A non-existence result for supersonic travelling waves in the Gross-Pitaevskii equation

Philippe Gravejat [1]

Abstract

We prove the non-existence of non-constant travelling waves of finite energy and of speed $c > \sqrt{2}$ in the Gross-Pitaevskii equation in dimension $N \geq 2$.

Introduction

In this paper, we will focus on the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2). \tag{1}$$

One of the motivations for this equation is the analysis of Bose-Einstein condensation, which describes the behaviour of interacting bosons near absolute zero. When condensation occurs, equation (1) might be used as a model for the Bose condensate (see [4] for more details). In particular, this model is relevant to describe Bose-condensed gases. The model is also sometimes proposed to describe the superfluid state of *Helium II*, though in this case the interactions between particles are important and cannot be neglected at temperature different from zero.

In order to describe this condensation, E.P. Gross [8] and L.P. Pitaevskii [12] considered a set of N bosons of mass m that fill a volume V: they then assumed almost all bosons are Bose-condensed in the fundamental state of energy. Therefore, they can be described by a macroscopic wave function Ψ. They then deduced the Gross-Pitaevskii equation satisfied by the function Ψ from a Hartree-Fock approach,

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\Delta\Psi - \Psi\int_V |\Psi(x',t)|^2 U(x-x')dx' = 0.$$

Here, $U(x-x')$ denotes the interaction between the bosons at positions x and x': this interaction being of very short range, it is often approached by $U_0\ \delta(x-x')$. Thus, denoting E_b, the average energy level per unit mass of a boson, and,

$$u(t,x) = e^{\frac{-imE_bt}{\hbar}}\Psi(t,x),$$

they computed the equation

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u + mE_b u - U_0 u|u|^2 = 0.$$

They finally rescaled the equation by taking the mean density $\rho_0 = \sqrt{\frac{mE_b}{U_0}}$ as unity, $\frac{\hbar}{\sqrt{2m^2E_b}}$ as unit length, and, $\frac{\hbar}{mE_b}$ as unit time, in order to obtain the dimensionless equation

$$i\partial_t u + \Delta u + u(1 - |u|^2) = 0. \tag{1}$$

[1] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte Courrier 187, 75252 Paris Cedex 05, France. E-mail: gravejat@clipper.ens.fr

At this point, we can write the hydrodynamic form of this equation by using the Madelung transform [11],

$$u = \sqrt{\rho} e^{i\theta},$$

which is only meaningful where ρ does not vanish. Denoting $v = 2\nabla\theta$, we deduce the equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v.\nabla v) + \nabla \rho^2 = \rho \nabla(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}). \end{cases}$$

Those equations are similar to the Euler equations for a irrotational ideal fluid with pressure $p(\rho) = \rho^2$: the term of the right member is then considered as a quantic pressure term. Here, we can remark that the sound speed is $c_s = \sqrt{2}$.

In this article, we will consider equation (1) in the space $\mathbb{R}^N$ for every integer $N \geq 2$: we can notice that this equation is associated to the energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2 = \int_{\mathbb{R}^N} e(u).$$

We will study the travelling waves of finite energy and of speed $c \geq 0$ for this equation i.e. the solutions u which are of the form

$$u(t, x) = v(x_1 - ct, \ldots, x_N).$$

The simplified equation for v, which we will consider now, is

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{2}$$

C.A. Jones, S.J. Putterman and P.H. Roberts [10, 9] first considered formally and numerically those particular solutions because they suppose they play an important role in the long time dynamics of general solutions: they conjectured that non-constant travelling waves only exist when their speed c is in the interval $]0, \sqrt{2}[$ ie they all are subsonic. They then noticed the apparition of vortices for those solutions when c tends to 0 in dimension two (two parallel oppositely directed vortices) and in dimension three (a vortex ring). They also gave for each value of c, the asymptotic development at infinity in dimension two,

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2}$$

and in dimension three,

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}},$$

where the constant α is the stretched dipole coefficient.

F. Béthuel and J.C. Saut [3, 2] first studied mathematically those travelling waves: they proved their existence in dimension two when c is small and the apparition of vortices in this case. They also gave a mathematical proof for their limit at infinity.
In dimension $N \geq 3$, F. Béthuel, G. Orlandi and D. Smets [1] showed their existence when c is small and the apparition of a vortex ring.
In every dimension, A. Farina [5] proved a universal bound for their modulus.
Finally, we proved their uniform convergence to a constant of modulus one in dimension $N \geq 3$ [6], and also studied their decay at infinity in dimension $N \geq 2$ [7].

In this paper, we will complete all those results by the following theorem.

Theorem 1. *In dimension $N \geq 2$, a solution of equation (2) of finite energy and speed $c > \sqrt{2}$ is constant.*

This paper will be organised around the proof of Theorem 1. In the first step, we will write the equation satisfied by $\eta = 1 - |v|^2$. Then, we will derive a new integral identity when $c > \sqrt{2}$: this is the crucial step of the proof of Theorem 1. Finally, we will write the Pohozaev identities in order to prove that the energy $E(v)$ vanishes and that the travelling wave v is constant.

1 Equation satisfied by η

In this part, we will write the equation satisfied by the variable $\eta = 1 - |v|^2$ for every $c \geq 0$: in particular, the results in this section (i.e. Propositions 1, 2 and 3) are valid for every $c \geq 0$. We first recall two useful propositions yet mentioned in [6, 7] and based on arguments taken from F. Béthuel and J.C. Saut [3, 2].

Proposition 1. *For every $c \geq 0$, if v is a solution of equation (2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy, then v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty]$.*

Thus, a travelling wave is a regular function and a classical solution of equation (2), which will simplify the following discussion.

Proposition 2. *The modulus ρ of v satisfies*

$$\rho(x) \underset{|x| \to +\infty}{\to} 1.$$

Proof. Indeed, the function η^2 is uniformly continuous because v is bounded and lipschitzian by Proposition 1. As $\int_{\mathbb{R}^N} \eta^2$ is finite, η converges uniformly to 0 at infinity, which completes the proof of this proposition. $\square$

Thus, the function ρ does not vanish at infinity, and we can define a regular function θ on a neighborhood of infinity such that v can be written

$$v = \rho e^{i\theta}.$$

Denoting ψ, a regular function from $\mathbb{R}^N$ to $[0, 1]$ such that $\psi = 0$ on a neighborhood of $Z = \{x \in \mathbb{R}^N, \rho(x) = 0\}$, and $\psi = 1$ on a neighbourhood of infinity, and denoting $v = v_1 + iv_2$, we can write the equation satisfied by the function η:

Proposition 3. *For every $c \geq 0$, the function η satisfies the equation*

$$\Delta^2 \eta - 2\Delta\eta + c^2 \partial^2_{1,1} \eta = -\Delta F + 2c\partial_1 \mathrm{div}(\mathrm{G}), \tag{3}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 + 2c(v_1 \partial_1 v_2 - v_2 \partial_1 v_1) - 2c\partial_1(\psi\theta)$$

and

$$G = v_1 \nabla v_2 - v_2 \nabla v_1 - \nabla(\psi\theta).$$

Proof. By equation (2), we have

$$\Delta v_1 - c\partial_1 v_2 + v_1(1-|v|^2) = 0 \tag{4}$$

$$\Delta v_2 + c\partial_1 v_1 + v_2(1-|v|^2) = 0. \tag{5}$$

We then compute

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -2\Delta|\nabla v|^2 - 2\Delta(v.\Delta v) - 2\Delta\eta + c^2\partial_{1,1}^2\eta,$$

and by equations (4)-(5), we have on one hand

$$v.\Delta v = v_1\Delta v_1 + v_2\Delta v_2 = c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - |v|^2\eta,$$

and, on the other hand,

$$c\partial_1\eta = -2c(v_1\partial_1 v_1 + v_2\partial_1 v_2) = 2(\Delta v_2 v_1 - \Delta v_1 v_2) = 2\mathrm{div}(\nabla v_2 v_1 - \nabla v_1 v_2).$$

Therefore, we finally get

$$\begin{aligned}\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta &= -2\Delta|\nabla v|^2 - 2\Delta\eta^2 - 2c\Delta(v_1\partial_1 v_2 - v_2\partial_1 v_1)\\ &+ 2c\partial_1\mathrm{div}(v_1\nabla v_2 - v_2\nabla v_1)\\ &= -\Delta(2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta))\\ &+ 2c\partial_1\mathrm{div}(v_1\nabla v_2 - v_2\nabla v_1 - \nabla(\psi\theta))\\ &= -\Delta F + 2c\partial_1\mathrm{div}(G),\end{aligned}$$

which is the desired equality. □

2 A new integral relation

We have

Proposition 4. *If $c > \sqrt{2}$, the travelling wave v satisfies the integral equation*

$$\int_{\mathbb{R}^N}(|\nabla v|^2 + \eta^2) = c(\frac{2}{c^2} - 1)\int_{\mathbb{R}^N}(v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\psi\theta)). \tag{6}$$

Remark 1. This is the only point where we use the assumption $c > \sqrt{2}$.

For the proof, we use

Lemma 1. *F and G belong to the space $W^{2,1}(\mathbb{R}^N)$.*

Proof. Indeed, G is regular and satisfies at infinity

$$G = (\rho^2 - 1)\nabla\theta.$$

By Proposition 1, the functions η and ∇v belong to $H^2(\mathbb{R}^N)\cap W^{2,\infty}(\mathbb{R}^N)$: since

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2,$$

and since ρ uniformly converges to 1 at infinity by Proposition 2, the function $\nabla\theta$ belongs to $H^2\cap W^{2,\infty}$ on a neighbourhood of infinity: thus, the function G belongs to the space $W^{2,1}(\mathbb{R}^N)$. Since

$$F = 2(|\nabla v|^2 + \eta^2) + 2cG_1,$$

the function F also belongs to this space, which completes the proof of Lemma 1. □

Proof of Proposition 4. By Proposition 1, the function η belongs to $H^4(\mathbb{R}^N)$, and we can write by taking the Fourier transformation of equation (3)

$$\forall \xi \in \mathbb{R}^N, (|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)\widehat{\eta}(\xi) = |\xi|^2\widehat{F}(\xi) - 2c\sum_{j=1}^{N} \xi_1\xi_j\widehat{G_j}(\xi) := H(\xi). \tag{7}$$

Consider the set

$$\Gamma = \{\xi \in \mathbb{R}^N, |\xi|^4 + 2|\xi|^2 - c^2\xi_1^2 = 0\}.$$

This set is reduced to $\{0\}$ when $c \leq \sqrt{2}$, but, when $c > \sqrt{2}$, it is a regular hypersurface of codimension 1 except at $\{0\}$: in dimension 2, it has the geometry of a bretzel, and in higher dimensions, it has the geometry of two spheres linked at some point. Indeed, Γ is a surface of revolution around axis x_1: in spherical coordinates $\xi = (r\cos(\alpha), r\sin(\alpha)\cos(\beta), \ldots)$, it is described by the equation

$$r^2 = c^2\cos^2(\alpha) - 2.$$

In particular, we notice that there are two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of points of $\Gamma \setminus \{0\}$ which tend to 0 when n tends to $+\infty$ and which satisfy

$$\frac{x_n}{|x_n|} \underset{n\to+\infty}{\to} (\frac{2}{c^2}, \sqrt{1-\frac{2}{c^2}}, 0, \ldots), \text{ and, } \frac{y_n}{|y_n|} \underset{n\to+\infty}{\to} (\frac{2}{c^2}, -\sqrt{1-\frac{2}{c^2}}, 0, \ldots). \tag{8}$$

Coming back to the study of equation (7), we claim that

Lemma 2. *The function H defined by equation* (7) *is continuous on $\mathbb{R}^N$ and satisfies*

$$H = 0 \text{ on } \Gamma.$$

Proof of Lemma 2. The first assertion follows from Lemma 1: indeed, since the functions F and G belong to the space $W^{2,1}(\mathbb{R}^N)$, the functions $\xi \mapsto |\xi|^2\widehat{F}(\xi)$ and $\xi \mapsto \xi_1\xi_j\widehat{G_j}(\xi)$ are continuous on $\mathbb{R}^N$, and therefore, the function H is continuous on $\mathbb{R}^N$ too.
In order to prove the second assertion, we argue by contradiction and assume there is some point $\xi_0 \in \Gamma \setminus \{0, (\sqrt{c^2-2}, 0, \ldots, 0)\}$ such that

$$H(\xi_0) \neq 0.$$

Since the function H is continuous on $\mathbb{R}^N$, there is some neighbourhood V of the point ξ_0 and some strictly positive number A such that

$$\forall \xi \in V, |H(\xi)| \geq A.$$

Hence, we have by equation (7)

$$\forall \xi \in V \setminus \Gamma, |\widehat{\eta}(\xi)|^2 \geq \frac{A^2}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2}.$$

Integrating this relation and using spherical coordinates, we get

$$\begin{aligned}\int_{V\setminus\Gamma} |\widehat{\eta}(\xi)|^2 d\xi &\geq A^2 \int_{V\setminus\Gamma} \frac{d\xi}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2} \\ &\geq A_N \int_{V\setminus\Gamma\cap\mathbb{R}\times\mathbb{R}_+} \frac{s^{N-2}dsd\xi_1}{((s^2+\xi_1^2)^2 + 2s^2 + (2-c^2)\xi_1^2)^2} \\ &\geq A_N \int_{V\setminus\Gamma\cap\mathbb{R}_+\times[0,\pi]} \frac{r^{N-1}\sin^{N-2}(\alpha)drd\alpha}{r^4(r^2+2-c^2\cos^2(\alpha))^2}.\end{aligned}$$

Thus, denoting

$$\xi_0 = (r_0 \cos(\alpha_0), r_0 \sin(\alpha_0)\cos(\beta_0), \ldots),$$

there is some real number $\epsilon > 0$ such that

$$\int_{V\setminus\Gamma} |\widehat{\eta}(\xi)|^2 d\xi \geq A_N \int_{r_0-\epsilon}^{r_0+\epsilon} \int_{\alpha_0-\epsilon}^{\alpha_0+\epsilon} \frac{r^{N-1}\sin^{N-2}(\alpha) dr d\alpha}{r^4(r^2+2-c^2\cos^2(\alpha))^2} := A_N I(\alpha_0, r_0, \epsilon).$$

Since $\xi_0 \in \Gamma \setminus \{0, (\sqrt{c^2-2}, 0, \ldots, 0)\}$, r_0 is different from 0 and α_0 is different from 0 and $\frac{\pi}{2}$, and so, we can compute for ϵ sufficiently small

$$I(\alpha_0, r_0, \epsilon) \geq A(\alpha_0, r_0, \epsilon) \int_{r_0-\epsilon}^{r_0+\epsilon} \int_{\alpha_0-\epsilon}^{\alpha_0+\epsilon} \frac{dr d\alpha}{(r^2+2-c^2\cos^2(\alpha))^2}.$$

By doing the change of variable $r = \sqrt{c^2 cos^2(\beta) - 2}$, we know that there is some real number $\delta > 0$ such that

$$I(\alpha_0, r_0, \epsilon) \geq A(\alpha_0, r_0, \epsilon) \int_{\alpha_0-\delta}^{\alpha_0+\delta} \int_{\alpha_0-\delta}^{\alpha_0+\delta} \frac{d\beta d\alpha}{(c^2\cos^2(\beta) - c^2\cos^2(\alpha))^2},$$

and finally, by denoting $a = \alpha - \alpha_0$ and $b = \beta - \alpha_0$, we get

$$\begin{aligned} I(\alpha_0, r_0, \epsilon) &\geq A(\alpha_0, r_0, \epsilon, c) \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{da db}{(\cos^2(b+\alpha_0) - \cos^2(a+\alpha_0))^2} \\ &\geq A(\alpha_0, r_0, \epsilon, c) \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{da db}{(\cos(2b+2\alpha_0) - \cos(2a+2\alpha_0))^2} \\ &\geq A(\alpha_0, r_0, \epsilon, c) \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{da db}{(\sin(b-a))^2}. \end{aligned}$$

Since the function $(a, b) \mapsto \frac{1}{(\sin(b-a))^2}$ is not integrable at the origin in $\mathbb{R}^2$, the integral $I(\alpha_0, r_0, \epsilon)$ is not finite and we can conclude that

$$\int_{V\setminus\Gamma} |\widehat{\eta}(\xi)|^2 d\xi = +\infty.$$

Since the energy of the function v is finite, so is the integral $\int_{\mathbb{R}^N} \eta^2$, and by Plancherel theorem, we deduce

$$\int_{\mathbb{R}^N} |\widehat{\eta}(\xi)|^2 d\xi < +\infty,$$

which leads to a contradiction and proves that H is identically equal to 0 on the set $\Gamma \setminus \{0, (\sqrt{c^2-2}, 0, \ldots, 0)\}$. The second assertion of Lemma 2 then follows from the continuity of the function H. □

End of the proof of Proposition 4. By Lemma 2, we now know that

$$\forall n \in \mathbb{N}, H(x_n) = 0,$$

which gives by dividing by $|x_n|^2$,

$$\forall n \in \mathbb{N}, \widehat{F}(x_n) = 2c \sum_{j=1}^{N} \frac{(x_n)_1}{|x_n|} \frac{(x_n)_j}{|x_n|} \widehat{G_j}(x_n).$$

By continuity of the functions $\widehat{F}$ and $\widehat{G_j}$, we can take the limit as $x_n \to 0$ of this expression and obtain by assertion 8

$$\widehat{F}(0) = \frac{4}{c}\widehat{G_1}(0) + 2c\sqrt{1 - \frac{2}{c^2}}\widehat{G_2}(0).$$

Likewise, we know that

$$\forall n \in \mathbb{N}, H(y_n) = 0,$$

which gives by the same method,

$$\widehat{F}(0) = \frac{4}{c}\widehat{G_1}(0) - 2c\sqrt{1 - \frac{2}{c^2}}\widehat{G_2}(0).$$

Finally, we have

$$\widehat{F}(0) = \frac{4}{c}\widehat{G_1}(0),$$

so that,

$$\int_{\mathbb{R}^N} F(x)dx = \frac{4}{c}\int_{\mathbb{R}^N} G_1(x)dx.$$

The conclusion follows from the expression of the functions F and G. □

3 Pohozaev identities

We now prove for sake of completeness two well-known identities based on the use of Pohozaev multipliers (See [10, 9, 3, 1] for more details). Those estimates do not use the fact that $c > \sqrt{2}$.

Proposition 5. *Let $c \geq 0$. A finite energy solution v to equation (2) satisfies the two identities*

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2, \tag{9}$$

$$\forall 2 \leq j \leq N, E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + \frac{c}{2}\int_{\mathbb{R}^N} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)). \tag{10}$$

Proof. We first fix some real number $R > 0$ and we multiply equation (2) by Pohozaev multiplier $x_1\partial_1 v$ on the ball $B(0, R)$

$$\int_{B(0,R)} (\Delta v.x_1\partial_1 v + x_1\partial_1 v.v(1 - |v|^2)) = 0. \tag{11}$$

Integrating by parts, we compute

$$\begin{aligned}\int_{B(0,R)} \Delta v.x_1\partial_1 v = &\int_{B(0,R)} \frac{|\nabla v|^2}{2} - \int_{B(0,R)} |\partial_1 v|^2 + \int_{S(0,R)} x_1\partial_1 v.\partial_\nu v \\ &- \int_{S(0,R)} \nu_1 x_1 \frac{|\nabla v|^2}{2},\end{aligned}$$

and

$$\int_{B(0,R)} x_1\partial_1 v.v(1 - |v|^2) = \int_{B(0,R)} \frac{(1 - |v|^2)^2}{4} - \int_{S(0,R)} x_1\nu_1\frac{(1 - |v|^2)^2}{4}.$$

By equation (11), we then get

$$\int_{B(0,R)} e(v) = \int_{B(0,R)} |\partial_1 v|^2 - \int_{S(0,R)} x_1\partial_1 v.\partial_\nu v + \int_{S(0,R)} \nu_1 x_1 e(v). \tag{12}$$

On one hand, by Proposition 1, we know that

$$\int_{B(0,R)} e(v) - \int_{B(0,R)} |\partial_1 v|^2 \underset{R\to+\infty}{\to} E(v) - \int_{\mathbb{R}^N} |\partial_1 v|^2.$$

On the other hand, we have

$$|\int_{S(0,R)} x_1\partial_1 v.\partial_\nu v - \nu_1 x_1 e(v)| \leq AR\int_{S(0,R)} e(v).$$

Since the integral $\int_{\mathbb{R}_+}(\int_{S(0,R)} e(v))dR$ is finite, there are some positive real numbers R_n such that $R_n \underset{n\to+\infty}{\to} +\infty$ and

$$\forall n \in \mathbb{N}, R_n \int_{S(0,R_n)} e(v) \leq \frac{1}{\ln(R_n)},$$

which gives

$$\int_{S(0,R_n)} (x_1\partial_1 v.\partial_\nu v - \nu_1 x_1 e(v)) \underset{n\to+\infty}{\to} 0,$$

and finally, by equation (12),

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2.$$

In order to prove the second identity, we multiply equation (2) by Pohozaev multiplier $x_j\partial_j v$ on the ball $B(0,R)$

$$\int_{B(0,R)} (\Delta v.x_j\partial_j v + ic\partial_1 v.x_j\partial_j v + x_j\partial_j v.v(1-|v|^2)) = 0. \tag{13}$$

Integrating by parts, we compute

$$\begin{aligned}\int_{B(0,R)} \Delta v.x_j\partial_j v = &\int_{B(0,R)} \frac{|\nabla v|^2}{2} - \int_{B(0,R)} |\partial_j v|^2 + \int_{S(0,R)} x_j\partial_j v.\partial_\nu v\\ &- \int_{S(0,R)} \nu_j x_j \frac{|\nabla v|^2}{2},\end{aligned}$$

and

$$\int_{B(0,R)} x_j\partial_j v.v(1-|v|^2) = \int_{B(0,R)} \frac{(1-|v|^2)^2}{4} - \int_{S(0,R)} x_j\nu_j\frac{(1-|v|^2)^2}{4}.$$

If R is sufficiently large such as $\psi = 1$ on $S(0,R)$, we also compute

$$\begin{aligned}\int_{B(0,R)} i\partial_1 v.x_j\partial_j v = &\frac{1}{2}(\int_{S(0,R)} x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta)\\ &- \int_{B(0,R)} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta))),\end{aligned}$$

which leads to

$$\begin{aligned}\int_{B(0,R)} e(v) = &\int_{B(0,R)} |\partial_j v|^2 + \frac{c}{2}\int_{B(0,R)} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta))\\ &+ \int_{S(0,R)} (x_j\nu_j e(v) - x_j\partial_j v.\partial_\nu v - x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta)).\end{aligned} \tag{14}$$

On one hand, by Proposition 1, we know that

$$\int_{B(0,R)} e(v) - \int_{B(0,R)} |\partial_j v|^2 - \frac{c}{2}\int_{B(0,R)} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta))$$
$$\underset{R\to+\infty}{\to} E(v) - \int_{\mathbb{R}^N} |\partial_j v|^2 - \frac{c}{2}\int_{\mathbb{R}^N} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)).$$

On the other hand, we have

$$|\int_{S(0,R)} (x_j\nu_j e(v) - x_j\partial_j v.\partial_\nu v - x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta))| \leq AR\int_{S(0,R)} e(v).$$

By using the sequence of positive real numbers R_n constructed for proving equality (9), we get

$$\int_{S(0,R_n)} (x_j\nu_j e(v) - x_j\partial_j v.\partial_\nu v - x_j(\rho^2-1)(\nu_j\partial_1\theta - \nu_1\partial_j\theta)) \underset{n\to+\infty}{\to} 0,$$

and finally, by equation (14),

$$E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + \frac{c}{2}\int_{\mathbb{R}^N} (v_2\partial_1 v_1 - v_1\partial_1 v_2 + \partial_1(\psi\theta)).$$

□

4 Conclusion

We now complete the proof of Theorem 1. By Proposition 5, we have

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2,$$

which gives by denoting $\nabla_\perp v = (\partial_2 v, \ldots, \partial_N v)$,

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \eta^2 = \frac{1}{2}\int_{\mathbb{R}^N} |\partial_1 v|^2 = \frac{E(v)}{2},$$

and

$$\int_{\mathbb{R}^N} \eta^2 = 2E(v) - 2\int_{\mathbb{R}^N} |\nabla_\perp v|^2. \tag{15}$$

We then compute

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = 3E(v) - \int_{\mathbb{R}^N} |\nabla_\perp v|^2, \tag{16}$$

and, by Proposition 5,

$$c\int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\psi\theta)) = \frac{2}{N-1}\int_{\mathbb{R}^N} |\nabla_\perp v|^2 - 2E(v). \tag{17}$$

Proposition 4 gives

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = c(\frac{2}{c^2} - 1)\int_{\mathbb{R}^N} (v_1\partial_1 v_2 - v_2\partial_1 v_1 - \partial_1(\psi\theta)),$$

which leads by equations (16)-(17) to

$$(c^2+4)(N-1)E(v) = ((N-3)c^2+4)\int_{\mathbb{R}^N} |\nabla_\perp v|^2. \tag{18}$$

If $N = 2$, we get

$$(c^2 + 4)E(v) = (4 - c^2) \int_{\mathbb{R}^N} |\nabla_\perp v|^2,$$

which gives by equation (15),

$$\frac{c^2 + 4}{2} \int_{\mathbb{R}^N} \eta^2 = -2c^2 \int_{\mathbb{R}^N} |\nabla_\perp v|^2 = 0.$$

Finally, we have involving equation (15) once more

$$E(v) = 0.$$

If $N \geq 3$, since by equation (15),

$$\int_{\mathbb{R}^N} |\nabla_\perp v|^2 \leq E(v),$$

equation (18) gives

$$(2c^2 + 4N - 8)E(v) \leq 0,$$

and finally, $E(v)$ is also equal to 0 in this case.

In conclusion, since $E(v) = 0$, the function ∇v vanishes on $\mathbb{R}^N$ and v is a constant (of modulus one since η also vanishes on $\mathbb{R}^N$).

Acknowledgements. The author is grateful to F. Béthuel, G. Orlandi, J.C. Saut and D. Smets for interesting and helpful discussions.

References

[1] F. Béthuel, G. Orlandi, and D. Smets. Vortex rings for the Gross-Pitaevskii equation. *J. Eur. Math. Soc.*, 6(1):17–94, 2004.

[2] F. Béthuel and J.-C. Saut. Travelling waves for the Gross-Pitaevskii equation II. *In preparation.*

[3] F. Béthuel and J.-C. Saut. Travelling waves for the Gross-Pitaevskii equation I. *Ann. Inst. Henri Poincaré, Physique Théorique*, 70(2):147–238, 1999.

[4] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons et atomes. Introduction à l'électrodynamique quantique.* Savoirs actuels. Interéditions, Éditions du CNRS, 1987.

[5] A. Farina. From Ginzburg-Landau to Gross-Pitaevskii. *Monatsh. Math.*, 139:265–269, 2003.

[6] P. Gravejat. Limit at infinity for travelling waves in the Gross-Pitaevskii equation. *C. R. Math. Acad. Sci. Paris*, 336(2):147–152, 2003.

[7] P. Gravejat. Decay for travelling waves in the Gross-Pitaevskii equation. *Ann. Inst. Henri Poincaré, Analyse Non Linéaire*, 21(5):591–637, 2004.

[8] E.P. Gross. Hydrodynamics of a superfluid condensate. *J. Math. Phys.*, 4(2):195–207, 1963.

[9] C.A. Jones, S.J. Putterman, and P.H. Roberts. Motions in a Bose condensate V. Stability of solitary wave solutions of nonlinear Schrödinger equations in two and three dimensions. *J. Phys. A, Math. Gen.*, 19:2991–3011, 1986.

[10] C.A. Jones and P.H. Roberts. Motions in a Bose condensate IV. Axisymmetric solitary waves. *J. Phys. A, Math. Gen.*, 15:2599–2619, 1982.

[11] E. Madelung. Quantumtheorie in Hydrodynamische form. *Zts. f. Phys.*, 40:322–326, 1926.

[12] L.P. Pitaevskii. Vortex lines in an imperfect Bose gas. *Sov. Phys. JETP*, 13(2):451–454, 1961.