψ_n are smooth. They also belong to E by (91). Moreover, we have $\psi_n - \psi = w_n - w$. Arguing as in the proof of Lemma B.1, we infer that

$$\|\psi_n - \psi\|_H \leq C \|w_n - w\|_{H^1} \rightarrow 0,$$

as $n \rightarrow +\infty$. Similarly, we have

$$1 - |\psi_n|^2 - (1 - |\psi|^2) = 2\langle U, w - w_n \rangle_{\mathbb{C}} + |w|^2 - |w_n|^2.$$

Invoking the Sobolev embedding theorem is enough to prove the convergence of this quantity towards 0 in $L^2(\mathbb{R}^2)$. This concludes the proof of Corollary B.1. \square

C On the Cauchy problem in the space E

The goal of this section is to make a link with the analysis of the Cauchy problem for the Gross-Pitaevskii equation in [5], and to prove Proposition 3. We recall that in [5], the Cauchy problem for (3) is proved to be globally well-posed in $U + H^1(\mathbb{R}^2, \mathbb{C})$ for any function $U \in \mathcal{U}$, where the space \mathcal{U} is given in (92). Moreover, (some slightly different version of) the renormalized energy \mathcal{E} is shown to be preserved along the flow. In order to prove Proposition 4, we need the following close extension of this result.

Proposition C.1. *Let $k \geq 1$ and $U \in \mathcal{U}$. The Cauchy problem for (3) is globally well-posed in $U + H^k(\mathbb{R}^2, \mathbb{C})$, and the renormalized energy \mathcal{E} is conserved by the flow.*

Proof. Global well-posedness for $k = 1$ and $k = 2$ was proved in [5]. Concerning the renormalized energy, it is defined in [5] as

$$\mathcal{E}_U(w) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla w|^2 - \int_{\mathbb{R}^2} \langle \Delta U, w \rangle_{\mathbb{C}} + \frac{1}{4} \int_{\mathbb{R}^2} (1 - |U + w|^2)^2,$$

for $\Psi = U + w \in U + H^1(\mathbb{R}^2, \mathbb{C})$. Moreover, it was shown that

$$\mathcal{E}_U(w) = \lim_{r \rightarrow +\infty} \int_{B_r} \left(e_{\text{GL}}(\Psi) - \frac{|\nabla U|^2}{2} \right).$$

In particular, we have

$$\mathcal{E}_U(w_t) = \mathcal{E}(\Psi_t) - \mathcal{E}(U) + \frac{1}{4} \int_{\mathbb{R}^2} (1 - |U|^2)^2,$$

and therefore, $\mathcal{E}(\Psi_t)$ is constant, since U is fixed and \mathcal{E}_U is preserved by the flow.

Concerning the cases $k \geq 3$,⁶ local well-posedness follows as for $k = 2$, since the nonlinearity is Lipschitz due to the Sobolev embedding theorem of $H^k(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$ when $k \geq 2$. Global existence for $k \geq 3$ is then a consequence of standard energy estimates and global existence for $k = 2$, again using the control of the uniform norm provided by the H^2 -norm. \square

Proposition 3 is finally a direct consequence of Lemma B.8 and Proposition C.1.

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⁶We have made use of the case $k = 3$ in the course of the proof of Proposition 4.

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