Limit at infinity for travelling waves in the Gross-Pitaevskii equation

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Abstract

We study the decay of the travelling waves of finite energy in the Gross-Pitaevskii equation in dimension greater than three and prove their uniform convergence to a constant of modulus one at infinity.

Résumé

Nous étudions la limite à l'infini des ondes progressives d'énergie finie dans l'équation de Gross-Pitaevskii en dimension supérieure ou égale à trois et nous montrons leur convergence uniforme vers une constante de module un.

Version française abrégée

Dans cet article, nous étudions les ondes progressives u de vitesse $c > 0$ pour l'équation de Gross-Pitaevskii $i\partial_t u = \Delta u + u(1 - |u|^2)$ de la forme $u(t, x) = v(x_1 - ct, \ldots, x_N)$. L'équation vérifiée par v que nous étudierons désormais est

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{1}$$

L'équation de Gross-Pitaevskii est un modèle physique qui décrit la supraconductivité et la superfluidité et qui est associé à l'énergie : $E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(1 - |v|^2)^2$.

C.A. Jones et P.H. Roberts [7] se sont intéressés aux ondes progressives d'énergie finie parce qu'elles sont supposées expliquer la dynamique en temps long des solutions générales : ils ont ainsi conjecturé qu'elles n'existent que lorsque $0 < c < \sqrt{2}$, ce que nous supposerons désormais, et qu'elles ont une limite à l'infini qui est une constante de module un.

F. Béthuel et J.C. Saut [3, 2] les ont étudiées sur le plan mathématique et ont notamment montré leur existence en dimension deux lorsque c est petit, et l'existence de leur limite à l'infini.

Théorème 1. *En dimension deux, une onde progressive pour l'équation de Gross-Pitaevskii de vitesse $0 < c < \sqrt{2}$ et d'énergie finie vérifie à une constante multiplicative de module un près*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

En dimension trois, F. Béthuel, G. Orlandi et D. Smets [1] ont prouvé leur existence lorsque c est petit, et, en toute dimension, A. Farina [5] a donné une borne universelle sur leur module. Dans cet article, nous allons compléter leurs travaux en dimension supérieure ou égale à trois par le théorème suivant.

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Théorème 2. *En dimension supérieure ou égale à trois, une onde progressive pour l'équation de Gross-Pitaevskii de vitesse $0 < c < \sqrt{2}$ et d'énergie finie vérifie à une constante multiplicative de module un près*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

Dans la suite, nous esquisserons la preuve de ce théorème : nous déterminerons d'abord la régularité des ondes progressives avant d'énoncer un argument général pour l'étude de la limite à l'infini d'une fonction.

Introduction

In this article, we will focus on the travelling waves of speed $c > 0$ in the Gross-Pitaevskii equation $i\partial_t u = \Delta u + u(1 - |u|^2)$ which are of the form $u(t, x) = v(x_1 - ct, \dots, x_N)$. The simplified equation for v, which we will study now, is

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \quad (1)$$

The Gross-Pitaevskii equation is a physical model for superconductivity and superfluidity which is associated to the energy: $E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(1 - |v|^2)^2$.

The travelling waves of finite energy are supposed to explain the long time dynamics of general solutions and were first considered by C.A. Jones and P.H. Roberts [7]: they conjectured that they only exist when $c < \sqrt{2}$, which will be supposed henceforth, and that they have a limit at infinity which is a constant of modulus one.

F. Béthuel and J.C. Saut [3, 2] first studied mathematically these travelling waves: they showed their existence in dimension two when c is small, and also gave a mathematical proof for their decay at infinity. In fact, they proved the following theorem.

Theorem 1. *In dimension two, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

In dimension three, F. Béthuel, G. Orlandi and D. Smets [1] showed their existence when c is small, and in every dimension, A. Farina [5] proved a universal bound for their modulus. In this paper, we will complete these results for the dimensions greater than three by proving the following theorem.

Theorem 2. *In dimension greater than three, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

This paper will be organized around the proof of this theorem: in a first part, we will study the local and Sobolev regularity of the travelling waves and, in a second part, we will give a general argument to study their decay at infinity.

1 Regularity of travelling waves

In this part, we will study the regularity of a travelling wave of finite energy and of speed $0 < c < \sqrt{2}$ in dimension N greater than two: we will proved the following proposition thanks to arguments from F. Béthuel and J.C. Saut [3, 2].

Proposition 1. *If v is a solution of the equation (1) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy, then v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, +\infty]$.*

Proof. We begin by establishing the following lemma which is valid even if $c \geq \sqrt{2}$.

Lemma 1. *v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty]$.*

The proof of this lemma is adapted from a bootstrap argument introduced in the article of F. Béthuel and J.C. Saut [3], so, we will only give its sketch, and only in dimension three because the general proof is identical with small changes of Sobolev indices.

We first consider a point z_0 in $\mathbb{R}^3$ and we denote Ω, the unit ball with center z_0. Then, we consider the solutions v_1 and v_2 of the equations

$$\begin{cases} \Delta v_1 = 0 \text{ on } \Omega \\ v_1 = v \text{ on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta v_2 = g(v) := v(1-|v|^2) + ic\partial_1 v \text{ on } \Omega \\ \qquad v_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Since the energy of v is finite, $g(v)$ is uniformly bounded in $L^{\frac{4}{3}}(\Omega)$, which means that $||g(v)||_{L^{\frac{4}{3}}(\Omega)}$ is bounded by a constant which only depends on c and $E(v)$ but not on z_0. By standard elliptic theory, and Sobolev embeddings, v_1 and v_2 are also uniformly bounded in $L^4(\Omega)$ and $W^{2,\frac{4}{3}}(\Omega)$ respectively.

If we denote ω, the ball with center z_0 and with radius $\frac{1}{2}$, by Caccioppoli inequalities, v_1 is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$ and in $W^{3,\frac{12}{11}}(\omega)$, so, v is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$. Furthermore, we compute $\nabla g(v) = \nabla v(1-|v|^2) - 2(v.\nabla v)v + ic\partial_1 \nabla v$, and then, $\nabla g(v)$ is uniformly bounded in $L^{\frac{12}{11}}(\omega)$. By standard elliptic theory, and Sobolev embeddings, we finally get that v is uniformly bounded in $C^{0,\frac{1}{12}}(\omega)$.

Thus, v is continuous and bounded on $\mathbb{R}^3$. Then, its gradient $w = \nabla v$ satisfies

$$-\Delta w - ic\partial_1 w + (\frac{c^2}{2} + 2)w = w(1-|v|^2) - 2(v.w)v + (\frac{c^2}{2} + 2)w = h(w),$$

so, from the preceding inequalities, $h(w)$ belongs to $L^2(\mathbb{R}^3)$, which proves that w belongs to $H^2(\mathbb{R}^3)$. So, w is continuous and bounded, and by iterating, we can conclude that v is regular, bounded and that all its derivatives belong to the spaces $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. Then, we end this proof by using a standard interpolation result between L^p-spaces.

We deduce from this first lemma the following lemma.

Lemma 2. *The modulus ρ of v satisfies*

$$\rho(x) \underset{|x| \to +\infty}{\to} 1.$$

Indeed, if we denote $\eta = 1 - \rho^2$, η^2 is uniformly continuous because v is bounded and lipschitzian by Lemma 1. As $\int_{\mathbb{R}^N} \eta^2$ is finite, η converges uniformly to 0 at infinity which ends the proof of this lemma.

Thus, ρ does not vanish at the neighborhood of infinity: so, we can write there $v = \rho e^{i\theta}$ and compute the following equations satisfied by ρ and θ:

$$\begin{cases} div(\rho^2 \nabla \theta) = -\frac{c}{2}\partial_1 \rho^2 \\ -\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2). \end{cases} \tag{1}$$

Thanks to this polar form, we can now conclude the proof of Proposition 1 by the following lemma.

Lemma 3. *The gradient of v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, 2[$.*

This proof is also adapted from an article of F. Béthuel and J.C. Saut [2], and so, we will only give its sketch. We first notice by Lemma 2 that ρ does not vanish at the neighborhood of infinity, and, in order to simplify, we will suppose that ρ does not vanish on $\mathbb{R}^N$: the general situation is technically slightly more involved, but follows essentially the same idea (See [6]).

So, we begin by denoting $F = 2\eta^2 - 2c\eta\partial_1\theta + 2|\nabla v|^2$ and $G = \eta\nabla\theta$. Because $|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2$, and by Lemma 1 and Lemma 2, we can establish that F and G are in all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [1, +\infty]$. Besides, we compute thanks to (1) using the Fourier transformation

$$\forall \xi \in \mathbb{R}^N, \begin{cases} (|\xi|^2 + 2)\widehat{\eta}(\xi) - 2ic\xi_1\widehat{\theta}(\xi) = \widehat{F}(\xi) \\ |\xi|^2\widehat{\theta}(\xi) + \frac{ic}{2}\xi_1\widehat{\eta}(\xi) = -i\sum\limits_{j=1}^{N}\xi_j\widehat{G_j}(\xi) \end{cases}$$

Denoting L_0 and $(L_{j,1})_{1 \leq j \leq N}$ the operators associated to the Fourier multipliers $\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}$, respectively $\widehat{R_{j,1}}(\xi) = \frac{\xi_j\xi_1}{|\xi|^2}$, we can assert

$$\eta = L_0(F + 2c\sum_{j=1}^{N} L_{j,1}(G)).$$

Furthermore, the Riesz operator theory checks that the operators $(L_{j,1})_{1 \leq j \leq N}$ are multipliers on all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$, and, $\widehat{K_0}$ is a regular bounded function on $\mathbb{R}^N \setminus \{0\}$ which satisfies

$$\prod_{j=1}^{N}(\xi_j^{k_j})\partial_1^{k_1} \ldots \partial_N^{k_N}\widehat{K_0}(\xi) \in L^\infty(\mathbb{R}^N)$$

as soon as $(k_1, \ldots, k_N) \in \{0,1\}^N$ satisfies $0 \leq \sum\limits_{j=1}^{N} k_j \leq N$. Therefore, by Lizorkin theorem [8] (See also [9]), L_0 is a multiplier on all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$ too. By the previous statements on F and G, we conclude that η is in all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$, and by the equation

$$\forall j \in \{1, \ldots, N\}, \partial_j\theta = -\frac{ic}{2}L_{j,1}(\eta) - i\sum_{k=1}^{N} L_{j,k}(G_k)$$

where $(L_{j,k})_{1 \leq j,k \leq N}$ is the operator associated to the Fourier multiplier $\widehat{R_{j,k}}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}$, $\nabla\theta$ is also in all the spaces $L^p(\mathbb{R}^N)$ for $p \in]1, +\infty[$.

By iterating this process to all the derivatives of η and $\nabla\theta$ by Lemma 1, we conclude that η and $\nabla\theta$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in]1, +\infty[$. Since $\eta = 1 - \rho^2$ and ρ is in all the spaces $W^{k,\infty}(\mathbb{R}^N)$ for $k \in \mathbb{N}$, and since $|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2$, Lemma 3 is proved as well as Proposition 1. □

2 Limit at infinity

Before concluding the proof of Theorem 2, we will establish the following general proposition concerning the limit of a function at infinity.

Proposition 2. *We consider a regular function v on $\mathbb{R}^N$: we suppose that N is greater than three and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ where $1 < p_0 < N-1 < p_1 < +\infty$. Then there is a constant $v_\infty \in \mathbb{C}$ which satisfies*

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

Proof. We begin by constructing the limit v_∞. Indeed, we have

$$\int_{\mathbb{S}^{N-1}} \int_1^{+\infty} |\partial_r v(r\xi)| dr d\sigma \leq \int_{\mathbb{S}^{N-1}} \left(\int_1^{+\infty} |\nabla v(r\xi)|^{p_0} r^{N-1} dr \right)^{\frac{1}{p_0}} \left(\int_1^{+\infty} r^{-\frac{N-1}{p_0-1}} dr \right)^{\frac{1}{p_0'}} d\sigma < +\infty$$

which gives $\int_1^{+\infty} |\partial_r v(r\xi)| dr < +\infty$ a.e. Thus, there is a function v_∞ defined on $\mathbb{S}^{N-1}$ such that

$$v(r\xi) \underset{r\to+\infty}{\to} v_\infty(\xi) \ a.e.$$

If we denote $\forall p \in [p_0, p_1], \forall r \in \mathbb{R}_+^*, \phi_p(r) = r^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p d\sigma$, this function is regular on $\mathbb{R}_+^*$ and its derivative satisfies

$$\int_0^{+\infty} |\phi_p'(r)| dr \leq C(||\nabla v||^p_{L^p(\mathbb{R}^N)} + ||\nabla v||^{p-1}_{L^p(\mathbb{R}^N)} ||\nabla v||_{W^{1,p}(\mathbb{R}^N)}) < +\infty.$$

Hence, the function ϕ_p has a limit at infinity, and since $\int_0^{+\infty} \phi_p(r) dr = ||\nabla v||^p_{L^p(\mathbb{R}^N)} < +\infty$, this limit is zero. Furthermore, if we denote $\forall (r, \xi) \in \mathbb{R}_+^* \times \mathbb{S}^{N-1}, v_r(\xi) = v(r\xi)$, we remark that $|\nabla v(r\xi)|^2 = |\partial_r v(r\xi)|^2 + r^{-2} |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)|^2$, which leads finally to

$$r^{N-1-p} \int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)|^p d\sigma \underset{r\to+\infty}{\to} 0.$$

Thus, if $N - 1 \leq q < \min\{p_1, N\}$, we get for every $r \in \mathbb{R}_+^*$

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} |v_r - v_\infty|^q d\sigma \leq \int_{\mathbb{S}^{N-1}} \left(\int_r^{+\infty} |\partial_r v(s\xi)| ds \right)^q d\sigma &\leq \int_{\mathbb{S}^{N-1}} r^{q-n} \int_r^{+\infty} |\nabla v(s\xi)|^q s^{N-1} ds d\sigma \\ &\leq C_{n,q} ||\nabla v||^q_{L^q(\mathbb{R}^N)} r^{q-N}, \end{aligned}$$

which gives

$$\begin{aligned} ||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} = C_N \int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}} |v_r - v_\infty|^*(t) dt &\leq C_{N,q} \left(\int_0^{|\mathbb{S}^{N-1}|} |v_r - v_\infty|^{*q}(t) dt \right)^{\frac{1}{q}} \\ &\leq C_{N,q} ||v_r - v_\infty||_{L^q(\mathbb{S}^{N-1})} \\ &\leq C_{N,q} ||\nabla v||^q_{L^q(\mathbb{R}^N)} r^{q-N} \end{aligned}$$

and proves that $||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})}$ tends to 0 when r tends to $+\infty$. Now, we fix $\epsilon > 0$ and we denote

$$\forall r \in \mathbb{R}_+, \begin{cases} \forall \lambda \in \mathbb{R}_+^*, a_r(\lambda) = |\{\xi \in \mathbb{S}^{N-1} / |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)| > \lambda\}| \\ \forall t \in \mathbb{R}_+^*, f_r(t) = |\nabla_{\mathbb{S}^{N-1}} v_r|^*(t) = \inf\{\lambda \in \mathbb{R}_+^* / a_r(\lambda) \leq t\}. \end{cases}$$

We have showed that there exists $r_\epsilon \in \mathbb{R}_+^*$ such that

$$\forall r > r_\epsilon, \forall i \in \{0,1\}, r^{N-1-p_i} \int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} v_r(\xi)|^{p_i} d\sigma \leq \epsilon^{p_i}.$$

This gives

$$\begin{cases} \forall \lambda \in \mathbb{R}_+^*, a_r(\lambda) \leq \min\{\frac{\epsilon^{p_0}}{r^{N-1-p_0}\lambda^{p_0}}, \frac{\epsilon^{p_1}}{r^{N-1-p_1}\lambda^{p_1}}\} \\ \forall t \in \mathbb{R}_+^*, f_r(t) \leq \min\{\frac{\epsilon}{r^{\frac{N-1}{p_0}-1} t^{\frac{1}{p_0}}}, \frac{\epsilon}{r^{\frac{N-1}{p_1}-1} t^{\frac{1}{p_1}}}\}. \end{cases}$$

Thus, we finally get

$$\begin{aligned} ||\nabla v_r||_{L^{N-1,1}(\mathbb{S}^{N-1})} &\leq C_N \epsilon \left(r^{1-\frac{N-1}{p_1}} \int_0^{r^{1-N}} t^{-\frac{N-2}{N-1}-\frac{1}{p_1}} dt + r^{1-\frac{N-1}{p_0}} \int_{r^{1-N}}^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}-\frac{1}{p_0}} dt \right) \\ &\leq C_{N,p_0,p_1} \epsilon. \end{aligned}$$

This proves that ∇v_r converges to 0 in $L^{N-1,1}(\mathbb{S}^{N-1})$ when r tends to $+\infty$. Since $(v_r)_{r>0}$ converges to v_∞ in $L^{N-1,1}(\mathbb{S}^{N-1})$ and the limit of its gradient is 0 in this same space, we conclude that the gradient of v_∞ is 0 and so, v_∞ is constant. Besides, by a theorem of A. Cianchi and L. Pick [4], we know that there is a constant C which satisfies for all $r > 0$

$$||v_r - v_\infty||_{L^\infty(\mathbb{S}^{N-1})} \leq C(||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} + ||\nabla(v_r - v_\infty)||_{L^{N-1,1}(\mathbb{S}^{N-1})}) \underset{r\to+\infty}{\to} 0$$

which ends the proof of this proposition. □

Now, we conclude the proof of Theorem 2: if v is a travelling wave of finite energy and of speed $c < \sqrt{2}$, it satisfies the hypothesis of Proposition 2 by Proposition 1. So there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

It remains to show that v_∞ has a modulus equal to one which is clear in view of Lemma 2.

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