Decay for travelling waves in the Gross-Pitaevskii equation

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Abstract

We study the limit at infinity of the travelling waves of finite energy in the Gross-Pitaevskii equation in dimension larger than two: their uniform convergence to a constant of modulus one and their asymptotic decay.

Résumé

Nous étudions la limite à l'infini des ondes progressives d'énergie finie pour les équations de Gross-Pitaevskii en dimension supérieure ou égale à deux: leur convergence uniforme vers une constante de module un et leur comportement asymptotique.

Introduction

In this article, we will focus on the travelling waves in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2)$$

of the form $u(t,x) = v(x_1 - ct, \ldots, x_N)$. The parameter $c > 0$ represents the speed of the travelling wave and the simplified equation for v, which we will study now, is

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{1}$$

The Gross-Pitaevskii equation is a physical model for superconductivity and superfluidity which is associated to the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |v|^2)^2 = \int_{\mathbb{R}^N} e(v).$$

The travelling waves of finite energy play an important role in the long time dynamics of general solutions and were first considered by C.A. Jones and P.H. Roberts [11]: they conjectured that they only exist when $c < \sqrt{2}$, which will be supposed henceforth, and that they are axisymmetric around the axis x_1. They also proposed an asymptotic development at infinity for these waves up to a multiplicative constant of modulus one. In dimension two, they conjectured that

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2} \tag{2}$$

and in dimension three,

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}}, \tag{3}$$

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where the constant α is supposed to be a relevant physical value, the stretched dipole coefficient.

F. Béthuel and J.C. Saut [4, 5] first studied mathematically these travelling waves: they showed their existence in dimension two when c is small, and also gave a mathematical proof for their limit at infinity.

Theorem 1. *In dimension two, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x| \to +\infty}{\to} 1.$$

In dimension larger than three, F. Béthuel, G. Orlandi and D. Smets [2] showed their existence when c is small, and in every dimension, A. Farina [9] proved a universal bound for their modulus, and we proved the non-existence of non-constant travelling waves for every $c > \sqrt{2}$ [10].

In this paper, we will complete those results by proving the convergence of the travelling waves at infinity in every dimension larger than three and by giving a first estimate of their decay, which is consistent with the conjectures (2)-(3) of C.A. Jones and P.H. Roberts [11]. We will precisely prove the following theorem.

Theorem 2. *In dimension larger than three, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x| \to +\infty}{\to} 1.$$

Moreover, in every dimension larger than two, the function

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

is bounded on $\mathbb{R}^N$.

By this theorem, we can characterize all the L^p-spaces to which the function $v - 1$ belongs.

Corollary 1. *The function $v - 1$ belongs to all the spaces $L^p(\mathbb{R}^N)$ for every*

$$\frac{N}{N-1} < p \leq +\infty.$$

This corollary is particularly interesting in dimension larger than three because in this case, the function $v - 1$ belongs to the space $L^2(\mathbb{R}^N)$, and therefore, in view of the energy bound, also to the space $H^1(\mathbb{R}^N)$: thus, the function $(x, t) \mapsto v(x_1 - ct, x_2, \ldots, x_N)$ is solution of the Cauchy problem

$$i\partial_t u = \Delta u + u(1 - |u|^2)$$

with the initial data

$$u(0, x) = v(x)$$

in the space $C^0(\mathbb{R}, 1 + H^1(\mathbb{R}^N))$. The following theorem due to F. Béthuel and J.C. Saut [4] asserts that this equation is well-posed in the space $1 + H^1(\mathbb{R}^N)$.

Theorem 3. *Let $v_0 \in 1 + H^1(\mathbb{R}^N)$. There is a unique solution $v \in C^0(\mathbb{R}, 1 + H^1(\mathbb{R}^N))$ of the time-dependant Gross-Pitaevskii equation. Moreover, the energy E is conserved, and the solution v depends continuously on the initial data v_0.*

Therefore, we can study the stability of a travelling wave in the space $1 + H^1(\mathbb{R}^N)$, and understand better the long time dynamics of the time dependant Gross-Pitaevskii equation.

The proof of Corollary 1 being an immediate consequence of Theorem 2, this paper will be organized around the proof of this theorem.

In a first part, we will study the local smoothness and the Sobolev regularity of a travelling wave v, by proving:

Theorem 4. *If v is a solution of (1) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy, then v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p \leq +\infty$.*

We will first prove that v is regular and belongs to all the L^p-spaces for $2 \leq p \leq +\infty$: this will be done by a bootstrap argument adapted from articles of F. Béthuel and J.C. Saut [4, 3]. We will deduce that the modulus ρ of v does not vanish at infinity, which leads to the equations

$$\begin{cases} div(\rho^2 \nabla\theta) = -\frac{c}{2}\partial_1 \rho^2 \\ -\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2). \end{cases} \tag{4}$$

by writing $v = \rho e^{i\theta}$. We will then write new equations in the more interesting variables $\eta = 1-\rho^2$ and $\nabla\theta$: denoting $F = 2\eta^2 - 2c\eta\partial_1\theta + 2|\nabla v|^2$ and $G = \eta\nabla\theta$, we observe that we can write the equations

$$\begin{cases} \Delta\eta - 2\eta + F + 2c\partial_1\theta = 0 \\ \Delta\theta - \frac{c}{2}\partial_1\eta = \sum\limits_{j=1}^{N} \partial_j G_j. \end{cases}$$

By taking the laplacian of the first line, and taking the operator ∂_1 in the second line, we have

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F - 2c\sum_{j=1}^{N} \partial_1\partial_j G_j \tag{5}$$

and, by taking the operator ∂_j in the second line, for every $j \in [\![1, N]\!]$,

$$\Delta\partial_j\theta = \frac{c}{2}\partial_1\partial_j\eta + \sum_{k=1}^{N} \partial_j\partial_k G_k. \tag{6}$$

Obviously, those equations are only valid at a neighborhood of infinity and we will have to introduce cut-off functions in order to obtain an equation valid on the whole space. Eluding for the moment this difficulty, we can develop an argument due to J.L. Bona and Yi A. Li [6], and A. de Bouard and J.C Saut [8] (Also see [14, 13] for many more details), which relies on the transformation of a PDE in a convolution equation. In fact, equations (5) and (6) can be written

$$\eta = K_0 * F + 2c\sum_{j=1}^{N} K_{1,j} * G_j \tag{7}$$

where K_0 and $K_{1,j}$ are the kernels of Fourier transformation,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2},$$

respectively,

$$\widehat{K_{1,j}}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2},$$

and, for every $j \in [\![1, N]\!]$,

$$\partial_j \theta = \frac{c}{2} K_{1,j} * F + c^2 \sum_{k=1}^{N} L_{1,j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k \tag{8}$$

where $L_{1,j,k}$ and $R_{j,k}$ are the kernels of Fourier transformation,

$$\widehat{L_{1,j,k}}(\xi) = \frac{\xi_1^2 \xi_j \xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)},$$

respectively,

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}.$$

Those equations seem more involved than the initial ones, but are more adapted in order to study the algebraic decay of the functions η and $\nabla\theta$.

We will understand why by studying a very simple example. Let us consider a convolution equation of the form

$$g = K * f,$$

where we suppose that the functions K and f are regular functions. We want to study the algebraic decay of the function g ie to determine all the indices α for which it belongs to the space

$$M_\alpha^\infty(\mathbb{R}^N) = \{u : \mathbb{R}^N \mapsto \mathbb{C} / \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} = \sup\{|x|^\alpha |u(x)|, x \in \mathbb{R}^N\} < +\infty\},$$

in function of the algebraic decay of K and f. We prove the following lemma.

Lemma 1. *Suppose that K and f belong to the space $M_{\alpha_1}^\infty(\mathbb{R}^N)$, respectively $M_{\alpha_2}^\infty(\mathbb{R}^N)$, where $\alpha_1 > N$ and $\alpha_2 > N$. Then the function g belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$ for*

$$\alpha \leq \min\{\alpha_1, \alpha_2\}.$$

Proof. The proof of this lemma relies on Young's inequalities:

$$\begin{aligned}
\forall x \in \mathbb{R}^N, |x|^\alpha |g(x)| &\leq |x|^\alpha \int_{\mathbb{R}^N} |K(x-y)||f(y)| dy \\
&\leq A \int_{\mathbb{R}^N} \left(|x-y|^\alpha |K(x-y)||f(y)| + |K(x-y)||y|^\alpha |f(y)|\right) dy \\
&\leq A \left(\|K\|_{M_\alpha^\infty(\mathbb{R}^N)} \|f\|_{L^1(\mathbb{R}^N)} + \|K\|_{L^1(\mathbb{R}^N)} \|f\|_{M_\alpha^\infty(\mathbb{R}^N)} \right).
\end{aligned}$$

Since $\alpha_1 > N$ and $\alpha_2 > N$, K and f belong to $L^1(\mathbb{R}^N)$: thus, if $\alpha \leq \min\{\alpha_1, \alpha_2\}$, the last term is finite, and, the function g belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$. □

Of course, the assumptions $\alpha_1 > N$ and $\alpha_2 > N$ are quite restrictive, but, we can easily generalize this method by using Young's inequalities involving not only the L^1-L^∞ duality, but, the L^p-$L^{p'}$ duality, and, prove easily the algebraic decay of functions which satisfy such a convolution equation.

Our situation is close to previous example. Indeed, equations (7))-(8) can be written

$$(\eta, \nabla\theta) = K * F(\eta, \nabla\theta),$$

where F behaves like a quadratic function in terms of the variables η and $\nabla\theta$.

In order to understand what happens in this case, we can consider the non-linear model

$$f = K * f^2,$$

where f and K are both regular functions. We get in this case

Lemma 2. *Suppose K and f belong to the space $M^\infty_{\alpha_1}(\mathbb{R}^N)$, respectively $M^\infty_{\alpha_2}(\mathbb{R}^N)$, where $\alpha_1 > N$ and $\alpha_1 > \alpha_2 > N$. Then, the function f belongs to the space $M^\infty_\alpha(\mathbb{R}^N)$ for*

$$\alpha \leq \alpha_1.$$

Proof. The proof of this lemma also relies on Young's inequalities:

$$\begin{aligned}\forall x \in \mathbb{R}^N, |x|^\alpha |f(x)| &\leq |x|^\alpha \int_{\mathbb{R}^N} |K(x-y)||f(y)|^2 dy \\ &\leq A \int_{\mathbb{R}^N} \left(|x-y|^\alpha |K(x-y)||f(y)|^2 + |K(x-y)||y|^\alpha |f(y)|^2\right) dy \\ &\leq A \left(\|K\|_{M^\infty_\alpha(\mathbb{R}^N)} \|f\|^2_{L^2(\mathbb{R}^N)} + \|K\|_{L^1(\mathbb{R}^N)} \|f\|^2_{M^\infty_{\frac{\alpha}{2}}(\mathbb{R}^N)} \right).\end{aligned}$$

Since $\alpha_1 > N$ and $\alpha_2 > N$, K and f belong to $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$: thus, if $\alpha \leq \min\{\alpha_1, 2\alpha_2\}$, the last term is finite, and, the function f belongs to the space $M^\infty_\alpha(\mathbb{R}^N)$. By iterating this step, we see that if $\alpha \leq \min\{\alpha_1, 2^k\alpha_2\}$ for every $k \in \mathbb{N}$, the function f belongs to the space $M^\infty_\alpha(\mathbb{R}^N)$, and we get the desired result. □

The situation of the functions η and $\nabla\theta$ is rather involved, but, this simple model shows that their decay is determined by the decay of the kernels.

This lemma provides a striking optimal decay property for super linear equation. Indeed, assuming f possesses *some* algebraic decay, then, if f is solution of such a convolution equation, it decays as fast as the kernel. However, some decay of f must be established *first*, in order to initiate the inductive argument.

In our particular case, we will determine the decay at infinity of the kernels K_0, $K_{1,j}$, $L_{1,j,k}$ and $R_{j,k}$, *some* decay at infinity for the functions η and $\nabla\theta$, before getting their optimal decay by the previous inductive argument.

Before doing so, we conclude this first part by showing that the kernels K_0, $K_{1,j}$, $L_{1,j,k}$ and $R_{j,k}$ are L^p-multipliers for every $1 < p < +\infty$, by using Lizorkin theorem [12] and standard arguments on Riesz operators. This will end the first part by establishing the L^p-regularity of v for $1 < p < 2$.

In the second part, we will begin our study of the decay of v by studying the decay of the kernels K_0, $K_{1,j}$, $L_{1,j,k}$ and $R_{j,k}$ at the origin, where they are singular, and at infinity. This will be done by three different ways:

- We will first use an L^1-L^∞ inequality, which generalizes the classical one between a function and its Fourier transformation, and which is presummably well-known to the experts. This follows from the next lemma:

  Lemma 3. *The following equality holds*

  $$\forall x \in \mathbb{R}^N, |x|^s f(x) = I_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\hat{f}(z) - \hat{f}(y)}{|z-y|^{N+s}} e^{ix.y} dy dz$$

  for every $f \in S(\mathbb{R}^N)$, and $0 < s < 1$, where we denote

  $$I_N = ((2\pi)^{N+1} \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}) u^{-\frac{N}{2}-s} du)^{-1},$$

$J_{\frac{N}{2}-1}$ *being the Bessel function defined by*

$$\forall u \in \mathbb{R}, J_{\frac{N}{2}-1}(u) = \Big(\frac{u}{2}\Big)^{\frac{N}{2}-1} \sum_{n=0}^{+\infty} \frac{(-1)^n u^{2n}}{4^n n! \Gamma(n+\frac{N}{2})}.$$

By this equality, we will prove the following theorem:

Theorem 5. *Let $N-2 < \alpha < N$, $n \in \mathbb{N}$, and $(j,k,l) \in [\![1,N]\!]^3$. The functions $d^n K_0$, $d^n K_{j,k}$ and $d^n L_{j,k,l}$ belong to $M^\infty_{\alpha+n}(\mathbb{R}^N)$.*

- We will then prove independantly that all those functions are bounded even in the critical case ie when $\alpha = N$. This will be done by another duality argument in $S'(\mathbb{R}^N)$, and by a classical integration by parts, and, this will give:

  Theorem 6. *Let $n \in \mathbb{N}$, and $(j,k,l) \in [\![1,N]\!]^3$. The functions $d^n K_0$, $d^n K_{j,k}$ and $d^n L_{j,k,l}$ belong to $M^\infty_{N+n}(\mathbb{R}^N)$.*

- Finally, we will study what we shall call the composed Riesz kernels ie the kernels $R_{j,k}$: in this case, by standard Riesz operator theory, we exactly know the form of the kernels $R_{j,k}$. Thus, if f is a regular function, and if we denote $g_{j,k} = R_{j,k} * f$ for every $(j,k) \in [\![1,N]\!]^2$, we can write the formula

$$\forall x \in \mathbb{R}^N, g_{j,k}(x) = A_N \int_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)dy$$
$$+ A_N \int_{|y|\leq 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} (f(x-y) - f(x))dy.$$

  Therefore, in this section, we will not study the decay of the kernels $R_{j,k}$ at infinity, but directly, the decay of the functions $g_{j,k}$, when the function f belongs to $L^1(\mathbb{R}^N)$, and the functions $|.|^\alpha f$ and $|.|^\alpha \nabla f$ are bounded for some positive number α.

In the third part, we will study the decay of η, $\nabla\eta$ and $\nabla\theta$ at infinity: we will first give a refined energy estimate due to F. Béthuel, G. Orlandi and D. Smets [2],

Lemma 4. *For every $c < \sqrt{2}$, there is a strictly positive constant α_c such that the function*

$$R \to R^{\alpha_c} \int_{B(0,R)^c} e(v)$$

is bounded on $\mathbb{R}_+$.

which is the starting point of the whole study of the decay of v at infinity. Indeed, it enables to prove *some* algebraic decay for the functions η and $\nabla\theta$, which leads to the following theorem by the inductive method yet mentionned.

Theorem 7. *The functions η, $\nabla\eta$ and $\nabla\theta$ belong to respectively $M^\infty_N(\mathbb{R}^N)$, $M^\infty_{N+1}(\mathbb{R}^N)$ and $M^\infty_N(\mathbb{R}^N)$.*

The key result of this theorem is that the algebraic decay of the functions η or $\nabla\theta$ is imposed by the kernels of the equations they satisfy, which was previously explained on a simple model.

Finally, in the last part, we will first prove the uniform convergence of v at infinity towards a constant of modulus one, v_∞, and then, we will conclude the proof of Theorem 2.

We recall once more that the previous discussion is only valid with the assumption that ρ does not vanish. In the case when ρ vanishes, the discussion can be carried on with some modifications involving cut-off functions. In particular, we will prove the following theorem.

Theorem 8. *Let v, a function from $\mathbb{R}^N$ to $\mathbb{C}$, where $N \geq 2$, and suppose that there is a bounded domain Ω of $\mathbb{R}^N$ such that*

- $\Delta v + ic\partial_1 v + v(1-|v|^2) = 0$ *on* ${}^c\Omega$, *with* $0 < c < \sqrt{2}$.
- $\int_{{}^c\Omega} e(v) = \frac{1}{2}\int_{{}^c\Omega} |\nabla v|^2 + \frac{1}{4}\int_{{}^c\Omega}(1-|v|^2)^2 < +\infty$.

Then, v satisfies up to a multiplicative constant of modulus one

$$v(x) \underset{|x|\to+\infty}{\to} 1,$$

and, the function

$$x \mapsto |x|^{N-1}(v(x)-1)$$

is bounded on $\mathbb{R}^N$.

Thus, the same results hold for equations of the form

$$\Delta v + ic\partial_1 v + v(1-|v|^2) - U.v = 0,$$

where U is a regular function with compact support, or for the equation on the complement of a bounded domain Ω,

$$\Delta v + ic\partial_1 v + v(1-|v|^2) = 0,$$

with the Dirichlet condition $v = 0$ on $\partial\Omega$ (See [5, 1] for existence results).

1 Regularity of travelling waves for the Gross-Pitaevskii equation

In this part, we study the regularity of a travelling wave v of finite energy and of speed $0 < c < \sqrt{2}$ in dimension $N \geq 2$: we prove the following theorem by arguments taken from F. Béthuel and J.C. Saut [4, 3].

Theorem 4. *If v is a solution of (1) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy, then v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p \leq +\infty$.*

In particular, we establish a more precise form of equations (7) and (8), in which the cut-off functions are included.

1.1 L^p-Regularity for $2 \leq p \leq +\infty$

We begin by establishing the following proposition, which is valid even if $c \geq \sqrt{2}$.

Proposition 1. *v is regular, bounded and its gradient belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $2 \leq p \leq +\infty$.*

Proof. We will only prove this proposition in dimension three because the general proof is identical with small changes of Sobolev indices. This proof is adapted from the article of F. Béthuel and J.C. Saut [4], where it is written in dimension two: it is based on a bootstrap method.

We first consider a point z_0 in $\mathbb{R}^3$ and we denote Ω, the unit ball with center z_0. Then, we consider the solutions v_1 and v_2 of the equations

$$\begin{cases} \Delta v_1 = 0 \text{ on } \Omega \\ v_1 = v \text{ on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta v_2 = v(1-|v|^2) + ic\partial_1 v := g(v) \text{ on } \Omega \\ v_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Since the energy $E(v)$ of v is finite, v is uniformly bounded in $L^4(\Omega)$, which means that the norm of v in $L^4(\Omega)$ is finite and bounded by a constant which only depends on c and $E(v)$ but not on z_0. Thus, $v(1-|v|^2)$ is uniformly bounded in $L^{\frac{4}{3}}(\Omega)$ and likewise, $\partial_1 v$ is also uniformly bounded in $L^{\frac{4}{3}}(\Omega)$, such as $g(v)$. By standard elliptic theory, v_2 is then uniformly bounded in $W^{2,\frac{4}{3}}(\Omega)$, and by Sobolev embeddings, v_1 is uniformly bounded in $L^4(\Omega)$.

If we denote ω, the ball with center z_0 and with radius $\frac{1}{2}$, and if we use the Caccioppoli inequalities, v_1 is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$ and in $W^{3,\frac{12}{11}}(\omega)$, which gives that v is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$.

Furthermore, we compute

$$\forall j \in [\![1,3]\!], \partial_j g(v) = \partial_j v(1-|v|^2) - 2(v.\partial_j v)v + ic\partial_{1,j}^2 v.$$

So, $\partial_j g(v)$ is uniformly bounded in $L^{\frac{12}{11}}(\omega)$, and, by standard elliptic theory, v_2 is uniformly bounded in $W^{3,\frac{12}{11}}(\omega)$ such as v. Finally, by Sobolev embeddings, v is uniformly bounded in $C^{0,\frac{1}{12}}(\omega)$: therefore, v is continuous and bounded on $\mathbb{R}^3$.

But, its gradient $w = \nabla v$ satisfies

$$-\Delta w - ic\partial_1 w + (\frac{c^2}{2}+2)w = w(1-|v|^2) - 2(v.w)v + (\frac{c^2}{2}+2)w := h(w),$$

and, $h(w)$ belongs to $L^2(\mathbb{R}^3)$, which proves that w belongs to $H^2(\mathbb{R}^3)$. So w is continuous and bounded, and by iterating, we can conclude that v is regular, bounded and that all its derivatives belong to the spaces $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. Then, we end this proof by using a standard interpolation result between L^p-spaces. $\square$

Remark 1. This proposition shows that every weak solution of finite energy of (1) is a classical solution.

1.2 Convolution equations

Now, we establish equations (7) and (8), in which we include the cut-off functions.

We first deduce from Proposition 1 the following lemma, which is valid even if $c \geq \sqrt{2}$.

Lemma 5. *The modulus ρ of v and all its derivatives $\partial^\alpha v$ satisfy*

$$\begin{cases} \rho(x) \underset{|x|\to+\infty}{\to} 1 \\ \partial^\alpha v(x) \underset{|x|\to+\infty}{\to} 0. \end{cases}$$

Proof. Indeed, if we denote $\eta = 1-\rho^2$, η^2 is uniformly continuous because v is bounded and lipschitzian by Proposition 1. As $\int_{\mathbb{R}^N} \eta^2$ is finite, we can conclude

$$\eta(x) \underset{|x|\to+\infty}{\to} 0.$$

By the same argument, we can prove that

$$\partial^\alpha v(x) \underset{|x|\to+\infty}{\to} 0.$$

□

Thus, ρ does not vanish at the neighborhood of infinity: so, we can write there $v = \rho e^{i\theta}$ and compute the following equations satisfied by ρ and θ,

$$\begin{cases} div(\rho^2\nabla\theta) = -\frac{c}{2}\partial_1\rho^2 \\ -\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2). \end{cases} \tag{4}$$

As it was previously mentionned, we begin by denoting $F = 2\eta^2 - 2c\eta\partial_1\theta + 2|\nabla v|^2$ and $G = \eta\nabla\theta$, and, we compute thanks to the second equation of (4)

$$\Delta\eta - 2\eta + F + 2c\partial_1\theta = 0,$$

and, thanks to the first one,

$$\Delta\theta - \frac{c}{2}\partial_1\eta = \sum_{j=1}^{N} \partial_j G_j.$$

By substituting this equality in the previous one, we finally have for the variable η

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F - 2c\sum_{j=1}^{N} \partial_1\partial_j G_j.$$

In fact, this equality is only valid at the points where the function ρ does not vanish. In particular, in order to obtain an equation on the entire space, we truncate each function which appears in this equality. Thus, we introduce a real number $R_0 > 0$ such that ρ does not vanish on $B(0,R_0)^c$, and a regular positive function ψ which is equal to 0 on $B(0,2R_0)$ and to 1 on $B(0,3R_0)^c$: with those notations, we calculate the new equation on the whole space

$$\Delta^2\breve{\eta} - 2\Delta\breve{\eta} + c^2\partial_{1,1}^2\breve{\eta} = -\Delta\breve{F} - 2c\sum_{j=1}^{N} \partial_1\partial_j\breve{G}_j + \Phi_0, \tag{9}$$

where we denote

$$\begin{aligned} \Phi_0 = {} & \eta\Delta^2\psi + 4\nabla\Delta\psi.\nabla\eta + 4\nabla\Delta\eta.\nabla\psi + 2\Delta\eta\Delta\psi + 4\nabla^2\eta.\nabla^2\psi \\ & - 4\nabla\eta.\nabla\psi - 2\eta\Delta\psi + c^2\eta\partial_{1,1}^2\psi + 2c^2\partial_1\psi\partial_1\eta + F\Delta\psi + 2\nabla\psi.\nabla F \\ & + 2c\sum_{j=1}^{N}(G_j\partial_1\partial_j\psi + \partial_1\psi\partial_j G_j + \partial_j\psi\partial_1 G_j). \end{aligned} \tag{10}$$

and, for every numerical function f defined on $\mathbb{R}^N$,

$$\breve{f} = \psi f.$$

To proceed further, we study the function Φ_0 and prove the next lemma.

Lemma 6. *Φ_0 is a regular function with a compact support included in the annulus*

$$\mathcal{A} = B(0,3R_0) \setminus B(0,2R_0),$$

which satisfies

$$\Phi_0 = -\sum_{j=1}^{N}\sum_{k=1}^{N} \partial_j \partial_k P_{j,k},$$

where the functions $P_{j,k}$ given by

$$\forall x \in \mathbb{R}^N, P_{j,k}(x) = -\int_0^1 (1-u)\Phi_0(\frac{x}{u})u^{-N-2}x_j x_k du,$$

belong to all the spaces $L^p(\mathbb{R}^N)$ for $1 \le p < \frac{N}{N-1}$.

Proof. By equation (10), Φ_0 is a regular function with a compact support included in the annulus

$$\mathcal{A} = B(0,3R_0) \setminus B(0,2R_0).$$

Thus, its Fourier transformation is also regular and we can write its Taylor formula in 0:

$$\forall \xi \in \mathbb{R}^N, \widehat{\Phi_0}(\xi) = \widehat{\Phi_0}(0) + \sum_{j=1}^{N} \xi_j \partial_j \widehat{\Phi_0}(0) + \sum_{j=1}^{N}\sum_{k=1}^{N} \xi_j \xi_k \int_0^1 \partial_j \partial_k \widehat{\Phi_0}(u\xi)(1-u)du. \tag{11}$$

In order to prove Lemma 6, we begin by proving

Step 1.

$$\widehat{\Phi_0}(0) = 0.$$

Indeed, we have for every $R \ge 4R_0$,

$$\widehat{\Phi_0}(0) = \int_{\mathbb{R}^N} \Phi_0 = \int_{B(0,R)} \Phi_0.$$

Therefore, by equation (10) and by some fastidious integrations by parts, we get

$$\begin{aligned}
\widehat{\Phi_0}(0) = & \int_{B(0,R)} \psi(-\Delta^2\eta + 2\Delta\eta - c^2\partial_{1,1}^2\eta - \Delta F - 2c\sum_{j=1}^{N} \partial_1 \partial_j G_j) \\
& + \int_{S(0,R)} (\partial_\nu \Delta\eta + c^2\nu_1\partial_1\eta - 2\partial_\nu\eta + \partial_\nu F + 2c\sum_{j=1}^{N} \nu_1 \partial_j G_j) \\
= & \int_{S(0,R)} (\partial_\nu \Delta\eta + c^2\nu_1\partial_1\eta - 2\partial_\nu\eta + \partial_\nu F + 2c\sum_{j=1}^{N} \nu_1 \partial_j G_j).
\end{aligned}$$

We then take a positive regular function ϕ_R with compact support such that

$$\phi_R = 1 \text{ on } B(0,R)$$

and,

$$\int_{\mathbb{R}^N} |\nabla \phi_R| \le A,$$

where A is a real number independant of R.

In order to construct such a function, we take a regular function

$$f : \mathbb{R} \mapsto [0, 1]$$

such that

$$\begin{cases} f = 0 \text{ on } \mathbb{R}_- \\ f = 1 \text{ on } [1, +\infty[, \end{cases}$$

and, we fix some strictly positive real number λ: we then denote

$$\forall x \in \mathbb{R}^N, \phi_R(x) = f\left(\frac{|x| - R}{\lambda R}\right).$$

This function satisfies all the properties mentionned above and we can compute:

$$\int_{\mathbb{R}^N} |\nabla \phi_R(x)| dx \leq A_N \lambda^{N-1} R^{2N-2} (\lambda + 1)^{N-1} \int_0^1 |f'(u)| du.$$

Therefore, if we take $\lambda = \frac{1}{R^2}$, the L^1-norm of the gradient of ϕ_R is bounded independantly of R for R sufficiently large, and we have constructed such a desired function.

By multiplying it with equation (5), we get

$$\int_{B(0,R)^c} \phi_R(\Delta^2 \eta - 2\Delta \eta + c^2 \partial_{1,1}^2 \eta + \Delta F + 2c \sum_{j=1}^N \partial_1 \partial_j G_j) =$$

$$\int_{B(0,R)^c} (-\nabla \phi_R . \nabla \Delta \eta + 2 \nabla \phi_R . \nabla \eta - \nabla \phi_R . \nabla F - c^2 \partial_1 \phi_R \partial_1 \eta - 2c \sum_{j=1}^N \partial_1 \phi_R \partial_j G_j) + \widehat{\Phi_0}(0),$$

so, by Holder inequality,

$$\begin{aligned} |\widehat{\Phi_0}(0)| \leq A(&\|\nabla \Delta \eta\|_{L^\infty(B(0,R)^c)} + 2\|\nabla \eta\|_{L^\infty(B(0,R)^c)} + \|\nabla F\|_{L^\infty(B(0,R)^c)} \\ &+ c^2 \|\partial_1 \eta\|_{L^\infty(B(0,R)^c)} + 2c \sum_{j=1}^N \|\partial_j G_j\|_{L^\infty(B(0,R)^c)}). \end{aligned}$$

By Lemma 5, we know that the right term tends to 0 when R tends to infinity and we can conclude that

$$\widehat{\Phi_0}(0) = 0.$$

We also prove

Step 2.

$$\forall 1 \leq j \leq N, \partial_j \widehat{\Phi_0}(0).$$

We have likewise

$$\partial_j \widehat{\Phi_0}(0) = -\int_{\mathbb{R}^N} i x_j \Phi_0(x) dx = -\int_{B(0,R)} i x_j \Phi_0(x) dx,$$

and, by some fastidious integration by parts,

$$\partial_j\widehat{\Phi_0}(0) = -i\int_{B(0,R)} x_j\psi(-\Delta^2\eta + 2\Delta\eta - c^2\partial_{1,1}^2\eta - \Delta F - 2c\sum_{k=1}^N \partial_1\partial_k G_k)$$

$$- i\int_{S(0,R)} (2\eta\nu_j - 2x_j\partial_\nu\eta - F\nu_j + x_j\partial_\nu F - c^2\delta_{j,1}\eta\nu_1 + c^2 x_j\nu_1\partial_1\eta + x_j\partial_\nu\Delta\eta$$

$$- \partial_j\partial_\nu\eta - 2c\delta_{j,1}\sum_{k=1}^N \nu_k G_k + 2c\sum_{k=1}^N \partial_k G_k\nu_1 x_j)$$

$$= -i\int_{S(0,R)} (2\eta\nu_j - 2x_j\partial_\nu\eta - F\nu_j + x_j\partial_\nu F - c^2\delta_{j,1}\eta\nu_1 + c^2 x_j\nu_1\partial_1\eta + x_j\partial_\nu\Delta\eta$$

$$- \partial_j\partial_\nu\eta - 2c\delta_{j,1}\sum_{k=1}^N \nu_k G_k + 2c\sum_{k=1}^N \partial_k G_k\nu_1 x_j).$$

We then also construct a positive regular function ϕ_R with compact support such that

$$\phi_R = 1 \text{ on } B(0,R),$$

but, such that

$$\int_{\mathbb{R}^N} |x||\nabla\phi_R(x)|dx \leq A,$$

where A is a real number independant of R: we can construct such a function by adaptating the previous construction.

By multiplying the function $x_j\phi_R$ with equation (5), we also get

$$\int_{B(0,R)^c} x_j\phi_R(\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta + \Delta F + 2c\sum_{k=1}^N \partial_1\partial_k G_k)$$

$$= \int_{B(0,R)^c} (-x_j\nabla\phi_R.\nabla\Delta\eta + \partial_j\nabla\eta.\nabla\phi_R + 2x_j\nabla\phi_R.\nabla\eta - 2\eta\partial_j\phi_R - c^2 x_j\partial_1\phi_R\partial_1\eta$$

$$+ c^2\delta_{1,j}\eta\partial_1\phi_R - x_j\nabla\phi_R.\nabla F + \partial_j\phi_R F - 2c\sum_{k=1}^N x_j\partial_1\phi_R\partial_k G_k + 2c\delta_{j,1}\sum_{k=1}^N \partial_k\phi_R G_k)$$

$$- i\partial_j\widehat{\Phi_0}(0),$$

and so, by Holder inequality,

$$|\partial_j\widehat{\Phi_0}(0)| \leq A(\|\nabla\Delta\eta\|_{L^\infty(B(0,R)^c)} + \|\partial_j\nabla\eta\|_{L^\infty(B(0,R)^c)} + 2\|\nabla\eta\|_{L^\infty(B(0,R)^c)}$$

$$+ 2\|\eta\|_{L^\infty(B(0,R)^c)} + c^2\|\partial_1\eta\|_{L^\infty(B(0,R)^c)} + c^2\|\eta\|_{L^\infty(B(0,R)^c)} + \|\nabla F\|_{L^\infty(B(0,R)^c)}$$

$$+ \|F\|_{L^\infty(B(0,R)^c)} + 2c\sum_{k=1}^N(\|\partial_k G_k\|_{L^\infty(B(0,R)^c)} + \|G_k\|_{L^\infty(B(0,R)^c)})).$$

By Lemma 5, we also know that the right term tends to 0 when R tends to infinity and we can conclude that

$$\partial_j\widehat{\Phi_0}(0) = 0.$$

Equation (11) then gives

$$\forall\xi\in\mathbb{R}^N, \widehat{\Phi_0}(\xi) = \sum_{j=1}^N\sum_{k=1}^N \xi_j\xi_k\widehat{P_{j,k}}(\xi),$$

where we denote

$$\widehat{P_{j,k}}(\xi) = \int_0^1 \partial_j\partial_k\widehat{\Phi_0}(u\xi)(1-u)du.$$

The inverse Fourier transformation leads to

$$\forall x \in \mathbb{R}^N, P_{j,k}(x) = -\int_0^1 (1-u)\Phi_0(\frac{x}{u})u^{-N-2}x_jx_kdu,$$

and, it only remains to prove

Step 3.

$$\forall 1 \leq p < \frac{N}{N-1}, P_{j,k} \in L^p(\mathbb{R}^N).$$

Indeed, we have the estimate

$$\forall x \in \mathbb{R}^N, |P_{j,k}(x)| \leq |x|^2 \int_0^1 |\Phi_0(\frac{x}{u})|u^{-N-2}du \leq A|x|^2 \int_{\frac{|x|}{3R_0}}^1 u^{-N-2}du \leq A|x|^{1-N},$$

and, since $P_{j,k}$ has a compact support included in the ball $B(0,3R_0)$,

$$\forall 1 \leq p < \frac{N}{N-1}, P_{j,k} \in L^p(\mathbb{R}^N),$$

which ends the proof of Lemma 6. $\square$

Finally, by equation (9), we get the following proposition:

Proposition 2. *The function $\breve{\eta}$ satisfies*

$$\Delta^2\breve{\eta} - 2\Delta\breve{\eta} + c^2\partial_{1,1}^2\breve{\eta} = -\Delta\breve{F} - 2c\sum_{j=1}^N \partial_1\partial_j\breve{G}_j - \sum_{j=1}^N\sum_{k=1}^N \partial_j\partial_k P_{j,k}. \tag{12}$$

Now, we are going to write the same equation for the variable $\nabla\theta$: by equations (4), we can write at infinity for every $1 \leq j \leq N$,

$$\Delta\partial_j\theta = \frac{c}{2}\partial_1\partial_j\eta + \sum_{k=1}^N \partial_j\partial_k G_k.$$

Using the same notations as for the variable η, we obtain the equation:

$$\Delta\partial_j\breve{\theta} = \frac{c}{2}\partial_1\partial_j\breve{\eta} + \sum_{k=1}^N \partial_j\partial_k\breve{G}_k + \Phi_j, \tag{13}$$

where we denote

$$\begin{aligned}\Phi_j &= 2\nabla\psi.\partial_j\nabla\theta + \Delta\psi\partial_j\theta - \frac{c}{2}\eta\partial_j\partial_1\psi - \frac{c}{2}\partial_j\psi\partial_1\eta \\ &\quad - \sum_{k=1}^N (G_k\partial_j\partial_k\psi + \partial_j\psi\partial_k G_k + \partial_k\psi\partial_j G_k).\end{aligned} \tag{14}$$

To proceed further, we also study the function Φ_j:

Lemma 7. *Φ_j is a regular function with a compact support included in the annulus*

$$\mathcal{A} = B(0, 3R_0) \setminus B(0, 2R_0),$$

which satisfies

$$\Phi_j = -\sum_{k=1}^{N}\sum_{l=1}^{N} \partial_k \partial_l Q_{k,l}^j,$$

where the functions $Q_{k,l}^j$ given by

$$\forall x \in \mathbb{R}^N, Q_{k,l}^j(x) = -\int_0^1 (1-u)\Phi_j(\frac{x}{u})u^{-N-2}x_k x_l du,$$

belong to all the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p < \frac{N}{N-1}$.

Proof. This proof is identical to the proof of Lemma 6: the only difference is the use of equations (6) and (14) instead of equations (5) and (10). So, we omit it. □

Finally, by equation (13), we deduce the following proposition:

Proposition 3. *For every $1 \leq j \leq N$, the function $\partial_j \breve{\theta}$ satisfies*

$$\Delta \partial_j \breve{\theta} = \frac{c}{2}\partial_1 \partial_j \breve{\eta} + \sum_{k=1}^{N} \partial_j \partial_k \breve{G}_k - \sum_{k=1}^{N}\sum_{l=1}^{N} \partial_l \partial_k Q_{k,l}^j. \tag{15}$$

In order to study those equations, we transform them in convolution equations:

Theorem 9. *The functions $\breve{\eta}$ and $\nabla\breve{\theta}$ satisfy*

$$\begin{aligned}
\breve{\eta} &= K_0 * \breve{F} + 2c\sum_{j=1}^{N} K_{1,j} * \breve{G}_j + \sum_{j=1}^{N}\sum_{k=1}^{N} K_{j,k} * P_{j,k} \\
\partial_j \breve{\theta} &= \frac{c}{2}K_{1,j} * \breve{F} + c^2\sum_{k=1}^{N} L_{1,j,k} * \breve{G}_k + \sum_{k=1}^{N} R_{j,k} * \breve{G}_k + \frac{c}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} L_{j,k,l} * P_{k,l} \\
&+ \sum_{k=1}^{N}\sum_{l=1}^{N} R_{k,l} * Q_{k,l}^j,
\end{aligned} \tag{16}$$

where K_0, $K_{j,k}$, $L_{j,k,l}$ and $R_{j,k}$ are the kernels of Fourier transformation,

$$\begin{aligned}
\widehat{K_0}(\xi) &= \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2} \\
\widehat{K_{j,k}}(\xi) &= \frac{\xi_j \xi_k}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2} \\
\widehat{L_{j,k,l}}(\xi) &= \frac{\xi_1 \xi_j \xi_k \xi_l}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)} \\
\widehat{R_{j,k}}(\xi) &= \frac{\xi_j \xi_k}{|\xi|^2}.
\end{aligned} \tag{17}$$

Though those equations look rather involved than the initial ones, they simplify a lot the study of the regularity and of the decay of v as we will see in the next part.

1.3 L^p-regularity for $1 < p < 2$

In order to complete the proof of Theorem 4, we establish the following proposition.

Proposition 4. *∇v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

Proof. This proof is adapted from an article of F. Béthuel and J.C. Saut [3] and based on the study of equations (16). Thus, we begin by studying the L^p-regularity of F and G:

Step 1. *F and G belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$.*

At the neighborhood of infinity, we know by Lemma 5 that

$$\frac{1}{2} \leq \rho \leq \frac{3}{2}.$$

Because

$$|\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2,$$

and because Proposition 1 asserts that η and ∇v belong to $L^2(\mathbb{R}^N)$, F and G are in $L^1(\mathbb{R}^N)$. Since Proposition 1 also asserts that η and ∇v belong to $L^\infty(\mathbb{R}^N)$, F and G are in $L^\infty(\mathbb{R}^N)$, and therefore, in all the spaces $L^p(\mathbb{R}^N)$ for $1 \leq p \leq +\infty$.

By iterating this process, we can establish that F and G are in all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$.

We then study the Gross-Pitaevskii kernels K_0, $K_{j,k}$, $L_{j,k,l}$ and $R_{j,k}$:

Step 2. *The kernels K_0, $K_{j,k}$, $L_{j,k,l}$ and $R_{j,k}$ are L^p-multipliers for every $1 < p < +\infty$.*

This step follows from Lizorkin theorem [12].

Lizorkin theorem. *Let $\widehat{K}$ a regular bounded function on $\mathbb{R}^N \setminus \{0\}$ which satisfies,*

$$\prod_{j=1}^{N} (\xi_j^{k_j}) \partial_1^{k_1} \dots \partial_N^{k_N} \widehat{K}(\xi) \in L^\infty(\mathbb{R}^N)$$

as soon as $(k_1, \dots, k_N) \in \{0,1\}^N$ satisfies

$$0 \leq \sum_{j=1}^{N} k_j \leq N.$$

Then, K is a L^p-multiplier for every $1 < p < +\infty$.

The functions K_0, $K_{j,k}$ and $L_{j,k,l}$ satisfy all the hypothesis of this theorem, and therefore, are L^p-multipliers for every $1 < p < +\infty$.

By standard Riesz operator theory, the functions $R_{j,k}$ are L^p-multipliers too.

By Steps 1 and 2, Lemmas 6 and 7, and Theorem 9, it follows that η and $\nabla \theta$ belong to $L^p(\mathbb{R}^N)$ for every $1 < p < \frac{N}{N-1}$, and by interpolation, for every $1 < p < 2$.

We can then iterate this process for all the derivatives of η and $\nabla \theta$: the only technical point is the use of the function ψ.

For example, for the function $\nabla \eta$, we do not derivate equation (12), but we derivate the equation

$$\Delta^2 \eta - 2\Delta \eta + c^2 \partial_{1,1}^2 \eta = -\Delta F - 2c \sum_{j=1}^{N} \partial_1 \partial_j G_j,$$

in order to obtain an equation of the form

$$\Delta^2(\psi\nabla\eta) - 2\Delta(\psi\nabla\eta) + c^2\partial_{1,1}^2(\psi\nabla\eta) = -\Delta(\psi\nabla F) - 2c\sum_{j=1}^{N}\partial_1\partial_j(\psi\nabla G_j) + \Psi_0$$

where

$$\Psi_0 = -\sum_{j=1}^{N}\sum_{k=1}^{N}\partial_j\partial_k S_{j,k}$$

satisfies analogous properties to Φ_0.

By using Proposition 1, we can perform exactly the same proof as for the functions η and $\nabla\theta$. Iterating this process, we then conclude that η and $\nabla\theta$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < +\infty$. Since $\eta = 1 - \rho^2$, ρ is in all the spaces $W^{k,\infty}(\mathbb{R}^N)$ for $k \in \mathbb{N}$, and,

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2,$$

Proposition 4 is proved as well as Theorem 4. □

Remark 2. In this part, we have also proved that the function η is in all the L^p-spaces for $1 < p \leq +\infty$.

2 Linear estimates for the Gross-Pitaevskii kernels

In this part, we study the algebraic decay of the kernels associated to the Gross-Pitaevskii equation K_0, $K_{j,k}$, $L_{j,k,l}$ and $R_{j,k}$, ie the exponents α for which the functions $|.|^\alpha K_0$, $|.|^\alpha K_{j,k}$, ... are bounded on the entire space. In particular, this will characterize all the L^p-spaces to which those kernels belong.

2.1 Inequality L^1-L^∞ and general estimates

We first estimate the kernels K_0, $K_{j,k}$ and $L_{j,k,l}$ by an L^1-L^∞ argument, essentially summed up in Lemma 3: these estimates are not optimal and we will see in the next section how to complete them. However, for sake of completeness, let us start by proving Lemma 3, which is presummably well-known to the experts.

Lemma 3. *The following equality holds*

$$\forall x \in \mathbb{R}^N, |x|^s f(x) = I_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z) - \hat{f}(y)}{|z-y|^{N+s}} e^{ix.y} dy dz$$

for every $f \in S(\mathbb{R}^N)$, and $0 < s < 1$, where we denote

$$I_N = ((2\pi)^{N+1}\int_0^{+\infty}(J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})}u^{\frac{N}{2}-1})u^{-\frac{N}{2}-s}du)^{-1},$$

$J_{\frac{N}{2}-1}$ being the Bessel function defined by

$$\forall u \in \mathbb{R}, J_{\frac{N}{2}-1}(u) = \left(\frac{u}{2}\right)^{\frac{N}{2}-1}\sum_{n=0}^{+\infty}\frac{(-1)^n u^{2n}}{4^n n!\Gamma(n+\frac{N}{2})}.$$

Proof. We fix $x \in \mathbb{R}^N$ and we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z)-\hat{f}(y)}{|z-y|^{N+s}} e^{ix.y} dydz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(y+t)-\hat{f}(y)}{|t|^{N+s}} e^{ix.y} dydt$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\sigma) e^{i(x.y-\sigma.y)} \frac{e^{-it.\sigma}-1}{|t|^{N+s}} dydtd\sigma.$$

We then compute

$$\int_{\mathbb{R}^N} \frac{e^{-it.\sigma}-1}{|t|^{N+s}} dt$$

by a general formula for the Fourier transformation of radial functions (See for example [15]):

$$\int_{\mathbb{R}^N} \frac{e^{-it.\sigma}-1}{|t|^{N+s}} d\sigma = 2\pi \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi r|\sigma|) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} (r|\sigma|)^{\frac{N}{2}-1}) r^{-s-\frac{N}{2}} |\sigma|^{1-\frac{N}{2}} dr$$
$$= 2\pi|\sigma|^s \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}) u^{-\frac{N}{2}-s} du.$$

So, if we denote

$$A_N = 2\pi \int_0^{+\infty} (J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}) u^{-\frac{N}{2}-s} du < 0,$$

we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z)-\hat{f}(y)}{|z-y|^{N+s}} e^{ix.y} dydz = A_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\sigma)|\sigma|^s e^{i(x.y-\sigma.y)} d\sigma dy$$
$$= A_N \int_{\mathbb{R}^N} \widehat{|.|^s f}(y) e^{iy.x} dy$$
$$= (2\pi)^N A_N f(x)|x|^s.$$

which gives the desired result. □

Remark 3. This lemma is not only valid for functions which are in $S(\mathbb{R}^N)$, but, by a standard argument of density, it can be generalized to all the measurable functions whose Fourier transformation satisfies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}} dydz < +\infty.$$

Moreover, in this case, the function $|.|^s f$ is not only bounded on $\mathbb{R}^N$, but, tends to 0 at infinity: we will see later that it is an important difference with the second kind of estimates we will prove.

By Lemma 3, we are now able to prove L^∞-estimates about the Gross-Pitaevskii kernels:

Theorem 5. *Let $N-2 < \alpha < N$, $n \in \mathbb{N}$, and $(j,k,l) \in [\![1,N]\!]^3$. The functions $d^n K_0$, $d^n K_{j,k}$ and $d^n L_{j,k,l}$ belong to $M^\infty_{\alpha+n}(\mathbb{R}^N)$.*

Proof. We first summarize some properties of the functions K_0, $K_{j,k}$, $L_{j,k,l}$ and of their derivatives.

Step 1. *Let $(n,p)\in\mathbb{N}^2$. Denoting f, either the function $d^p\widehat{d^nK_0}$, $d^p\widehat{d^nK_{j,k}}$ or, $d^p\widehat{d^nL_{j,k,l}}$, f is a rationnal fraction on $\mathbb{R}^N$, whose denominator only vanishes in 0 and such that*

$$|.|^{p-n}f\in L^\infty(B(0,1)),$$

and,

$$|.|^{p-n+2}f\in L^\infty(B(0,1)^c).$$

This step can be proved by a straightforward recursive argument, based on formulas (17), so we omit it.

Remark 4. Thanks to this first step, we realize that the behaviour of all those kernels is identical, and in order to simplify the proof, we will focus on the function d^nK_0.

We first notice that the functions $d^{N-1+n}\widehat{d^nK_0}$ belong to $L^1(\mathbb{R}^N)$, so, by the standard L^1-L^∞ inequality, the functions $|.|^{N-1+n}d^nK_0$ are bounded on the whole space.

In order to prove the other estimates, we then prove:

Step 2. *Let $s\in]0,1[$, and $n\in\mathbb{N}$. The functions*

$$|.|^{N-2+s+n}d^nK_0$$

are bounded on $\mathbb{R}^N$.

Indeed, we use Lemma 3 with the function $\widehat{f}$ equal to $d^{N-2+n}\widehat{d^nK_0}$. Thus, we compute

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat f(z)-\hat f(y)|}{|z-y|^{N+s}}dydz &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}dydt\\ &= \int_{\mathbb{R}^N}\int_{|t|\le 1}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}dydt\\ &+\int_{|t|>1}\int_{|y|>2|t|}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}dydt\\ &+\int_{|t|>1}\int_{|y|\le 2|t|}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}dydt.\end{aligned}$$

For the first integral, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{|t|\le 1}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}dydt &\le \int_0^1\int_{\mathbb{R}^N}\int_{|t|\le 1}\frac{|\nabla\hat f(y+\sigma t)|}{|t|^{N+s-1}}dydtd\sigma\\ &\le \int_{\mathbb{R}^N}|\nabla\widehat{f}(z)|dz\int_{|t|\le 1}\frac{dt}{|t|^{N+s-1}}\\ &\le A\int_{\mathbb{R}^N}|d^{N-1}\widehat{K_0}(\xi)|d\xi<+\infty\end{aligned}$$

and, for the second one,

$$\begin{aligned}\int_{|t|>1}\int_{|y|>2|t|}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}dydt &\le \int_0^1\int_{|t|>1}\int_{|y|>2|t|}\frac{|\nabla\hat f(y+\sigma t)|}{|t|^{N+s-1}}dydtd\sigma\\ &\le A\int_0^1\int_{|t|>1}\int_{|y|>2|t|}\frac{dt}{|t|^{N+s-1}}\frac{dyd\sigma}{|y+\sigma t|^{N+1}}\\ &\le A\int_{|t|>1}\int_{|y|>2|t|}\frac{dt}{|t|^{N+s-1}}\frac{dy}{(|y|-|t|)^{N+1}}\\ &\le A\int_{|t|>1}\frac{dt}{|t|^{N+s}}\int_{|u|>2}\frac{du}{(|u|-1)^{N+1}}<+\infty\end{aligned}$$

and, for the last one,

$$\int_{|t|>1}\int_{|y|\le 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt \le 2\int_{|t|>1}\frac{dt}{|t|^{N+s}}\int_{|y|\le 3|t|}|\widehat{f}(y)|dy$$
$$\le A\int_{|t|>1}\frac{dt}{|t|^{N+s}}\left(\int_{|y|\le 1}\frac{dy}{|y|^{N-2}}+\int_{1<|y|\le 3|t|}\frac{dy}{|y|^{N}}\right)$$
$$\le A\int_{|t|>1}\int_{|y|\le 1}\frac{dt}{|t|^{N+s}}\frac{dy}{|y|^{N-2}}+A\int_{|t|>1}\frac{ln(3|t|)}{|t|^{N+s}}dt$$
$$< +\infty.$$

Thus, we can conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz < +\infty,$$

and, by Lemma 3, $|.|^{N-2+s+n}d^nK_0$ is bounded on $\mathbb{R}^N$ for every $0<s<1$.

We finally complete this proof by the next similar step.

Step 3. *Let $s\in]0,1[$, and $n\in\mathbb{N}$. The functions*

$$|.|^{N-1+s+n}d^nK_0$$

are bounded on $\mathbb{R}^N$.

For this step, we also use Lemma 3, but, for the function

$$\hat{f}=d^{N-1+n}\widehat{d^nK_0}.$$

Likewise, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt$$
$$=\int_{\mathbb{R}^N}\int_{|t|\ge 1}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt$$
$$+\int_{|t|<1}\int_{|y|\le 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt$$
$$+\int_{|t|<1}\int_{|y|>2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt.$$

For the first integral, we have similarly

$$\int_{\mathbb{R}^N}\int_{|t|\ge 1}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt \le 2\int_{\mathbb{R}^N}|\hat{f}(z)|dz\int_{|t|\ge 1}\frac{dt}{|t|^{N+s}}$$
$$\le \int_{\mathbb{R}^N}|d^{N-1}\widehat{K_0}(z)|dz\int_{|t|\ge 1}\frac{dt}{|t|^{N+s}} < +\infty$$

for the second one,

$$\int_{|t|<1}\int_{|y|\le 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt \le 2\int_{|t|<1}\int_{|y|\le 3|t|}|\hat{f}(y)|dy\frac{dt}{|t|^{N+s}}$$
$$\le A\int_{|t|<1}\int_{|y|\le 3|t|}\frac{dt}{|t|^{N+s}}\frac{dy}{|y|^{N-1}}$$
$$\le A\int_{|t|<1}\int_{|u|\le 3}\frac{dt}{|t|^{N+s-1}}\frac{du}{|u|^{N-1}} < +\infty$$

and, for the last one,

$$\begin{aligned}
\int_{|t|<1}\int_{|y|>2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}dydt &\leq \int_0^1\int_{|t|<1}\int_{|y|>2|t|}\frac{dt}{|t|^{N+s-1}}|\nabla\widehat{f}(y+\sigma t)|dyd\sigma\\
&\leq A\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\Big(\int_{2>|y|>2|t|}\frac{dy}{(|y|-|t|)^N}\\
&+\int_{|y|>2}\frac{dy}{(|y|-|t|)^{N+2}}\Big)\\
&\leq A\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\int_2^{\frac{2}{|t|}}\frac{u^{N-1}}{(u-1)^N}du\\
&+A\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\int_{|y|>2}\frac{dy}{(|y|-1)^{N+2}}\\
&\leq A\int_{|t|<1}\frac{|\ln(t)|dt}{|t|^{N+s-1}}\\
&+A\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\int_{|y|>2}\frac{dy}{(|y|-1)^{N+2}}\\
&<+\infty.
\end{aligned}$$

Thus, we can also conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}dydz<+\infty,$$

and that, by Lemma 3, $|.|^{N+s-1+n}d^nK_0$ is bounded on $\mathbb{R}^N$ for every $0<s<1$, which completes the proofs of this last step and of Theorem 5. □

Remark 5. The key ingredient of those proofs is the form of the kernels Fourier transformation $\widehat{K}$:

- $\widehat{K}$ is a rationnal fraction.
- $\widehat{K}$ is only singular at the origin, where the singularity is of the form $\underset{\xi\to 0}{O}(\frac{1}{|\xi|^\alpha})$.
- At infinity, $\widehat{K}$ is of the form $\underset{|\xi|\to+\infty}{O}(\frac{1}{|\xi|^\beta})$, where $\beta>\alpha$.

Thus, we can perform the same proof for all the kernels whose Fourier transformation satisfies such assumptions in order to get their algebraic decay.

Before improving those first estimates, we can prove the next corollary which will not be improved in the following:

Corollary 2. *Let $(j,k,l)\in[\![1,N]\!]^3$. The functions K_0, $K_{j,k}$ and $L_{j,k,l}$ belong to all the spaces $L^p(\mathbb{R}^N)$ for*

$$1<p<\frac{N}{N-2},$$

and their gradient to all the spaces $L^p(\mathbb{R}^N)$ for

$$1\leq p<\frac{N}{N-1}.$$

Proof. This follows from the estimates of Theorem 5. □

2.2 Critical estimates

Actually, we can improve Theorem 5 by proving:

Theorem 6. *Let $n \in \mathbb{N}$, and $(j,k,l) \in [\![1,N]\!]^3$. The functions d^nK_0, $d^nK_{j,k}$ and $d^nL_{j,k,l}$ belong to $M^\infty_{N+n}(\mathbb{R}^N)$.*

This theorem seems very similar to Theorem 5, but its proof is quite different: in this case, we conjecture that the functions $|.|^{N+n}d^nK_0$, $|.|^{N+n}d^nK_{j,k}$ and $|.|^{N+n}d^nL_{j,k,l}$ are bounded on $\mathbb{R}^N$, but does not tend to 0 at infinity. We cannot hope to prove it from a general inequality deduced by using the density of regular functions, because it will mean that those functions tend to 0 at infinity.

Moreover, those estimates are supposed to be critical, because we also conjecture that the functions $|.|^{\alpha+n}d^nK_0$, $|.|^{\alpha+n}d^nK_{j,k}$ and $|.|^{\alpha+n}d^nL_{j,k,l}$ are not bounded on $\mathbb{R}^N$ for $\alpha > N$.

Proof. This proof relies on the following lemma:

Lemma 8. *Let $1 \leq j \leq N$. The function*

$$x \mapsto x_jf(x)$$

is bounded on $B(0,1)^c$ for every $f \in S'(\mathbb{R}^N)$ such that $\widehat{f}$ is a regular function on $\mathbb{R}^N \setminus \{0\}$ and

(i) $(|.|^{N+1} + |.|^{N-1})\widehat{f}$ is bounded on $\mathbb{R}^N$.

(ii) $(|.|^{N+2} + |.|^{N})\partial_j\widehat{f}$ is bounded on $\mathbb{R}^N$.

(iii) $(|.|^{N+3} + |.|^{N+1})\partial_j\partial_k\widehat{f}$ are bounded on $\mathbb{R}^N$ for every $1 \leq k \leq N$.

Indeed, we begin by showing the next general formula:

Step 1. *Let $\lambda > 0$. The following equality holds almost everywhere*

$$\begin{aligned} x_jf(x) =&(2\pi)^{-N}i(\int_{B(0,\lambda)^c} \partial_j\widehat{f}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j\widehat{f}(\xi)(e^{ix.\xi}-1)d\xi \\ &+ \int_{S(0,\lambda)} \xi_j\widehat{f}(\xi)e^{ix.\xi}d\xi). \end{aligned} \tag{18}$$

Let take $g \in S(\mathbb{R}^N)$. We have

$$< x_jf, \widehat{g} >=< f, x_j\widehat{g} >= -i < f, \widehat{\partial_jg} >= -i < \widehat{f}, \partial_jg > .$$

Since $\widehat{f}$ is in $L^1(\mathbb{R}^N)$, we can write

$$< x_jf, \widehat{g} >= -i\int_{\mathbb{R}^N} \widehat{f}(\xi)\partial_jg(\xi)d\xi,$$

and, by integrating by parts, we deduce

$$\begin{aligned} < x_jf, \widehat{g} >= -i < \widehat{f}, \partial_jg > = i\int_{B(0,\lambda)^c} \partial_j\widehat{f}(\xi)g(\xi)d\xi + i\int_{B(0,\lambda)} \partial_j\widehat{f}(\xi)(g(\xi)-g(0))d\xi \\ + ig(0)\int_{S(0,\lambda)} \xi_j\widehat{f}(\xi)d\xi. \end{aligned}$$

Since g is in $S(\mathbb{R}^N)$, we can write

$$g(\xi) = (2\pi)^{-N} \int_{\mathbb{R}^N} \widehat{g}(x) e^{ix.\xi} dx,$$

and get

$$< x_j f, \widehat{g} > = (2\pi)^{-N} i \int_{\mathbb{R}^N} \widehat{g}(x) (\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi$$
$$+ \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi) dx.$$

As the function

$$x \mapsto \int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi + \int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) d\xi$$

belongs to $L^1_{loc}(\mathbb{R}^N)$, by standard duality, we conclude that formula (18) is valid almost everywhere.

To proceed further, we evaluate each integral which appears in formula (18).

Step 2. *The following inequalities hold for every $x \in \mathbb{R}^N$ and $\lambda > 0$*

$$\begin{cases} |\int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi| \leq A\lambda |x| \\ |\int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) e^{ix.\xi} d\xi| \leq A\lambda, \end{cases}$$

where A is a real number independant of x and λ.

Indeed, on one hand, we know that

$$\forall u \in \mathbb{R}, |e^{iu} - 1| \leq A|u|,$$

and therefore,

$$|\int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi| \leq A|x| \int_{B(0,\lambda)} |\partial_j \widehat{f}(\xi)||\xi| d\xi.$$

By assumption (ii), we get

$$|\int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi| \leq A|x| \int_{B(0,\lambda)} \frac{d\xi}{|\xi|^{N-1}} \leq A\lambda |x|.$$

On the other hand, we deduce by assumption (i),

$$|\int_{S(0,\lambda)} \xi_j \widehat{f}(\xi) e^{ix.\xi} d\xi| \leq A \int_{S(0,\lambda)} \frac{d\xi}{|\xi|^{N-2}} \leq A\lambda.$$

It remains only a single integral to evaluate:

Step 3. *The following inequality holds for every $x \in B(0,1)^c$ and $0 < \lambda < 1$*

$$|\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi| \leq A(1 + \frac{1}{\lambda |x|}),$$

where A is a real number independant of x and λ.

Indeed, we have

$$\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi = \int_{B(0,1)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi.$$

For the first integral, by assumption (ii), we deduce

$$|\int_{B(0,1)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi| \leq \int_{B(0,1)^c} |\partial_j \widehat{f}(\xi)| d\xi \leq A.$$

For the second one, since $|x| > 1$, there is some $1 \leq k \leq N$ such that

$$|x_k| \geq \frac{|x|}{N}.$$

By integrating by parts this integral, we then get

$$\begin{aligned}
\int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi &= \frac{1}{ix_k} \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) \partial_k (e^{ix.\xi}) d\xi \\
&= \frac{1}{ix_k} (- \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \partial_k \widehat{f}(\xi) e^{ix.\xi} d\xi \\
&+ \int_{S(0,1)} \partial_j \widehat{f}(\xi) e^{ix.\xi} \xi_k d\xi \\
&- \int_{S(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} \xi_k d\xi),
\end{aligned}$$

and, by assumptions (ii) and (iii),

$$\begin{aligned}
|\int_{B(0,1)\setminus B(0,\lambda)} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi| &\leq \frac{N}{|x|} (A \int_{B(0,1)\setminus B(0,\lambda)} \frac{d\xi}{|\xi|^{N+1}} + A + A \int_{S(0,\lambda)} \frac{d\xi}{|\xi|^{N-1}}) \\
&\leq \frac{A}{\lambda |x|} + A.
\end{aligned}$$

Finally, we get

$$|\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi) e^{ix.\xi} d\xi| \leq A(1 + \frac{1}{\lambda |x|}) \leq \frac{A}{\lambda |x|} + A,$$

which is the desired result.

By Steps 1, 2 and 3, we finally get for every $x \in B(0,1)^c$ and $0 < \lambda < 1$,

$$|x_j f(x)| \leq A\lambda |x| + A\lambda + \frac{A}{\lambda |x|} + A.$$

By choosing

$$\lambda = \frac{1}{|x|},$$

we conclude the proof of Lemma 8.

In order to complete the proof of Theorem 6, we notice by the first step of the proof of Theorem 5 that the functions $d^{N-1+n} \widehat{d^n K_0}$, $d^{N-1+n} \widehat{d^n K_{j,k}}$ and $d^{N-1+n} \widehat{d^n L_{j,k,l}}$ satisfy the three assumptions of Lemma 8. Thus, by an immediate application of this lemma, we get the results of Theorem 6. $\square$

2.3 Estimates for the composed Riesz kernels

We focus next on the kernels $R_{j,k}$: as they are very simple, we know their exact expression. Thus, if f is a regular function, and if we denote $g_{j,k}$, the function defined by

$$\forall \xi \in \mathbb{R}^N, \widehat{g_{j,k}}(\xi) = \widehat{R_{j,k}}(\xi)\hat{f}(\xi),$$

we can write the exact formula

$$\begin{aligned}\forall x \in \mathbb{R}^N, g_{j,k}(x) = A_N &\int_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)dy \\ + A_N &\int_{|y|\leq 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} (f(x-y) - f(x))dy.\end{aligned}$$

Therefore, we do not have to study the decay of the kernels $R_{j,k}$ directly, and instead, we may restrict ourselves to the decay of the functions $g_{j,k}$ with suitable assumptions on f. In that context, we recall some useful facts, which are presummably well-known to the experts. For sake of completeness, we also mention the proofs.

Proposition 5. *Let f be a regular function which belongs to $L^p(\mathbb{R}^N)$ for every $p \in]1, +\infty]$, and, suppose that there is*

$$\delta \in]0, N]$$

such that for every $\beta \in [0, \delta[$,

$$\begin{cases} |.|^\beta f \in L^\infty(\mathbb{R}^N) \\ |.|^\beta \nabla f \in L^\infty(\mathbb{R}^N). \end{cases}$$

Then, the functions

$$|.|^\beta g_{j,k} \in L^\infty(\mathbb{R}^N)$$

for every $(j,k) \in [\![1, N]\!]^2$ and for every $\beta \in [0, \delta[$.

Proof. We first denote

$$\begin{aligned} g_{j,k}(x) = A_N &\int_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)dy \\ + A_N &\int_{|y|\leq 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} (f(x-y) - f(x))dy \\ = I_1&(x) + I_2(x). \end{aligned}$$

Therefore, if we fix $\beta \in [0, \delta[$, we get

$$|x|^\beta |I_1(x)| \leq A \int_{|y|>1} |x-y|^\beta |f(x-y)| \frac{dy}{|y|^N} + A \int_{|y|>1} |f(x-y)| \frac{dy}{|y|^{N-\beta}}.$$

Hence, if p is sufficiently large, we have

$$\int_{|y|>1} |f(x-y)| \frac{dy}{|y|^{N-\beta}} \leq ||f||_{L^{p'}(\mathbb{R}^N)} \left(\int_{|y|>1} \frac{dy}{|y|^{p(N-\beta)}} \right)^{\frac{1}{p}} < +\infty,$$

and, if $\beta < \delta - \epsilon$ and $|x| > 4$, then

$$\begin{aligned}
\int_{|y|>1} |x-y|^\beta |f(x-y)| \frac{dy}{|y|^N} &\leq A \int_{|y|>1} \frac{dy}{|y|^N |x-y|^\epsilon} \\
&\leq \frac{A}{|x|^\epsilon} \int_{|t|>\frac{1}{|x|}} \frac{dt}{|t|^N |\frac{x}{|x|} - t|^\epsilon} \\
&\leq \frac{A}{|x|^\epsilon} \int_{\frac{1}{|x|}<|t|<\frac{1}{2}} \frac{dt}{|t|^N} + \frac{A}{|x|^\epsilon} \int_{\frac{1}{2}<|t|<\frac{3}{2}} \frac{dt}{|\frac{x}{|x|} - t|^\epsilon} \\
&+ \frac{A}{|x|^\epsilon} \int_{|t|>\frac{3}{2}} \frac{dt}{|t|^N (|t|-1)^\epsilon} \\
&\leq \frac{A \ln |x|}{|x|^\epsilon} + A + \frac{A}{|x|^\epsilon} \int_{|t - \frac{x}{|x|}|<\frac{1}{2}} \frac{dt}{|\frac{x}{|x|} - t|^\epsilon} \\
&\leq \frac{A \ln |x|}{|x|^\epsilon} + A < +\infty,
\end{aligned}$$

and, if $|x| \leq 4$, we get

$$\int_{|y|>1} |x-y|^\beta |f(x-y)| \frac{dy}{|y|^N} \leq A \int_{1<|y|<5} \frac{dy}{|y|^N} + A \int_{|y|>5} \frac{dy}{|y|^N (|y|-4)^\epsilon} < +\infty.$$

Thus, the function $|.|^\beta I_1$ is bounded on $\mathbb{R}^N$, and likewise, we have for the function I_2:

$$|x|^\beta I_2(x) \leq A \int_{|y|\leq 1} |x-y|^\beta |f(x-y) - f(x)| \frac{dy}{|y|^N} + A \int_{|y|\leq 1} |f(x-y) - f(x)| \frac{dy}{|y|^{N-\beta}}.$$

On one hand, if $\beta < \delta - \epsilon$, we have

$$\begin{aligned}
\int_{|y|\leq 1} |x-y|^\beta |f(x-y) - f(x)| \frac{dy}{|y|^N} &\leq ||\nabla f||_{L^\infty(B(x,1))} (|x|+1)^\beta \int_{|y|\leq 1} \frac{dy}{|y|^{N-1}} \\
&\leq \frac{A}{(1+|x|)^\epsilon} < +\infty
\end{aligned}$$

and, on the other hand, we have if $\beta = 0$

$$\int_{|y|\leq 1} |f(x-y) - f(x)| \frac{dy}{|y|^N} \leq A \int_{|y|\leq 1} \frac{dy}{|y|^{N-1}} < +\infty$$

and, if $\beta > 0$, we get

$$\int_{|y|\leq 1} |f(x-y) - f(x)| \frac{dy}{|y|^{N-\beta}} \leq A \int_{|y|\leq 1} \frac{dy}{|y|^{N-\beta}}.$$

Therefore, the function $|.|^\beta I_2$ is also bounded on $\mathbb{R}^N$, such as the function $|.|^\beta g_{j,k}$. $\square$

Remark 6. In fact, we can prove a similar proposition for the Riesz kernels.

Actually, we will also make use of the next more precise proposition in the critical case, which is also presummably well-known to the experts. For sake of completeness, we also mention the proofs.

Proposition 6. *Let f be a regular function which belongs to $L^1(\mathbb{R}^N)$, and, suppose that*

$$\left\{\begin{array}{c}(1+|.|^N)f \in L^\infty(\mathbb{R}^N)\\(1+|.|^{N+1})\nabla f \in L^\infty(\mathbb{R}^N).\end{array}\right.$$

Then, the functions

$$|.|^N g_{j,k} \in L^\infty(\mathbb{R}^N)$$

for every $(j,k) \in [\![1,N]\!]^2$.

Proof. We notice that we have

$$\begin{aligned}g_{j,k}(x) &= A_N \int_{|y|>\frac{|x|}{4},|x-y|>\frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} f(x-y)dy\\&+ A_N \int_{|x-y|\leq \frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} f(x-y)dy\\&+ A_N \int_{|y|\leq \frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} (f(x-y)-f(x))dy\\&= I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

For the first integral, we have

$$\begin{aligned}|I_1(x)| &\leq A_N \int_{|y|>\frac{|x|}{4},|x-y|>\frac{|x|}{4}} \frac{dy}{|y|^N|x-y|^N}\\&\leq \frac{A_N}{|x|^N} \int_{|z|>\frac{1}{4},|\frac{x}{|x|}-z|>\frac{1}{4}} \frac{dz}{|z|^N|\frac{x}{|x|}-z|^N}\\&\leq \frac{A_N}{|x|^N} \int_{|z|>\frac{1}{4},|e_1-z|>\frac{1}{4}} \frac{dz}{|z|^N|e_1-z|^N}\\&\leq \frac{A_N}{|x|^N},\end{aligned}$$

for the second one,

$$\begin{aligned}|I_2(x)| &\leq \frac{A_N}{|x|^N} \int_{|x-y|\leq \frac{|x|}{4}} |f(x-y)|dy\\&\leq \frac{A_N}{|x|^N} \int_{|z|<\frac{|x|}{4}} |f(t)|dt\\&\leq \frac{A_N}{|x|^N},\end{aligned}$$

and, for the last one,

$$\begin{aligned}|I_3(x)| &\leq A_N \int_{|y|\leq \frac{|x|}{4}} |y|^{1-N}|x|^{-N-1}dy\\&\leq \frac{A_N}{|x|^N}.\end{aligned}$$

Finally, the function $|.|^N g_{j,k}$ is bounded on $\mathbb{R}^N$. $\square$

3 Decay at infinity

In this part, we study the algebraic decay of the functions η and $\nabla\theta$, and of some of their derivatives, by the method yet explained in the introduction, which was introduced by J.L. Bona and Yi A. Li [6], and A. de Bouard and J.C. Saut [8], and relies on the transformation of a PDE in a convolution equation (See also [14, 13] for many more details).

3.1 A refined energy estimate

We first give an energy estimate for v by arguments from F. Béthuel, G. Orlandi and D. Smets [2].

Proposition 7. *There is some real number $\alpha > 0$ such that the integral*

$$\int_{\mathbb{R}^N} |x|^\beta e(v)(x)dx$$

is finite for every $0 \leq \beta < \alpha$.

Proof. This proof relies on the following lemma proved by F. Béthuel, G. Orlandi and D. Smets [2].

Lemma 4. *The function*

$$R \to R^{\alpha_c} \int_{B(0,R)^c} e(v)$$

is bounded on $\mathbb{R}_+$ for some real number $\alpha_c > 0$.

In order to prove this lemma, by Lemma 5, we will choose R so large that ρ does not vanish on $B(0,R)^c$.

We already know that ρ and θ satisfy both the equations

$$\begin{cases} div(\rho^2\nabla\theta) = -\frac{c}{2}\partial_1\rho^2 \\ -\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2) \end{cases} \tag{4}$$

on the domain $B(0,R)^c$.

Thus, we take $\lambda > R$ and denote $\Omega = B(0,\lambda) \setminus B(0,R)$, and, $\theta_R = \frac{1}{|\mathbb{S}_R|}\int_{\mathbb{S}_R}\theta$. We begin by multiplying the first equation by $\rho^2 - 1$, which gives by integrating by parts,

$$\begin{aligned} 2\int_\Omega \rho|\nabla\rho|^2 - \int_{\mathbb{S}_\lambda} \partial_\nu\rho(\rho^2-1) + \int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) \\ + \int_\Omega \rho(\rho^2-1)|\nabla\theta|^2 + c\int_\Omega \rho(\rho^2-1)\partial_1\theta = \int_\Omega \rho(\rho^2-1)^2. \end{aligned}$$

We then know that $\partial_\nu\rho(\rho^2-1)$ belongs to $L^1(\mathbb{R}^N)$, so, we can construct an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$, and such that $\int_{\mathbb{S}_{\lambda_n}} \partial_\nu\rho(\rho^2-1)$ tends to 0: by taking the limit at infinity in the previous equality, we get

$$\begin{aligned} 2\int_{B(0,R)^c} \rho|\nabla\rho|^2 + \int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) + \int_{B(0,R)^c} \rho(\rho^2-1)|\nabla\theta|^2 \\ + c\int_{B(0,R)^c} \rho(\rho^2-1)\partial_1\theta = \int_{B(0,R)^c} \rho(\rho^2-1)^2. \end{aligned}$$

We also get such a result by multiplying the second equation by $\theta-\theta_R$, and, by integrating by parts,

$$\int_\Omega \rho^2|\nabla\theta|^2-\int_{\mathbb{S}_\lambda}\rho^2\partial_\nu\theta(\theta-\theta_R)+\int_{\mathbb{S}_R}\rho^2\partial_\nu\theta(\theta-\theta_R)$$
$$=-\frac{c}{2}\int_\Omega(\rho^2-1)\partial_1\theta+\frac{c}{2}\int_{\mathbb{S}_\lambda}(\rho^2-1)\nu_1(\theta-\theta_R)-\frac{c}{2}\int_{\mathbb{S}_R}(\rho^2-1)\nu_1(\theta-\theta_R).$$

By Theorem 4, we know that $\nabla\theta$ and $1-\rho^2$ belong to $L^{\frac{N}{N-1}}(\mathbb{R}^N)$. Since

$$\begin{cases} |\int_{\mathbb{S}_\lambda}\rho^2\partial_\nu\theta(\theta-\theta_R)|\leq A\int_{\mathbb{S}_\lambda}|\partial_\nu\theta|\leq A(\lambda\int_{\mathbb{S}_\lambda}|\partial_\nu\theta|^{\frac{N}{N-1}})^{\frac{N-1}{N}} \\ |\int_{\mathbb{S}_\lambda}(\rho^2-1)\nu_1(\theta-\theta_R)|\leq A\int_{\mathbb{S}_\lambda}|1-\rho^2|\leq A(\lambda\int_{\mathbb{S}_\lambda}|1-\rho^2|^{\frac{N}{N-1}})^{\frac{N-1}{N}}, \end{cases}$$

we can also construct an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$, and such that

$$\lambda_n\int_{\mathbb{S}_{\lambda_n}}(|\nabla\theta|^{\frac{N}{N-1}}+|1-\rho^2|^{\frac{N}{N-1}})\underset{n\to\infty}{\to}0,$$

and get

$$\int_{B(0,R)^c}\rho^2|\nabla\theta|^2+\int_{\mathbb{S}_R}\rho^2\partial_\nu\theta(\theta-\theta_R)=-\frac{c}{2}\int_{B(0,R)^c}(\rho^2-1)\partial_1\theta-\frac{c}{2}\int_{\mathbb{S}_R}(\rho^2-1)\nu_1(\theta-\theta_R).$$

By adding the previous equality, it gives

$$\begin{aligned}\int_{B(0,R)^c}e(v)=&-\frac{c}{2}\int_{B(0,R)^c}\rho(\rho^2-1)\partial_1\theta-\frac{1}{2}\int_{\mathbb{S}_R}\rho^2\partial_\nu\theta(\theta-\theta_R)\\ &-\frac{c}{4}\int_{\mathbb{S}_R}(\theta-\theta_R)(\rho^2-1)\nu_1+\int_{B(0,R)^c}(1-\rho)(\frac{|\nabla\rho|^2}{2}+\frac{(1-\rho^2)^2}{4})\\ &-\frac{c}{4}\int_{B(0,R)^c}(1-\rho)(\rho^2-1)\partial_1\theta-\frac{1}{4}\int_{\mathbb{S}_R}\partial_\nu\rho(\rho^2-1)\\ &+\frac{1}{4}\int_{B(0,R)^c}\rho(1-\rho^2)|\nabla\theta|^2.\end{aligned}$$

It remains to evaluate each term in the right member of this equality. For the first one, we can write

$$|\frac{c}{2}\int_{B(0,R)^c}\rho(\rho^2-1)\partial_1\theta|\leq\frac{c}{\sqrt{2}}\int_{B(0,R)^c}(\frac{\rho^2\partial_1\theta^2}{2}+\frac{(1-\rho^2)^2}{4})\leq\frac{c}{\sqrt{2}}\int_{B(0,R)^c}e(v).$$

For the next one, we get by Sobolev-Poincaré inequality,

$$\begin{aligned}|\frac{1}{2}\int_{\mathbb{S}_R}\rho^2\partial_\nu\theta(\theta-\theta_R)|&\leq A\left(\int_{\mathbb{S}_R}\rho^2\partial\nu\theta^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}_R}(\theta-\theta_R)^2\right)^{\frac{1}{2}}\\ &\leq AR\left(\int_{\mathbb{S}_R}\rho^2\partial\nu\theta^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}_R}\partial_\nu\theta^2\right)^{\frac{1}{2}}\\ &\leq AR\int_{\mathbb{S}_R}e(v),\end{aligned}$$

and, likewise,

$$\begin{cases} |\frac{c}{4}\int_{\mathbb{S}_R}(\theta-\theta_R)(\rho^2-1)|\leq AR\int_{\mathbb{S}_R}e(v) \\ |\int_{\mathbb{S}_R}\partial_\nu\rho(\rho^2-1)|\leq A\int_{\mathbb{S}_R}e(v). \end{cases}$$

In order to estimate the other terms, we fix $\epsilon > 0$ and choose by Lemma 5, R sufficiently large such as $|\rho - 1|$ and $|\nabla\theta|$ are less than ϵ on the domain $B(0,R)^c$. For such an R, we have

$$\begin{cases} |\int_{B(0,R)^c}(1-\rho)(\frac{|\nabla\rho|^2}{2}+\frac{(1-\rho^2)^2}{4})| \leq \epsilon\int_{B(0,R)^c}e(v) \\ |\frac{c}{4}\int_{B(0,R)^c}(1-\rho)(\rho^2-1)\partial_1\theta| \leq A\epsilon\int_{B(0,R)^c}e(v) \\ |\frac{1}{4}\int_{B(0,R)^c}\rho(1-\rho^2)|\nabla\theta|^2| \leq A\epsilon\int_{B(0,R)^c}e(v) \end{cases}$$

which finally gives,

$$\int_{B(0,R)^c} e(v) \leq (\frac{c}{\sqrt{2}} + A\epsilon)\int_{B(0,R)^c} e(v) + AR\int_{\mathbb{S}_R} e(v).$$

Thus, if ϵ is sufficiently small such as

$$\frac{c}{\sqrt{2}} + A\epsilon < 1,$$

we have

$$\int_{B(0,R)^c} e(v) \leq A_c\int_{\mathbb{S}_R} e(v).$$

Denoting $J(R) = \int_{B(0,R)^c} e(v)$, we have for R sufficiently large

$$J(R) \leq -A_c R J'(R)$$

which gives

$$J(R) \leq \frac{B}{R^{\frac{1}{A_c}}}.$$

Thus, Lemma 4 is proved with $\alpha_c = \frac{1}{A_c}$.

Now, we can conclude the proof of Proposition 7: we choose $\beta \in [0, \alpha_c[$ and we compute

$$\begin{aligned}\int_{\mathbb{R}^N}|x|^\beta e(v)(x)dx &= \int_0^{+\infty} r^\beta\int_{\mathbb{S}_r} e(v)dr \\ &= -[r^\beta\int_r^{+\infty}\int_{\mathbb{S}_\rho} e(v)d\rho]_0^{+\infty} + \beta\int_0^{+\infty} r^{\beta-1}(\int_r^{+\infty}\int_{\mathbb{S}_\rho} e(v)d\rho)dr \\ &= \beta\int_0^{+\infty} r^{\beta-1}(\int_r^{+\infty}\int_{\mathbb{S}_\rho} e(v)d\rho)dr < +\infty.\end{aligned}$$

□

Remark 7. We will see in the following that this proposition is not optimal but is essential to begin the argument of the next section.

3.2 Decay of the functions η, $\nabla\eta$ and $\nabla\theta$

In order to estimate this decay, we prove the following theorem

Theorem 7. *The functions η, $\nabla\eta$ and $\nabla\theta$ are in $M_N^\infty(\mathbb{R}^N)$, $M_{N+1}^\infty(\mathbb{R}^N)$ and $M_N^\infty(\mathbb{R}^N)$ respectively.*

Proof. By Theorem 4, we first notice that the functions η, $\nabla\eta$ and $\nabla\theta$ are bounded on $\mathbb{R}^N$. Thus, we only have to prove that the functions $|.|^N\eta$, $|.|^{N+1}\nabla\eta$ and $|.|^N\nabla\theta$ are bounded at infinity, and more simply, that the functions $|.|^N\breve{\eta}$, $|.|^{N+1}\nabla\breve{\eta}$ and $|.|^N\nabla\breve{\theta}$ are bounded at infinity.

This remark done, by Proposition 7, we already know that there is a strictly positive constant α such that

$$\int_{\mathbb{R}^N} |x|^\beta e(v)(x)dx$$

is finite for every $0 \leq \beta < \alpha$. Using this result, we are going to show this first proposition.

Proposition 8. *There is some real number $\alpha > 0$ such that*

$$(\breve{\eta}, \nabla\breve{\eta}, \nabla\breve{\theta}) \in M^\infty_\beta(\mathbb{R}^N)^3$$

for every $0 \leq \beta < \alpha$.

The proof of this step relies on equations (16) of Theorem 9

$$\breve{\eta} = K_0 * \breve{F} + 2c\sum_{j=1}^N K_{1,j} * \breve{G}_j + \sum_{j=1}^N\sum_{k=1}^N K_{j,k} * P_{j,k},$$

$$\partial_j\breve{\theta} = \frac{c}{2}K_{1,j} * \breve{F} + c^2\sum_{k=1}^N L_{1,j,k} * \breve{G}_k + \sum_{k=1}^N R_{j,k} * \breve{G}_k + \frac{c}{2}\sum_{k=1}^N\sum_{l=1}^N L_{j,k,l} * P_{k,l}$$
$$+ \sum_{k=1}^N\sum_{l=1}^N R_{k,l} * Q^j_{k,l}.$$

We are going to study each term of those equations beginning by

Step 1. *Let $j \in [\![1, N]\!]$. Then,*

(1.1) $K_0 * \breve{F} \in M^\infty_\beta(\mathbb{R}^N)$

(1.2) $K_{1,j} * \breve{G}_j \in M^\infty_\beta(\mathbb{R}^N)$

for every β sufficiently small.

Indeed, we have for every $\beta > 0$, and every $x \in \mathbb{R}^N$,

$$|x|^\beta|K_0 * \breve{F}(x)| \leq A(\int_{\mathbb{R}^N} |x-y|^\beta|K_0(x-y)||\breve{F}(y)|dy + \int_{\mathbb{R}^N} |K_0(x-y)||y|^\beta|\breve{F}(y)|dy).$$

On one hand, by Theorem 6, we know that for every $\beta \in [0, N[$, there is some real number $p \in [1, +\infty]$ such that

$$|.|^\beta K_0 \in L^p(\mathbb{R}^N),$$

and, since by Theorem 4, $\breve{F}$ is in $L^{p'}(\mathbb{R}^N)$, we get by Young's inequality,

$$\|(|.|^\beta K_0) * \breve{F}\|_{L^\infty(\mathbb{R}^N)} \leq \|K_0\|_{M^p_\beta(\mathbb{R}^N)}\|\breve{F}\|_{L^{p'}(\mathbb{R}^N)} < +\infty,$$

where we denote

$$\|K_0\|_{M^p_\beta(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |K_0(y)|^p|y|^{p\beta}dy\right)^{\frac{1}{p}}.$$

On the other hand, by Proposition 7, we notice that there is some real number $\alpha > 0$ such that

$$\forall \beta \in [0, \alpha[, \int_{\mathbb{R}^N} |.|^\beta (|\breve{F}| + |\breve{G}|) < +\infty.$$

Since by Corollary 2, we have

$$K_0 \in L^q(\mathbb{R}^N)$$

for every $1 < q < \frac{N}{N-2}$, we get for every $\beta \in [0, \frac{2\alpha}{N}[$,

$$\|K_0 * (|.|^\beta \breve{F})\|_{L^\infty(\mathbb{R}^N)} \leq \|K_0\|_{L^q(\mathbb{R}^N)} \|\breve{F}\|_{M_\beta^{q'}(\mathbb{R}^N)}.$$

As there is $1 < q < \frac{N}{N-2}$ such that

$$\beta q' < \alpha,$$

and, by Lemma 5, $\breve{F}$ tends to 0 at infinity, we are led to

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |\breve{F}|^{q'} \leq A \int_{\mathbb{R}^N} |.|^{\beta q'} |\breve{F}| < +\infty,$$

and, the function $K_0 * (|.|^\beta \breve{F})$ is bounded on $\mathbb{R}^N$, such as the function $|.|^\beta K_0 * \breve{F}$: the proof being identical for the functions $|.|^\beta K_{1,j} * \breve{G}_j$ by replacing F by G_j, we omit it.

We now study the last term of the first equation:

Step 2. *Let $(j,k) \in [\![1, N]\!]^2$, and with the notations of the first part, $R = 4R_0$. Then,*

$$(1.3)\ K_{j,k} * P_{j,k} \in M_N^\infty(B(0,R)^c).$$

Indeed, by Lemma 6, we have

$$\forall x \in B(0,R)^c, K_{j,k} * P_{j,k}(x) = \int_{B(0,3R_0)} K_{j,k}(x-y) P_{j,k}(y) dy,$$

and so, by Theorem 6,

$$|K_{j,k} * P_{j,k}(x)| \leq A \int_{B(0,3R_0)} |x-y|^{-N} |P_{j,k}(y)| dy \leq \frac{A}{|x|^N} \|P_{j,k}\|_{L^1(\mathbb{R}^N)},$$

which completes the proof of this step.

The result of Proposition 8 concerning the function $\breve{\eta}$ follows from combining Steps 1 and 2, and Theorem 4.

In order to prove the remaining results, we first study the function $\nabla \breve{\eta}$ which satisfies the equation

$$\nabla \breve{\eta} = \nabla K_0 * \breve{F} + 2c \sum_{j=1}^N \nabla K_{1,j} * \breve{G}_j + \sum_{j=1}^N \sum_{k=1}^N \nabla K_{j,k} * P_{j,k}, \tag{19}$$

and establish results similar to the previous steps,

Step 3. *Let $(j,k) \in [\![1, N]\!]^2$, and with the notations of the first part, $R = 4R_0$. Then,*

(1.4) $\nabla K_0 * \breve{F} \in M_\beta^\infty(\mathbb{R}^N)$

(1.5) $\nabla K_{1,j} * \breve{G}_j \in M_\beta^\infty(\mathbb{R}^N)$

(1.6) $\nabla K_{j,k} * P_{j,k} \in M_{N+1}^\infty(B(0,R)^c)$

for every real number β sufficiently small.

Indeed, for (1.4), we have for every $\beta > 0$, and, every $x \in \mathbb{R}^N$,

$$|x|^\beta |\nabla K_0 * \breve{F}(x)| \leq A(\int_{\mathbb{R}^N} |x-y|^\beta |\nabla K_0(x-y)||\breve{F}(y)|dy + \int_{\mathbb{R}^N} |\nabla K_0(x-y)||y|^\beta |\breve{F}(y)|dy).$$

On one hand, we deduce from Theorem 6 that for every $\beta \in [0, N+1[$,

$$\|(|.|^\beta \nabla K_0) * \breve{F}\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla K_0\|_{M^p_\beta(\mathbb{R}^N)} \|\breve{F}\|_{L^{p'}(\mathbb{R}^N)} < +\infty,$$

where p is chosen such that $|.|^\beta \nabla K_0$ belongs to $L^p(\mathbb{R}^N)$.

On the other hand, by Corollary 2, for every $q \in [1, \frac{N}{N-1}[$ and every $\beta \in [0, \frac{\alpha}{N}[$,

$$\|\nabla K_0 * (|.|^\beta \breve{F})\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \|\breve{F}\|_{M^{q'}_\beta(\mathbb{R}^N)}$$

and, since there is $q \in [1, \frac{N}{N-1}[$ such that $\beta q' < \alpha$,

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |\breve{F}|^{q'} \leq A \int_{\mathbb{R}^N} |\ |^{\beta q'} |\breve{F}| < +\infty,$$

and, the function $\nabla K_0 * (|.|^\beta \breve{F})$ is bounded on $\mathbb{R}^N$, such as the function $|.|^\beta \nabla K_0 * \breve{F}$: similarly, the functions $|.|^\beta \nabla K_{1,j} * \breve{G}_j$ are bounded as soon as β is sufficiently small.

And, likewise for (1.6), we have the formula

$$\forall x \in B(0,R)^c, \nabla K_{j,k} * P_{j,k}(x) = \int_{B(0,3R_0)} \nabla K_{j,k}(x-y) P_{j,k}(y) dy,$$

and so, by Theorem 6,

$$|\nabla K_{j,k} * P_{j,k}(x)| \leq A \int_{B(0,3R_0)} |x-y|^{-N-1} |P_{j,k}(y)| dy \leq \frac{A}{|x|^{N+1}} \|P_{j,k}\|_{L^1(\mathbb{R}^N)}.$$

Therefore, the function $|.|^{N+1} \nabla K_{j,k} * P_{j,k}$ is bounded at infinity, which ends the proof of this step, and shows that the results of Proposition 8 for the function $\nabla \breve{\eta}$ are valid.

It only remains to study the function $\nabla \breve{\theta}$ thanks to equations (16)

$$\partial_j \breve{\theta} = \frac{c}{2} K_{1,j} * \breve{F} + c^2 \sum_{k=1}^N L_{1,j,k} * \breve{G}_k + \sum_{k=1}^N R_{j,k} * \breve{G}_k + \frac{c}{2} \sum_{k=1}^N \sum_{l=1}^N L_{j,k,l} * P_{k,l}$$
$$+ \sum_{k=1}^N \sum_{l=1}^N R_{k,l} * Q^j_{k,l}.$$

The study of the terms involving the kernels $K_{1,j}$, $L_{1,j,k}$ and $L_{j,k,l}$ is strictly identical to those of Steps 1, 2 and 3, and leads to

Step 4. *Let $(j,k,l) \in [\![1, N]\!]^3$, and with the notations of the first part, $R = 4R_0$. Then,*

(1.7) $K_{1,j} * \breve{F} \in M^\infty_\beta(\mathbb{R}^N)$

(1.8) $L_{1,j,k} * \breve{G}_k \in M^\infty_\beta(\mathbb{R}^N)$

(1.9) $L_{j,k,l} * P_{k,l} \in M^\infty_N(B(0,R)^c)$

for every real number β sufficiently small.

It remains to evaluate the functions $R_{j,k} * \breve{G}_k$ and $R_{k,l} * Q^j_{k,l}$: for the first one, we prove

Step 5. *Let $(j,k) \in [\![1,N]\!]^2$. We have*

$$(1.10)\ R_{j,k} * \breve{G}_k \in M^\infty_\beta(\mathbb{R}^N),$$

for every real number β sufficiently small.

Indeed, by the previous steps of this proof, the functions $|.|^\beta \breve{\eta}$ and $|.|^\beta \nabla \breve{\eta}$ are bounded on $\mathbb{R}^N$ for β sufficiently small: so, the functions $|.|^\beta \breve{G}$ and $|.|^\beta \nabla \breve{G}$ are also bounded on $\mathbb{R}^N$ for β sufficiently small. Since the functions G and ∇G belong to all the spaces $L^p(\mathbb{R}^N)$, by Proposition 5, we can conclude that the functions $|.|^\beta R_{j,k} * \breve{G}_k$ are bounded for β sufficiently small.

For the second one, we show

Step 6. *Let $(j,k,l) \in [\![1,N]\!]^3$, and with the notations of the first part, $R = 4R_0$. Then,*

$$(1.11)\ R_{k,l} * Q^j_{k,l} \in M^\infty_N(B(0,R)^c).$$

By Lemma 7, we have likewise

$$\forall x \in B(0,R)^c, R_{k,l} * Q^j_{k,l}(x) = A_N \int_{B(0,3R_0)} \frac{\delta_{k,l}|x-y|^2 - N(x_k - y_k)(x_l - y_l)}{|x-y|^{N+2}} Q^j_{k,l}(y) dy,$$

and so,

$$|R_{k,l} * Q^j_{k,l}(x)| \leq A \int_{B(0,3R_0)} |x-y|^{-N} |Q^j_{k,l}(y)| dy \leq \frac{A}{|x|^N} \|Q^j_{k,l}\|_{L^1(\mathbb{R}^N)}.$$

Therefore, the function $|.|^N R_{k,l} * Q^j_{k,l}$ is bounded at infinity, which concludes the proof of this step and of Proposition 8.

The proof of Theorem 7 now relies on the following iterative argument.

Proposition 9. *We suppose that there is some real number $\alpha > 0$ such that*

$$(\breve{\eta}, \nabla \breve{\eta}, \nabla \breve{\theta}) \in M^\infty_\beta(\mathbb{R}^N)^3,$$

for every $\beta \in [0, \alpha[$. Then,

$$(\breve{\eta}, \nabla \breve{\theta}) \in M^\infty_\beta(\mathbb{R}^N)^2,$$

for every

$$\beta \in [0, \min\{N, 2\alpha\}[,$$

and,

$$\nabla \breve{\eta} \in M^\infty_\beta(\mathbb{R}^N),$$

for every

$$\beta \in [0, \min\{N+1, 2\alpha\}[.$$

This proof is very similar to the previous one: we begin by using the quadratic form of the functions F and G.

Step 7. *The function*

$$|.|^\beta (|\breve{F}| + |\breve{G}|)$$

is bounded for every

$$\beta \in [0, 2\alpha[.$$

This step is clear, but as we will see soon, it is the key ingredient of the iterative argument.

Thanks to it and to equation (16),

$$\breve{\eta} = K_0 * \breve{F} + 2c\sum_{j=1}^{N} K_{1,j} * \breve{G}_j + \sum_{j=1}^{N}\sum_{k=1}^{N} K_{j,k} * P_{j,k},$$

we begin by studying the function $\breve{\eta}$:

Step 8. *For every $\beta \in [0, \min\{N, 2\alpha\}[$, we have*

(1.12) $K_0 * \breve{F} \in M^{\infty}_{\beta}(\mathbb{R}^N)$

(1.13) $K_{1,j} * \breve{G}_j \in M^{\infty}_{\beta}(\mathbb{R}^N)$.

We have likewise for any $\beta > 0$, and any $x \in \mathbb{R}^N$,

$$|x|^{\beta}|K_0 * \breve{F}(x)| \leq A(\int_{\mathbb{R}^N} |x-y|^{\beta}|K_0(x-y)||\breve{F}(y)|dy + \int_{\mathbb{R}^N} |K_0(x-y)||y|^{\beta}|\breve{F}(y)|dy).$$

On one hand, by Theorem 5, we know that for every $\beta \in [0, N[$, there is some real number $p \in [1, +\infty]$ such that

$$|.|^{\beta} K_0 \in L^p(\mathbb{R}^N),$$

which gives by Theorem 4

$$\|(|.|^{\beta} K_0) * \breve{F}\|_{L^{\infty}(\mathbb{R}^N)} \leq \|K_0\|_{M^p_{\beta}(\mathbb{R}^N)} \|\breve{F}\|_{L^{p'}(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 2,

$$K_0 \in L^q(\mathbb{R}^N)$$

for every $1 < q < \frac{N}{N-2}$: so, we get for every $\beta \in [0, 2\alpha[$,

$$\|K_0 * (|.|^{\beta} \breve{F})\|_{L^{\infty}(\mathbb{R}^N)} \leq \|K_0\|_{L^q(\mathbb{R}^N)} \|\breve{F}\|_{M^{q'}_{\beta}(\mathbb{R}^N)}.$$

As there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |\breve{F}|^{q'} < +\infty,$$

the function $K_0 * (|.|^{\beta} \breve{F})$ is bounded on $\mathbb{R}^N$, such as the function $|.|^{\beta} K_0 * \breve{F}$: the proof being identical for the functions $|.|^{\beta} K_{1,j} * \breve{G}_j$ by replacing F by G_j, we omit it.

By statement (1.3), the result of Proposition 9 for the function $\breve{\eta}$ is valid. Now, let us study the function $\nabla\breve{\eta}$, which satisfies equation (19):

$$\nabla\breve{\eta} = \nabla K_0 * \breve{F} + 2c\sum_{j=1}^{N} \nabla K_{1,j} * \breve{G}_j + \sum_{j=1}^{N}\sum_{k=1}^{N} \nabla K_{j,k} * P_{j,k}.$$

We then establish

Step 9. *For every $\beta \in [0, \min\{2\alpha, N+1\}[$, we have*

(1.14) $\nabla K_0 * \breve{F} \in M^{\infty}_{\beta}(\mathbb{R}^N)$

(1.15) $\nabla K_{1,j} * \breve{G}_j \in M^\infty_\beta(\mathbb{R}^N)$

We can establish likewise to statement (1.12) by Theorem 5 that

$$(|.|^\beta \nabla K_0) * \breve{F} \in L^\infty(\mathbb{R}^N),$$

for every $\beta \in [0, N+1[$. We also deduce from Corollary 2 that for every $q \in [1, \frac{N}{N-1}[$ sufficiently small and every $\beta \in [0, 2\alpha[$,

$$\|\nabla K_0 * (|.|^\beta \breve{F})\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \|\breve{F}\|_{M^{q'}_\beta(\mathbb{R}^N)} < +\infty.$$

By doing similarly, the functions $\nabla K_{1,j} * (|.|^\beta \breve{G}_j)$ and $(|.|^\beta \nabla K_{1,j}) * \breve{G}_j$ are also bounded for $\beta \in [0, \min\{N+1, 2\alpha\}[$, which completes the proof of this step.

The result of Proposition 9 for the function $\nabla \breve{\eta}$ follows from statement (1.6), and, it only remains to study the function $\nabla \breve{\theta}$, which satisfies equation (16),

$$\partial_j \breve{\theta} = \frac{c}{2} K_{1,j} * \breve{F} + c^2 \sum_{k=1}^N L_{1,j,k} * \breve{G}_k + \sum_{k=1}^N R_{j,k} * \breve{G}_k + \frac{c}{2} \sum_{k=1}^N \sum_{l=1}^N L_{j,k,l} * P_{k,l}$$
$$+ \sum_{k=1}^N \sum_{l=1}^N R_{k,l} * Q^j_{k,l}.$$

The study of the terms involving the kernels $K_{1,j}$, $L_{1,j,k}$ and $L_{j,k,l}$ is strictly identical to those of Steps 8 and 9.

Step 10. *Let* $(j,k,l) \in [\![1, N]\!]^3$. *Then,*

(1.16) $K_{1,j} * \breve{F} \in M^\infty_\beta(\mathbb{R}^N)$

(1.17) $L_{1,j,k} * \breve{G}_k \in M^\infty_\beta(\mathbb{R}^N)$

for every $\beta \in [0, \min\{N, 2\alpha\}[$.

Since statements (1.9) and (1.11) are still valid, it only remains to evaluate the function $R_{j,k} * \breve{G}_k$:

Step 11. *For every* $\beta \in [0, \min\{N, 2\alpha\}[$,

$$(1.18)\ R_{j,k} * \breve{G}_k \in M^\infty_\beta(\mathbb{R}^N).$$

Indeed, by the previous steps of this proof, the functions $|.|^\beta \breve{\eta}$ and $|.|^\beta \nabla \breve{\eta}$ are bounded on $\mathbb{R}^N$ for $\beta \in [0, \min\{N, 2\alpha\}[$: so, the functions $|.|^\beta \breve{G}$ and $|.|^\beta \nabla \breve{G}$ are also bounded on $\mathbb{R}^N$ for β in this range. Since the functions $\breve{G}$ and $\nabla \breve{G}$ belong to all the spaces $L^p(\mathbb{R}^N)$, by Proposition 5, we can conclude that the functions $|.|^\beta R_{j,k} * \breve{G}_k$ are bounded for β in this range. This ends the proof of Proposition 9.

By combining Propositions 8 and 9, we know that

$$(\breve{\eta}, \nabla \breve{\theta}) \in M^\infty_\beta(\mathbb{R}^N)^2,$$

for every

$$\beta \in [0, N[,$$

and,

$$\nabla \breve{\eta} \in M^\infty_\beta(\mathbb{R}^N),$$

for every

$$\beta \in [0, N+1[.$$

Thus, it only remains to study the case $\beta = N$ or $\beta = N+1$. We begin by the asymptotic decay of the functions $\breve{F}$ and $\breve{G}$.

Step 12. *The function*

$$|.|^{N+1}(|\breve{F}| + |\breve{G}|)$$

is bounded on $\mathbb{R}^N$.

This step follows from the quadratic form of $\breve{F}$ and $\breve{G}$, and from the previous statements.

We then study the function $\breve{\eta}$ i.e. equation (16)

$$\breve{\eta} = K_0 * \breve{F} + 2c \sum_{j=1}^{N} K_{1,j} * \breve{G}_j + \sum_{j=1}^{N} \sum_{k=1}^{N} K_{j,k} * P_{j,k}.$$

Since we already have (1.3), we only need to prove

Step 13. *We have*

(1.19) $K_0 * \breve{F} \in M^\infty_N(\mathbb{R}^N)$

(1.20) $K_{1,j} * \breve{G}_j \in M^\infty_N(\mathbb{R}^N)$

which follows from repeating the proof of Step 8, using Theorem 6 and Step 12 instead of Theorem 5 and Step 7.

For the function $\nabla \breve{\eta}$, by equation (19) and statement (1.6), we also only have to prove

Step 14. *We have*

(1.21) $\nabla K_0 * \breve{F} \in M^\infty_{N+1}(\mathbb{R}^N)$

(1.22) $\nabla K_{1,j} * \breve{G}_j \in M^\infty_{N+1}(\mathbb{R}^N)$

which also follows from repeating the proof of Step 9, using Theorem 6 and Step 12 instead of Theorem 5 and Step 7.

Finally, we have the same kind of result for the function $\nabla \breve{\theta}$ ie

Step 15. *Let* $(j, k, l) \in [\![1, N]\!]^3$. *Then,*

(1.23) $K_{1,j} * \breve{F} \in M^\infty_N(\mathbb{R}^N)$

(1.24) $L_{1,j,k} * \breve{G}_k \in M^\infty_N(\mathbb{R}^N)$,

which is clear by repeating the proof of Step 10, using Theorem 6 and Step 12 instead of Theorem 5 and Step 7. Since statements (1.9) and (1.11) are still valid, it only remains to evaluate the function $R_{j,k} * \breve{G}_k$:

Step 16. *We have*

$$(1.25)\ R_{j,k} * \breve{G}_k \in M^\infty_N(\mathbb{R}^N).$$

Indeed, by the previous steps of this proof, the functions $|.|^N \breve{\eta}$ and $|.|^{N+1} \nabla \breve{\eta}$ are bounded on $\mathbb{R}^N$: so, the functions $|.|^N \breve{G}$ and $|.|^{N+1} \nabla \breve{G}$ are also bounded on $\mathbb{R}^N$. Since the functions $\breve{G}$ and $\nabla \breve{G}$ belong to $L^1(\mathbb{R}^N)$, by Proposition 6, we can conclude that the functions $|.|^N R_{j,k} * \breve{G}_k$ are bounded on $\mathbb{R}^N$, which completes the proofs of this step and of Theorem 7. $\square$

4 Convergence and asymptotic decay at infinity

In this last part, we first prove the convergence at infinity of v towards a complex number of modulus one, v_∞, in dimension larger than three and then, complete the proof of Theorem 2.

4.1 Convergence at infinity in dimension larger than three

Before concluding the proof of Theorem 2, we will study the convergence at infinity of v by establishing the following general proposition concerning the limit of a function at infinity.

Proposition 10. *We consider a regular function v on $\mathbb{R}^N$: we suppose that N is greater than three and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ where*

$$1 < p_0 < N-1 < p_1 < +\infty.$$

Then there is a constant $v_\infty \in \mathbb{C}$ which satisfies

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

Proof. We first construct the limit v_∞. Indeed, we have

$$\int_{\mathbb{S}^{N-1}} \int_1^{+\infty} |\partial_r v(r\xi)| dr d\sigma \leq \int_{\mathbb{S}^{N-1}} \left(\int_1^{+\infty} |\nabla v(r\xi)|^{p_0} r^{N-1} dr \right)^{\frac{1}{p_0}} \left(\int_1^{+\infty} r^{-\frac{N-1}{p_0-1}} dr \right)^{\frac{1}{p_0'}} d\sigma$$
$$< +\infty$$

which gives

$$\int_1^{+\infty} |\partial_r v(r\xi)| dr < +\infty \ a.e.$$

Therefore, there is a measurable function v_∞ defined on $\mathbb{S}^{N-1}$ such that

$$v(r\xi) \underset{r\to+\infty}{\to} v_\infty(\xi) \ a.e.$$

Now, let us denote

$$\forall p \in [p_0, p_1], \forall r \in \mathbb{R}_+^*, I_p(r) = r^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p d\sigma.$$

This function is regular on $\mathbb{R}_+^*$ and its derivative satisfies

$$\forall r \in \mathbb{R}_+^*, |I_p'(r)| \leq (N-1) r^{N-2} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^p d\sigma + p r^{N-1} \int_{\mathbb{S}^{N-1}} |\nabla v(r\xi)|^{p-1} |\partial_r \nabla v(r\xi)| d\sigma,$$

which gives

$$\int_0^{+\infty} |I_p'(r)| dr \leq A(||\nabla v||_{L^p(\mathbb{R}^N)}^p + ||\nabla v||_{L^p(\mathbb{R}^N)}^{p-1} ||\nabla v||_{W^{1,p}(\mathbb{R}^N)}) < +\infty.$$

So the function I_p has a limit at $+\infty$, and since

$$\int_0^{+\infty} I_p(r) dr = ||\nabla v||_{L^p(\mathbb{R}^N)}^p < +\infty,$$

this limit is zero. Furthermore, if we denote

$$\forall r \in \mathbb{R}_+^*, \forall \xi \in \mathbb{S}^{N-1}, v_r(\xi) = v(r\xi),$$

we remark that

$$|\nabla v(r\xi)|^2 = |\partial_r v(r\xi)|^2 + r^{-2}|\nabla^{\mathbb{S}^{N-1}} v_r(\xi)|^2$$

which leads finally to

$$r^{N-1-p}\int_{\mathbb{S}^{N-1}} |\nabla^{\mathbb{S}^{N-1}} v_r(\xi)|^p d\sigma \underset{r\to+\infty}{\to} 0.$$

Thus, if $N-1 \leq q < \min\{p_1, N\}$, we get for all $r \in \mathbb{R}_+^*$

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |v_r(\xi) - v_\infty(\xi)|^q d\sigma &\leq \int_{\mathbb{S}^{N-1}} \left(\int_r^{+\infty} |\partial_r v(s\xi)| ds\right)^q d\sigma \\
&\leq \int_{\mathbb{S}^{N-1}} r^{q-N} \int_r^{+\infty} |\nabla v(s\xi)|^q s^{N-1} ds d\sigma \\
&\leq A_{N,q}||\nabla v||^q_{L^q(\mathbb{R}^N)} r^{q-N},
\end{aligned}$$

which gives

$$\begin{aligned}
||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} &= \frac{1}{N-1}\int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}} |v_r - v_\infty|^*(t) dt \\
&\leq A_N \left(\int_0^{|\mathbb{S}^{N-1}|} |v_r - v_\infty|^{*q}(t) dt\right)^{\frac{1}{q}} \left(\int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{q'(N-2)}{N-1}} dt\right)^{\frac{1}{q'}} \\
&\leq A_{N,q}||v_r - v_\infty||_{L^q(\mathbb{S}^{N-1})} \\
&\leq A_{N,q}||\nabla v||^q_{L^q(\mathbb{R}^N)} r^{q-N} \\
&\underset{r\to+\infty}{\to} 0.
\end{aligned}$$

Now, we fix $\epsilon > 0$ and we denote

$$\forall r \in \mathbb{R}_+, \begin{cases} \forall \lambda \in \mathbb{R}_+^*, a_r(\lambda) = |\{\xi \in \mathbb{S}^{N-1} / |\nabla^{\mathbb{S}^{N-1}} v_r(\xi)| > \lambda\}| \\ \forall t \in \mathbb{R}_+^*, f_r(t) = |\nabla^{\mathbb{S}^{N-1}} v_r|^*(t) = \inf\{\lambda \in \mathbb{R}_+^* / a_r(\lambda) \leq t\}. \end{cases}$$

We have just showed that there is $r_\epsilon > 0$ such that

$$\forall r > r_\epsilon, \forall q \in \{p_0, p_1\}, r^{N-1-q}\int_{\mathbb{S}^{N-1}} |\nabla^{\mathbb{S}^{N-1}} v_r(\xi)|^q d\sigma \leq \epsilon^q,$$

which gives

$$\forall \lambda \in \mathbb{R}_+^*, a_r(\lambda) \leq \min\{\frac{\epsilon^{p_0}}{r^{N-1-p_0}\lambda^{p_0}}, \frac{\epsilon^{p_1}}{r^{N-1-p_1}\lambda^{p_1}}\},$$

and,

$$\forall t \in \mathbb{R}_+^*, f_r(t) \leq \min\{\frac{\epsilon}{r^{\frac{N-1}{p_0}-1} t^{\frac{1}{p_0}}}, \frac{\epsilon}{r^{\frac{N-1}{p_1}-1} t^{\frac{1}{p_1}}}\}.$$

Thus, we finally get

$$\begin{aligned}
||\nabla^{\mathbb{S}^{N-1}} v_r||_{L^{N-1,1}(\mathbb{S}^{N-1})} &= \frac{1}{N-1}\int_0^{|\mathbb{S}^{N-1}|} f_r(t) t^{-\frac{N-2}{N-1}} dt \\
&\leq \frac{1}{N-1}(\epsilon r^{1-\frac{N-1}{p_1}} \int_0^{r^{1-N}} t^{-\frac{N-2}{N-1}-\frac{1}{p_1}} dt \\
&+ \epsilon r^{1-\frac{N-1}{p_0}} \int_{r^{1-N}}^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}-\frac{1}{p_0}} dt) \\
&\leq A_{N,p_0,p_1}\epsilon
\end{aligned}$$

which proves that $\nabla^{\mathbb{S}^{N-1}} v_r$ converges to 0 in $L^{N-1,1}(\mathbb{S}^{N-1})$ when r tends to $+\infty$. But, v_r converges to v_∞ in $L^{N-1,1}(\mathbb{S}^{N-1})$, so, the gradient of v_∞ is zero and v_∞ is constant. Besides, by a theorem of A. Cianchi and L. Pick [7], we know that there is a constant A which satisfies for all $r > 0$

$$||v_r - v_\infty||_{L^\infty(\mathbb{S}^{N-1})} \leq A(||v_r - v_\infty||_{L^{N-1,1}(\mathbb{S}^{N-1})} + ||\nabla^{\mathbb{S}^{N-1}}(v_r - v_\infty)||_{L^{N-1,1}(\mathbb{S}^{N-1})})$$

which gives

$$||v_r - v_\infty||_{L^\infty(\mathbb{S}^{N-1})} \underset{r\to+\infty}{\to} 0$$

and ends the proof of this proposition. □

Now, we can prove the first part of Theorem 2: if v is a travelling wave of finite energy and of speed $c < \sqrt{2}$, it satisfies the hypothesis of Proposition 10 thanks to Theorem 4. So there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

It remains to show that this constant has a modulus equal to one, which follows from Lemma 5.

Remark 8. In order to simplify the notations, and since the solutions are defined up to a rotation, we will suppose in the following that

$$v_\infty = 1.$$

Hence, ρ and θ uniformly tend to 1, respectively 0, at infinity.

4.2 Asymptotic decay for θ and v

We are now in position to prove the second part of Theorem 2: in order to do so, we show the following proposition:

Proposition 11. *The function $(1 + |.|^{N-1})\breve{\theta}$ is bounded on $\mathbb{R}^N$.*

Proof. Indeed, thanks to the previous paragraph, $\breve{\theta}$ is bounded, and, by Theorem 7, we know that the function $|.|^N \nabla\breve{\theta}$ is bounded on $\mathbb{R}^N$. As we have

$$\forall x \in \mathbb{R}^N \setminus \{0\}, \breve{\theta}(x) = -\int_{|x|}^{+\infty} \partial_r \breve{\theta}(\frac{sx}{|x|})ds,$$

we get

$$\forall x \in \mathbb{R}^N \setminus \{0\}, |\breve{\theta}(x)| \leq A\int_{|x|}^{+\infty} \frac{ds}{s^N} \leq \frac{A}{|x|^{N-1}}.$$

□

Finally, we also know by Theorem 7 that the function $|.|^N \nabla v$ is bounded on $\mathbb{R}^N$, and similarly, we deduce that the function $|.|^{N-1}(v-1)$ is bounded on $\mathbb{R}^N$, which ends the proof of Theorem 2.

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