Asymptotics for the travelling waves in the Gross-Pitaevskii equation

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Abstract

We investigate the asymptotic behaviour of the subsonic travelling waves of finite energy in the Gross-Pitaevskii equation in dimension larger than two. In particular, we give their first order development at infinity in the case they are axisymmetric, and link it to their energy and momentum.

Introduction

1 Motivations

In this article, we focus on the travelling waves in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2) \tag{1}$$

of the form $u(t, x) = v(x_1 - ct, \ldots, x_N)$. The parameter $c \geq 0$ represents the speed of the travelling wave, which moves in direction x_1. The equation for v, which we will consider now, writes

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{2}$$

The Gross-Pitaevskii equation is a physical model for the Bose-Einstein condensation, which is associated at least formally to the so-called Ginzburg-Landau energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |v|^2)^2, \tag{3}$$

and to the momentum

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla v.v. \tag{4}$$

Equation (1) presents an hydrodynamic form

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathrm{v}) = 0, \\ \rho(\partial_t \mathrm{v} + \mathrm{v}.\nabla \mathrm{v}) + \nabla \rho^2 = \rho \nabla \left(\frac{\Delta \rho}{\rho} - \frac{|\nabla \rho|^2}{2\rho^2} \right), \end{cases} \tag{5}$$

obtained by using the Madelung transform [14]

$$u = \sqrt{\rho} e^{i\theta},$$

and denoting

$$\mathrm{v} = 2\nabla\theta.$$

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Equations (5) are similar to Euler equations for an irrotational ideal fluid with pressure $p(\rho) = \rho^2$. In particular, the speed of the sound waves near the constant solution $u = 1$ is

$$c_s = \sqrt{2}.$$

The travelling waves of finite energy play an important role in the long time dynamics of general solutions and were thoroughly studied by C.A. Jones, S.J. Putterman and P.H. Roberts [13, 12]. They conjectured that there exist non-constant travelling waves of finite energy only in the subsonic case

$$0 < c < \sqrt{2}.$$

F. Béthuel and J.C. Saut [3, 2] first investigated mathematically this conjecture. In dimension two, they showed the existence of non-constant travelling waves of finite energy for small values of c, and for a sequence of values of $c < \sqrt{2}$ tending to $\sqrt{2}$. They also proved their non-existence for $c = 0$ in every dimension. Their work was complemented in dimension larger than three by F. Béthuel, G. Orlandi and D. Smets [1], who also showed their existence when c is small. On the other hand, we proved their non-existence for every $c > \sqrt{2}$ [8]. Thus, the problem of their non-existence only remains open in the sonic case $c = \sqrt{2}$ (see [10] however for more details). We will deliberately omit this case and only consider from now on the subsonic travelling waves, i.e. we will assume

$$0 < c < \sqrt{2}.$$

Under this assumption and the additional hypothesis the travelling waves are axisymmetric around axis x_1, C.A. Jones, S.J. Putterman and P.H. Roberts [13, 12] characterised their behaviour at infinity by giving their first order development up to a multiplicative constant of modulus one. In dimension two, they derived a formal asymptotic expansion

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2} \tag{6}$$

and in dimension three,

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}}. \tag{7}$$

Here, the constant α is the stretched dipole coefficient linked to the energy $E(v)$ and to the scalar momentum in direction x_1, $p(v) = P_1(v)$, by the formulae

$$2\pi\alpha\sqrt{1 - \frac{c^2}{2}} = cE(v) + 2\left(1 - \frac{c^2}{4}\right)p(v) \tag{8}$$

in dimension two, and

$$4\pi\alpha = \frac{c}{2}E(v) + 2p(v) \tag{9}$$

in dimension three.

The goal of this paper is to provide a rigorous derivation of the asymptotic behaviour described in (6), (7), (8) and (9), and a generalisation to every dimension $N \geq 2$.

2 Main results

Our main results are summed up in the next three theorems. The first one is the most general. We consider any subsonic travelling waves of finite energy in any dimension $N \geq 2$, and prove the existence of their first order development at infinity (which is consistent with conjectures (6)) and (7) in dimensions two and three). Moreover, we compute a linear partial differential equation satisfied by the first order term of their asymptotic expansion.

Theorem 1. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$. There exist a complex number λ_∞ of modulus one and a smooth function v_∞ defined from the sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}$ such that*

$$|x|^{N-1}(v(x) - \lambda_\infty) - i\lambda_\infty v_\infty \left(\frac{x}{|x|}\right) \underset{|x|\to+\infty}{\to} 0 \text{ uniformly.}$$

Moreover, the function v_∞ satisfies the following linear partial differential equation on $\mathbb{S}^{N-1}$

$$\Delta_{\mathbb{S}^{N-1}} v_\infty - \frac{c^2}{2}\partial_1^{\mathbb{S}^{N-1}}(\partial_1^{\mathbb{S}^{N-1}} v_\infty) + c^2(N-1)\sigma_1 \partial_1^{\mathbb{S}^{N-1}} v_\infty + (N-1)(1 + \frac{c^2}{2}(1-(N+1)\sigma_1^2))v_\infty = 0. \tag{10}$$

Remarks 1. 1. Subsequently, we will always assume that

$$\lambda_\infty = 1.$$

Indeed, if this is not the case, we can study the function $\lambda_\infty^{-1} v$ instead of v: it is also a travelling wave of finite energy and of speed c which satisfies equation (2).

2. Equation (10) is defined on the sphere $\mathbb{S}^{N-1}$ immersed in the space $\mathbb{R}^N$. In order to clarify its sense, we need to explicit some notations for derivations on $\mathbb{S}^{N-1}$. Thus, consider some function $f \in C^\infty(\mathbb{S}^{N-1}, \mathbb{C})$: the notation $\partial_i^{\mathbb{S}^{N-1}}$ is defined by

$$\forall i \in \{1, \ldots, N\}, \forall x \in \mathbb{S}^{N-1}, \partial_i^{\mathbb{S}^{N-1}} f(x) = \lim_{t\to 0} \frac{f(\frac{x+te_i}{|x+te_i|}) - f(x)}{t},$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^N$. The operator $\Delta_{\mathbb{S}^{N-1}}$ is the usual Laplace-Beltrami operator on the sphere $\mathbb{S}^{N-1}$, given by

$$\forall x \in \mathbb{S}^{N-1}, \Delta_{\mathbb{S}^{N-1}} f(x) = \sum_{i=1}^{N} \partial_i^{\mathbb{S}^{N-1}}(\partial_i^{\mathbb{S}^{N-1}} f)(x).$$

Our next theorems specify the form of the limit function v_∞ in two cases: for the axisymmetric travelling waves, which only depend on the variables x_1 and

$$x_\perp = \sqrt{\sum_{i=2}^{N} x_i^2},$$

in every dimension $N \geq 2$, and for every travelling wave in dimension $N = 2$. In both cases, equation (10) reduces to an ordinary differential equation of second order, which is entirely integrable. In particular, it yields a proof of conjectures (6), (7), (8) and (9) in the axisymmetric case.

Theorem 2. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$, axisymmetric around axis x_1. Then, there exists some constant α such that the function v_∞ is given by*

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}. \tag{11}$$

Moreover, the constant α is equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \left(1 - \frac{c^2}{2}\right)^{\frac{N-3}{2}} \left(\frac{4-N}{2} cE(v) + (2 + \frac{N-3}{2}c^2)p(v)\right). \tag{12}$$

Likewise, in dimension two, we can describe explicitly the asymptotic behaviour of every travelling wave.

Theorem 3. *Let v, a travelling wave for the Gross-Pitaevskii equation in dimension two of finite energy and speed $0 < c < \sqrt{2}$. Then, there exist some constants α and β such that the function v_∞ is given by*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}}. \tag{13}$$

Moreover, the constants α and β are equal to

$$\begin{aligned} \alpha &= \frac{1}{2\pi\sqrt{1-\frac{c^2}{2}}}(cE(v) + (2 - \frac{c^2}{2})p(v)), \\ \beta &= \frac{\sqrt{1-\frac{c^2}{2}}}{\pi} P_2(v). \end{aligned} \tag{14}$$

Remarks 2. 1. There is a difficulty in the definition of $\vec{P}(v)$. Indeed, the integral which appears in definition (4) is not always convergent for a travelling wave of finite energy. In order to state formulae (12) and (14) rigorously, we define the momentum $\vec{P}(v)$ as

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla v.(v-1), \tag{15}$$

and the scalar momentum in direction x_1 by

$$p(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\partial_1 v.(v-1). \tag{16}$$

By [9], those integrals are well-defined in the case of travelling waves of finite energy. However, we will give another equivalent definition of the momentum which is more suitable in our context (see Subsection 3.1 of the introduction).

2. Theorem 3 is consistent with the axisymmetric case: assuming $\beta = 0$, we recover the axisymmetric solution of Theorem 2 with the same value of the stretched dipole coefficient α.

The integration of equation (10) seems rather involved in dimension $N \geq 3$: we are not able to compute an explicit formula for the function v_∞ from equation (10). However, we conjecture its expression as follows.

Conjecture 1. *Let v be a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$. Then, there exist some constants α, β_2, ..., β_N such that the function v_∞ is equal to*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + \sum_{j=2}^{N} \beta_j \frac{\sigma_j}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}.$$

Moreover, the constants α and β_j are equal to

$$\begin{aligned} \alpha &= \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\left(1 - \frac{c^2}{2}\right)^{\frac{N-3}{2}}\left(\frac{4-N}{2}cE(v) + (2 + \frac{N-3}{2}c^2)p(v)\right), \\ \beta_j &= \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}}\left(1 - \frac{c^2}{2}\right)^{\frac{N-1}{2}} P_j(v). \end{aligned}$$

Remark 1. In the second part, we will motivate this conjecture. Notice that, in case Conjecture 1 holds, it yields every possible asymptotic behaviour of a travelling wave v of finite energy in the Gross-Pitaevskii equation. In particular, the first order term v_∞ of the asymptotic expansion of v is completely determined by some integral quantities α, β_2, ..., β_N, related to the energy $E(v)$ and the momentum $\vec{P}(v)$.
This raises an interesting question. Consider N real numbers a_1, ..., a_N: is it possible to construct a travelling wave v such that the values of the integral quantities α, β_2, ..., β_N are exactly equal to a_1, ..., a_N ? In other words, is it possible to construct travelling waves v whose asymptotic behaviour correspond to any possible one given by Conjecture 1, or are there other restrictions for the possible asymptotic behaviours ?
To our knowledge, those questions remain open problems. Indeed, the existence results of F. Béthuel and J.C. Saut [3, 2] in dimension two and F. Béthuel, G. Orlandi and D. Smets [1] in dimension $N \geq 3$ prove the existence of travelling waves which are assumed to be axisymmetric. However, in this case, we can show that the constants β_2, ..., β_N are all equal to 0 (which is consistent with Theorem 2). Therefore, we do not know any travelling wave for which the values of β_2, ..., β_N are not 0. Thus, a first step to answer to our questions could be to prove the existence of travelling waves which are not axisymmetric.

One of the main interests of Theorems 1, 2 and 3 is their **sharpness**. In order to clarify this claim, we must recall some recent mathematical results. F. Béthuel and J.C. Saut [3, 2] first investigated the asymptotic behaviour of subsonic travelling waves in dimension two. They gave a mathematical evidence for their convergence towards a constant of modulus one at infinity. We complemented their work in [7] by proving the same convergence in every dimension $N \geq 3$. Finally, in [9], we gave a first estimate of their decay at infinity (which is moreover an important starting point of the analysis in this paper).

Theorem 4 ([9])**.** *In dimension $N \geq 2$, for every travelling wave v for the Gross-Pitaevskii equation of finite energy and speed $0 < c < \sqrt{2}$, the function*

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

is bounded on $\mathbb{R}^N$.

Theorems 1, 2, 3 and 4 are then sharp because the decay rate at infinity they give is optimal. There exist some travelling waves v such that the function

$$x \mapsto |x|^{\beta}(v(x) - 1)$$

is not bounded on $\mathbb{R}^N$ for any $\beta > N-1$: the decay exponent $N-1$ is the best possible in general (although some travelling waves, the constant ones for instance, can decay faster at infinity). The proof of the existence of such travelling waves v follows from two arguments. The first one is the proof of the existence of non-constant axisymmetric travelling waves by F. Béthuel and J.C. Saut [3, 2] in dimension two, and F. Béthuel, G. Orlandi and D. Smets [1] in dimension $N \geq 3$. The second one relies on the next corollary of Theorem 2.

Corollary 1. *Let v be a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$, axisymmetric around axis x_1. Then, the constant α is equal to 0 if and only if v is a constant travelling wave.*

Therefore, if we now consider a non-constant axisymmetric travelling wave v, by Theorem 2 and Corollary 1, the function v_∞ is not identically equal to 0 on $\mathbb{S}^{N-1}$. In particular, by Theorem 1, it means that the function

$$x \mapsto |x|^{\beta}(v(x) - 1)$$

is not bounded on $\mathbb{R}^N$ for any $\beta > N-1$, which shows the sharpness of Theorems 1, 2, 3 and 4.

Now, in the hope of clarifying the proof of Theorem 1 and in order to specify general arguments which could prove fruitful for other equations, we are going to explain how to infer such a theorem.

3 Sketch of the proof of Theorem 1

Theorem 1 deals with the asymptotic expansion of a travelling wave. We construct the limit at infinity of some function, in our case the function

$$x \mapsto |x|^{N-1}(v(x)-1),$$

prove that the convergence is uniform and then compute a partial differential equation satisfied by the limit function.

3.1 A new formulation of equation (2)

In [9], we already investigated the asymptotic behaviour of the travelling waves v in the Gross-Pitaevskii equation. In particular, we derived Theorem 4 just mentioned above. The proof of this theorem relies on a new formulation of equation (2), also relevant here, which we are going to recall concisely. The first argument is to state the local smoothness and the Sobolev regularity of a subsonic travelling wave v (see also the articles of F. Béthuel and J.C. Saut in dimension two [3, 2], and of A. Farina [6]).

Proposition 1 ([9])**.** *If v is a solution of finite energy of equation (2) in $L^1_{loc}(\mathbb{R}^N)$, then, v is C^∞, bounded, and the functions $\eta := 1-|v|^2$ and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p \leq +\infty$.*

It follows that the modulus ρ of v converges to 1 at infinity. In particular, there is some real number R_0 such that

$$\rho \geq \frac{1}{2} \text{ on } B(0,R_0)^c.$$

Since the energy $E(v)$ is finite, it follows (up to a standard degree argument in dimension two) that we may construct a lifting θ of v on $B(0,R_0)^c$, that is a function in $C^\infty(B(0,R_0)^c,\mathbb{R})$ such that

$$v = \rho e^{i\theta}.$$

We next compute new equations for the new functions η and $\nabla\theta$: since θ is not well-defined on $\mathbb{R}^N$, we introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N,[0,1])$ such that

$$\begin{cases} \psi = 0 \text{ on } \mathrm{B}(0,2\mathrm{R}_0), \\ \psi = 1 \text{ on } \mathrm{B}(0,3\mathrm{R}_0)^c. \end{cases}$$

All the asymptotic estimates obtained in [7, 9] are independent of the choice of R_0 and ψ, and it will also be the case here. We finally deduce

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}) \tag{17}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{18}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta) \tag{19}$$

and

$$G = i\nabla v.v + \nabla(\psi\theta). \tag{20}$$

Remark 2. At this stage, we can state another definition of the momentum

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} (i\nabla v.v + \nabla(\psi\theta)),$$

and of the scalar momentum in direction x_1

$$p(v) = \frac{1}{2}\int_{\mathbb{R}^N} (i\partial_1 v.v + \partial_1(\psi\theta)).$$

A straightforward computation shows that those new definitions are equivalent to the previous ones given by formulae (15) and (16). In the following, we will always use them in preference to formulae (15) and (16) since they seem more suitable in our context.

It follows from those new definitions and from formulae (19) and (20) that the functions F and G are almost quadratic functions of η and ∇v, related to the density of energy and of momentum. This is an important aspect of equations (17) and (18): they link our new functions η and θ to some superlinear quantities F and G, which have a relevant interpretation in terms of quantities conserved by the Gross-Pitaevskii equations. In particular, the superlinear nature of the nonlinearities is a key ingredient to establish the asymptotic properties of the travelling waves. It motivates the introduction of the new variables η and θ.

3.2 Convolution equations

It is well-known that the asymptotic properties of solutions to linear partial differential equations are related to the behaviour at infinity of their kernels, and this, for a large deal, also remains valid for many nonlinear problems. Our approach is reminiscent of the article of J.L. Bona and Yi A. Li [4], and also appeared in the articles of A. de Bouard and J.C Saut [5], and M. Maris [15, 16]. It consists in transforming the partial differential equations satisfied by the travelling wave (equation (2) in our context) in some convolution equations. In the case of the travelling waves for the Gross-Pitaevskii equation, we already computed such convolution equations in [7, 9]. They follow from equations (17) and (18) and write

$$\eta = K_0 * F + 2c\sum_{j=1}^{N} K_j * G_j \tag{21}$$

where K_0 and K_j are the kernels of Fourier transform,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{22}$$

respectively

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{23}$$

and for every $j \in \{1, \ldots, N\}$,

$$\partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k \tag{24}$$

where $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transform,

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2 \xi_j \xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{25}$$

respectively

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}. \tag{26}$$

Equations (21) and (24) are convolution equations with terms of the form $K * f$. The functions K are kernels with explicit Fourier transforms which are rational fractions. The functions f are nonlinear functions of η, ∇v and $\nabla(\psi\theta)$.

Our purpose is now to compute the limit at infinity of various weighted functions, for instance

$$x \mapsto |x|^N \eta(x).$$

By the previous convolution equations, it reduces to get the limit at infinity of functions of the type

$$x \mapsto |x|^p K * f(x) = \int_{\mathbb{R}^N} |x|^p K(x-y) f(y) dy, \tag{27}$$

where p is equal to N, K refers to one of the kernels K_0, K_j, $L_{j,k}$ or $R_{j,k}$ and f to the functions F or G. We will handle this problem, which is of independent interest[1], by invoking the dominated convergence theorem. Here, a main part of the analysis is devoted to study the properties of the kernel K, leaving the nonlinear nature of the function f aside for the moment.[2]

3.3 Main properties of the kernels and pointwise convergence at infinity

In this section, we derive a number of results for our model function (27), which enter directly in the proof of Theorem 1 and which rely on the dominated convergence theorem as mentioned above. More precisely, we wish to establish limits of functions of the form (27), as $|x| \to +\infty$, depending on the value of p and the form of K and f.

Step 1. *Pointwise convergence of the kernels.*

The first step is to prove the pointwise convergence when $|x|$ tends to $+\infty$ of the integrand, i.e.

$$y \mapsto |x|^p K(x-y),$$

where the function K is a kernel whose Fourier transform is known explicitly, actually in our case a rational fraction (the second step being the domination of the integrand).

Remark 3. It can depend on the direction of the convergence $\sigma = \frac{x}{|x|}$: denoting $x = R\sigma$ where $R > 0$ and $\sigma \in \mathbb{S}^{N-1}$, we are reduced to study the pointwise convergence of the functions

$$y \mapsto R^p K(R\sigma - y)$$

when R tends to $+\infty$ for every $\sigma \in \mathbb{S}^{N-1}$.

[1] A similar analysis will be carried out on the solitary waves for the Kadomtsev-Petviashvili equation (see [11]).

[2] If the function f had compact support, then the limit at infinity of $K * f$ would be directly deduced from the limit of K. In our subsequent analysis, we also have to take into account the decay of f using nonlinear arguments.

Our argument relies on the properties of the Fourier transform of the kernel K. Indeed, we introduce the space of functions

$$\widehat{\mathcal{K}}(\mathbb{R}^N) = \{u \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}), \forall i \in \mathbb{N}, d^i u \in M_i^\infty(\mathbb{R}^N) \cap M_{i+2}^\infty(\mathbb{R}^N)\},$$

where $M_\alpha^\infty(\mathbb{R}^N)$ is defined by

$$M_\alpha^\infty(\mathbb{R}^N) = \{u : \mathbb{R}^N \mapsto \mathbb{C} \ / \ \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} = \sup\{|x|^\alpha |u(x)|, x \in \mathbb{R}^N\} < +\infty\},$$

for every $\alpha > 0$.

Remark 4. The choice of the spaces is suggested by the form of the Fourier transforms of the kernels K_0, K_j and $L_{j,k}$. They belong to $\widehat{\mathcal{K}}(\mathbb{R}^N)$ by formulae (22), (23) and (25). However, we can introduce some variants for other equations.

Now, we can specify the pointwise convergence of some functions whose Fourier transforms are in $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Indeed, we claim

Theorem 5. *Let $\alpha \in \mathbb{N}^N$ and $K \in S'(\mathbb{R}^N, \mathbb{C})$. Assume its Fourier transform $\widehat{K}$ is a rational fraction*

$$\widehat{K} = \frac{P}{Q},$$

which belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$ and such that

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, Q(\xi) \neq 0.$$

Then, there exists a measurable function $K_\infty^\alpha \in L^\infty(\mathbb{S}^{N-1}, \mathbb{C})$ such that

$$\forall (\sigma, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N, R^{N+|\alpha|} \partial^\alpha K(R\sigma - y) \underset{R \to +\infty}{\to} K_\infty^\alpha(\sigma). \tag{28}$$

Remark 5. In particular, we prove the pointwise convergence of all the derivatives of the kernels K which satisfy the assumptions of Theorem 5: it will be very useful in the following.

As previously mentioned, Theorem 5 relies on the Fourier transform of the kernels K through the next lemma which already appeared in [9].

Lemma 1. *Let $(\sigma, y, R) \in \mathbb{S}^{N-1} \times \mathbb{R}^N \times \mathbb{R}_+^*$ and assume $|y| < R$ and $\sigma_j \neq 0$ for some integer $1 \leq j \leq N$. Consider a tempered distribution $K \in S'(\mathbb{R}^N, \mathbb{C})$ such that its Fourier transform is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Then, we have*

$$\begin{aligned} R^N K(R\sigma - y) = & \frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N} \Bigg(\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma - y)} d\xi + \int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi) \\ & (e^{i\xi.(R\sigma-y)} - 1) d\xi + R \int_{S(0,\frac{1}{R})} \xi_j \partial_j^{N-1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \Bigg). \end{aligned} \tag{29}$$

The proof of Theorem 5 then follows from applying the dominated convergence theorem to formula (29).

There are many other ways to study the convergences as in (28), but the use of the Fourier transforms of the kernels seems well-adapted to the context of partial differential equations, where we know them explicitly. However, in some cases, we know the explicit expression of the kernel K. It allows to bypass Theorem 5 for the computation of the limit of (27) by direct computations. This is the case for the so-called composed Riesz kernels $R_{j,k}$. Indeed, if f is a smooth function and if we denote $g_{j,k} = R_{j,k} * f$, we compute

$$\forall x \in \mathbb{R}^N, g_{j,k}(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \int_{|x-y|>1} \frac{\delta_{j,k}|x-y|^2 - N(x-y)_j(x-y)_k}{|x-y|^{N+2}} f(y)dy + \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \int_{|x-y|\leq 1} \frac{\delta_{j,k}|x-y|^2 - N(x-y)_j(x-y)_k}{|x-y|^{N+2}} (f(y)-f(x))dy. \tag{30}$$

Here, the difficulty to apply the dominated convergence theorem does not come from the limit at infinity of the kernels, but instead, from the domination of this convergence.

Step 2. *Domination of the convergence.*

The second step is to dominate the integrand, given by

$$y \mapsto |x|^p K(x-y)f(y),$$

independently of $x \in \mathbb{R}^N$. In order to do so, we assume for instance that f is a smooth function on $\mathbb{R}^N$ with some algebraic decay, i.e. f and some of its derivatives belong to some space $C^\infty(\mathbb{R}^N) \cap M_\alpha^\infty(\mathbb{R}^N)$ for some real number $\alpha > 0$.

Remark 6. The choice of such assumptions is suggested by the algebraic decay of the functions F and G. Indeed, in [9], we computed the algebraic decay of the functions η, $\nabla(\psi\theta)$ and ∇v by an argument due to J.L. Bona and Yi A. Li [4], and A. de Bouard and J.C Saut [5] (see also the articles of M. Maris [15, 16] for many more details).

Proposition 2 ([9])**.** *Let $\alpha \in \mathbb{N}^N$. Then, the functions η, $\nabla(\psi\theta)$ and ∇v satisfy*

- $(\eta, \partial^\alpha\nabla(\psi\theta), \partial^\alpha\nabla v) \in M_N^\infty(\mathbb{R}^N)^3$,
- $\partial^\alpha\nabla\eta \in M_{N+1}^\infty(\mathbb{R}^N)$.

By Propositions 1 and 2, and formulae (19) and (20), the functions F and G are smooth on $\mathbb{R}^N$ and belong to $M_{2N}^\infty(\mathbb{R}^N)$, which explains the choice of the assumptions on f. However, it is possible to introduce some variants for other equations.

Under such assumptions for the function f, it remains to dominate the kernel K. It may be straightforward when we know its exact expression (for instance, in the case of the composed Riesz kernels by formula (30)). However, a suitable approach seems once more to estimate the algebraic decay of K. In many cases, we know the Fourier transform of K. Therefore, we can invoke some formula like (29) to obtain their algebraic decay. In [9], we handled this difficulty for the so-called Gross-Pitaevskii kernels K_0, K_j and $L_{j,k}$, and for their derivatives.

Proposition 3 ([9])**.** *Let $N-2 < \alpha \leq N$, $n \in \mathbb{N}$ and $(j,k) \in \{1,\dots,N\}^2$. The functions d^nK_0, d^nK_j and $d^nL_{j,k}$ belong to $M_{\alpha+n}^\infty(\mathbb{R}^N)$.*

Proving such a proposition for the kernel K (with possible different rates of decay) and using the assumptions on the function f with a suitable value of α enables to dominate the function

$$y \mapsto |x|^p K(x-y)f(y)$$

on $\mathbb{R}^N$. We can then apply the dominated convergence theorem to get the pointwise convergence at infinity of (27), that is the existence of the limit of the function

$$R \mapsto R^p K * f(R\sigma)$$

when R tends to $+\infty$ for every $\sigma \in \mathbb{S}^{N-1}$.

We can illustrate this argument for the travelling waves for Gross-Pitaevskii equation, where it can be applied to equations (21) and (24). In this case, the kernels K_0, K_j and $L_{j,k}$ satisfy the assumptions of Theorem 5 by formulae (22), (23) and (25). Therefore, we can compute their limit at infinity by Theorem 5. Moreover, they belong to the space of functions

$$\mathcal{K}(\mathbb{R}^N) = \{u \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C}), \forall n \in \mathbb{N}, \forall \alpha \in]N-2, N], d^n u \in M^\infty_{\alpha+n}(\mathbb{R}^N)\}$$

by Proposition 3. Therefore, by the argument of domination just above, all of those kernels satisfy

Lemma 2. *Let $1 \leq j, k \leq N$ and assume the function $K : \mathbb{R}^N \mapsto \mathbb{C}$ is in $\mathcal{K}(\mathbb{R}^N)$ and its Fourier transform is a rational fraction which is only singular at the origin and belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. We consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N \cap M^\infty_{2N+1}(\mathbb{R}^N)^N$,

and we denote $g = K * f$. Then, we have for every $\sigma \in \mathbb{S}^{N-1}$,

- $R^N g(R\sigma) \underset{R \to +\infty}{\to} K_\infty(\sigma) \int_{\mathbb{R}^N} f(x)dx$,
- $R^{N+1} \partial_j g(R\sigma) \underset{R \to +\infty}{\to} K^j_\infty(\sigma) \int_{\mathbb{R}^N} f(x)dx$.
- $R^{N+2} \partial^2_{j,k} g(R\sigma) \underset{R \to +\infty}{\to} K^{j,k}_\infty(\sigma) \int_{\mathbb{R}^N} f(x)dx$.

Remarks 3. 1. We do not need to assume (ii) to prove the assertions on the pointwise convergence of the functions g and $\partial_j g$: we just need to suppose (ii) in the case of the functions $\partial^2_{j,k} g$.

2. The notations K_∞, K^j_∞ and $K^{j,k}_\infty$ denote the limits at infinity of the kernels K, $\partial_j K$ and $\partial^2_{j,k} K$ given by Theorem 5. In particular, we prove the pointwise convergence at infinity of some derivatives of g towards those limits. It will be very useful to compute some partial differential equations like equation (10). However, it introduces some technical difficulties on which we will come back in Subsections 3.5 and 3.6.

3. For other equations, we can obtain the domination very differently. In particular, the algebraic decay conditions appearing in (i) and (ii) are suitable for our equations, but they can be modified in another context. In the article of J.L. Bona and Yi A. Li [4], domination for a different class of equations in dimension one is obtained using a different type of argument.

The following lemma yields another illustration of the above argument for the composed Riesz kernels. It will also be useful to prove Theorem 1.

Lemma 3. *Let $1 \leq j, k, l \leq N$ and $\sigma \in \mathbb{S}^{N-1}$. We consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N+1}(\mathbb{R}^N)$,

(iii) $d^2 f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N+2}(\mathbb{R}^N)$,

and we denote $g = R_{j,k} * f$. Then, we have

- $R^N g(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k)\int_{\mathbb{R}^N} f(x)dx.$

- $R^{N+1}\partial_l g(R\sigma) \underset{R\to+\infty}{\to} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(-(\delta_{j,k}\sigma_l + \delta_{j,l}\sigma_k + \delta_{k,l}\sigma_j) + (N+2)\sigma_j\sigma_k\sigma_l)\int_{\mathbb{R}^N} f(x)dx.$

Remarks 4. 1. We do not need to assume (iii) to show the existence of the pointwise limit of the function g. Moreover, the algebraic decay conditions appearing in (i), (ii) and (iii) should be fixed appropriately for different equations.

2. In Lemma 3 like in Lemma 2, we also prove the pointwise convergence at infinity of the gradient of g. It also introduces some technical difficulties on which we will come back in Subsections 3.5 and 3.6.

Finally, by convolution equations (21) and (24), Lemmas 2 and 3 yield the pointwise convergence at infinity of the functions η and θ.

Proposition 4. *Let $\sigma \in \mathbb{S}^{N-1}$ and $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$. Then, there exist some bounded measurable functions η_∞^α and θ_∞^α on $\mathbb{S}^{N-1}$ such that*

$$\begin{cases} R^{N+|\alpha|}\partial^\alpha\eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty^\alpha(\sigma), \\ R^{N-1+|\alpha|}\partial^\alpha\theta(R\sigma) \underset{R\to+\infty}{\to} \theta_\infty^\alpha(\sigma). \end{cases}$$

Remark 7. In particular, we prove the pointwise convergence at infinity of some derivatives of η and θ. Though it introduces some technical difficulties on which we will come back in Subsections 3.5 and 3.6, it is a decisive step to derive equation (10).

On the other hand, in Theorem 1, we would like rather more than the pointwise convergence of the function

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

towards its limit v_∞. We would like to prove its uniform convergence, i.e. whether the function

$$\sigma \mapsto R^{N-1}(v(R\sigma) - 1)$$

converges to v_∞ in $L^\infty(\mathbb{S}^{N-1})$ when R tends to $+\infty$. Coming back to our model problem (27), it means that we must prove whether the function

$$\sigma \mapsto R^p K * f(R\sigma) = R^p \int_{\mathbb{R}^N} K(R\sigma - y)f(y)dy$$

converges in $L^\infty(\mathbb{S}^{N-1})$ when R tends to $+\infty$.

3.4 Uniformity of the convergence

To solve this difficulty, our argument relies on Ascoli-Arzela's theorem. Indeed, we already know the existence of a pointwise limit at infinity, so, it will give the uniformity of the convergence. However, Ascoli-Arzela's theorem requires some compactness: we deduce it from the algebraic decay of the gradient of the function $K * f$. For instance, the sequence of functions

$$\sigma \mapsto R^p \nabla^{\mathbb{S}^{N-1}}(K * f)(R\sigma)$$

is uniformly bounded on $\mathbb{S}^{N-1}$, which yields the desired compactness.

Thus, in the context of Gross-Pitaevskii equation, we convert the pointwise convergence of Proposition 4 in a uniform one.

Proposition 5. *There exist some functions* $(\eta_\infty, v_\infty) \in C^1(\mathbb{S}^{N-1})^2$ *and* $\theta_\infty \in C^2(\mathbb{S}^{N-1})$ *such that*

- $R^N \eta(R\sigma) \underset{R\to+\infty}{\to} \eta_\infty(\sigma)$ in $C^1(\mathbb{S}^{N-1})$,
- $R^{N-1}\theta(R\sigma) \underset{R\to+\infty}{\to} \theta_\infty(\sigma)$ in $C^2(\mathbb{S}^{N-1})$,
- $R^{N-1}(v(R\sigma) - 1) \underset{R\to+\infty}{\to} v_\infty(\sigma)$ in $C^1(\mathbb{S}^{N-1})$.

Remark 8. Actually, we prove the convergence at infinity of η, θ and v in some spaces $C^1(\mathbb{S}^{N-1})$ or $C^2(\mathbb{S}^{N-1})$, better than $L^\infty(\mathbb{S}^{N-1})$. It will be fruitful to derive equation (10).

The main difficulty here is to compute the gradient of the function $K*f$. Indeed, the gradient of such a convolution is not always the convolution $(\nabla K) * f$, in particular if the kernel K is not sufficiently smooth. We will see how to overcome such a difficulty in the next subsection.

3.5 Derivation of equation (10)

In the previous subsections, we obtained a uniform limit at infinity, denoted $L_\infty : \mathbb{S}^{N-1} \to \mathbb{C}$, for the function

$$x \mapsto |x|^p K * f(x).$$

An ultimate goal for this equation and similar ones would be to obtain an explicit formula for L_∞. However, this seems rather difficult, though presumably not completely out of reach (see Conjecture 1 for the Gross-Pitaevskii equation). Instead, we compute an elliptic partial differential equation satisfied by L_∞, namely equation (10) in our context. In some cases, for instance assuming L_∞ is axisymmetric, this equation may lead to the explicit form of L_∞ (see Theorems 2 and 3).

In order to derive such an equation, we take the limit at infinity of the partial differential equation satisfied by the function $K*f$ on $\mathbb{R}^N$ (equation (2) in our case). The implementation of this argument requires some precise knowledge of the convergence at infinity of some derivatives of the convolution $K * f$ to the corresponding derivatives of L_∞. In order to obtain it, we face a new difficulty related to the singularity at the origin of the kernels. Indeed, many of the derivatives of our kernels present a non-integrable singularity at the origin, and therefore, we are not allowed to derivate the convolution equation without additional care. The method to overcome this difficulty is reminiscent of some classical arguments in distribution theory, using integral formulae. More precisely, consider a kernel K which belongs to $\mathcal{K}(\mathbb{R}^N)$. Its gradient K is in $L^1(\mathbb{R}^N)$, which yields

$$\nabla(K * f) = (\nabla K) * f,$$

provided that f belongs for instance to some space $L^p(\mathbb{R}^N)$. However, we *cannot* write

$$d^2(K * f) = (d^2 K) * f,$$

mainly since we do not know enough integrability for the second derivative of K. Yet, we can find an explicit expression for the second derivative of $K * f$, provided that f is sufficiently smooth.

Lemma 4. *Let* $1 \leq j, k \leq N$ *and* $K \in \mathcal{K}(\mathbb{R}^N)$*. Consider a function* $f \in C^\infty(\mathbb{R}^N)$ *such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N$,

and denote $g = K * f$. Then, the second order partial derivative $\partial^2_{j,k} g$ of g is continuous on $\mathbb{R}^N$ and satisfies

$$\begin{aligned}\forall x \in \mathbb{R}^N, \partial^2_{j,k} g(x) = &\int_{B(0,1)^c} \partial^2_{j,k} K(y) f(x-y) dy + \int_{B(0,1)} \partial^2_{j,k} K(y)(f(x-y) - f(x)) dy \\ &+ \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy \right) f(x).\end{aligned} \tag{31}$$

Remarks 5. 1. Conditions (i) and (ii) are suitable in our context, since the functions F and G previously defined in equations (19) and (20) satisfy such conditions. However, they can be chosen differently for other equations.

2. Formula (31) is quite similar to the expected expression $(\partial^2_{j,k} K) * f$, which cannot hold since the function $\partial^2_{j,k} K$ presents a singularity at the origin. Indeed, the function K has a double partial derivative $D^2_{j,k} K$ in the sense of distributions, which is equal to

$$D^2_{j,k} K = \partial^2_{j,k} K 1_{B(0,1)^c} + PV(\partial^2_{j,k} K 1_{B(0,1)}) + \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy \right) \delta_0,$$

where $PV(\partial^2_{j,k} K 1_{B(0,1)})$ is the principal value at the origin of the function $\partial^2_{j,k} K$, given by

$$\forall \phi \in C^\infty_c(B(0,1)), < PV(\partial^2_{j,k} K 1_{B(0,1)}), \phi >= \int_{B(0,1)} \partial^2_{j,k} K(x)(\phi(x) - \phi(0)) dx.$$

Then, the double partial derivative in the sense of distribution of $K * f$ is equal to the distribution $D^2_{j,k} K * f$, which yields formula (31).

Likewise, we can compute explicit formulae for the first and second order derivatives of the composed Riesz kernels.

Lemma 5. *Let $1 \leq j, k, l, m \leq N$ and denote*

$$\forall y \in \mathbb{R}^N \setminus \{0\}, R_{j,k}(y) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}}.$$

We consider a function $f \in C^\infty(\mathbb{R}^N)$ such that

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(iii) $d^2 f \in L^\infty(\mathbb{R}^N)$,

and we set $g = R_{j,k} * f$. Then, g is C^1 on $\mathbb{R}^N$ and satisfies for every $x \in \mathbb{R}^N$,

$$\begin{aligned}\partial_l g(x) = &\int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y) dy + \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x)) dy \\ &+ \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x) - y.\nabla f(x)) dy.\end{aligned} \tag{32}$$

Moreover, if f verifies

(iv) $d^3 f \in L^\infty(\mathbb{R}^N)$,

g is C^2 on $\mathbb{R}^N$ and verifies for every $x \in \mathbb{R}^N$,

$$\begin{aligned}
\partial^2_{l,m} g(x) = & \int_{B(0,1)^c} \partial^2_{l,m} R_{j,k}(y) f(x-y)dy + \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) \\
& - \frac{1}{2} d^2 f(x)(y,y))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (\partial_m f(x) - y.\nabla \partial_m f(x))dy + \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) \\
& (f(x) - y.\nabla f(x) + \frac{1}{2} d^2 f(x)(y,y)) y_m dy.
\end{aligned} \tag{33}$$

Remarks 6. 1. The algebraic decay conditions appearing in (i) and (ii) should be adapted for various other kernels.

2. The derivatives and double derivatives of the composed Riesz kernels present singularities at the origin, which are finite parts of the functions $\partial_l R_{j,k}$ and $\partial^2_{l,m} R_{j,k}$, and some derivatives of the Dirac mass δ_0. They both appear in formulae (32) and (33) as they previously appeared in formula (31).

Formulae (31), (32) and (33) suitably replace convolution equations to prove the convergence at infinity of some derivatives of the convolution $K * f$. Indeed, instead of computing the pointwise limit at infinity of (27), we now compute the limit at infinity of functions such as

$$x \mapsto |x|^p \int_{B(x,1)} \partial^2_{j,k} K(x-y)(f(y) - f(x))dy.$$

However, the argument is the same as in Subsection 3.3. We first use Theorem 5 to prove the convergence at infinity of the derivatives of the kernel K, and then, Propositions 2 and 3 to dominate the convergence and get its uniformity. It yields the convergence at infinity of some derivatives of the convolution $K * f$, which was yet mentioned in Lemmas 2 and 3. Finally, by the above argument, we obtain some partial differential equation for the function L_∞, which completes the study of the asymptotics at infinity of a function given by a convolution equation. In particular, in our context, by equations (21) and (24), it yields a system of linear partial differential equations on the sphere $\mathbb{S}^{N-1}$ for the functions η_∞ and θ_∞, from which we can deduce equation (10).

Proposition 6. *The functions η_∞ and θ_∞ are in $C^\infty(\mathbb{S}^{N-1})$ and satisfy for every $\sigma \in \mathbb{S}^{N-1}$*

$$\eta_\infty(\sigma) = c(\partial_1^{\mathbb{S}^{N-1}} \theta_\infty(\sigma) - (N-1)\sigma_1 \theta_\infty(\sigma)), \tag{34}$$

$$\Delta^{\mathbb{S}^{N-1}} \theta_\infty(\sigma) + (N-1)\theta_\infty(\sigma) = \frac{c}{2}(\partial_1^{\mathbb{S}^{N-1}} \eta_\infty(\sigma) - N\sigma_1 \eta_\infty(\sigma)). \tag{35}$$

3.6 Completing the proof of Theorem 1

Theorem 1 is a consequence of Proposition 5, which yields the uniform convergence of the function

$$x \mapsto |x|^{N-1}(v(x) - 1)$$

towards v_∞, and of Proposition 6, which specifies the partial differential equation (10) satisfied by v_∞.

However, in order to complete its proof, we must mention some technical difficulties. In the case of the travelling waves for the Gross-Pitaevskii equation, the decay estimates obtained in Proposition 2 for the functions η, $\psi\theta$ and v are not sufficient to dominate the convergence at infinity of the functions $d^2\eta$, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$ and to prove the uniformity of the convergences of $\nabla\eta$, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$. They are neither sufficient to apply Lemmas 2 and 3, nor to prove Proposition 5.

Thus, we improve Proposition 2 for the functions $d^2\eta$, $d^2(\psi\theta)$, $d^2 v$ and $d^3(\psi\theta)$ in the following theorem.

Theorem 6. *Let v, a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$. Then, we have*

- $(d^2(\psi\theta), d^2 v) \in M^\infty_{N+1}(\mathbb{R}^N)^2$,
- $(d^2\eta, d^3(\psi\theta)) \in M^\infty_{N+2}(\mathbb{R}^N)^2$.

This improvement relies on the method introduced by J.L. Bona and Yi A. Li [4], A. de Bouard and J.C Saut [5] and M. Maris [15, 16]. To get a feeling for the idea of this method, let us compute for instance the algebraic decay of the function $d^2\eta$. By equation (21), we must estimate the algebraic decay of the function $d^2(K_0 * F)$, which reduces by equation (31) to prove in particular that the function

$$x \mapsto \int_{B(0,1)^c} \partial^2_{j,k} K_0(y) F(x-y) dy$$

belongs to $M^\infty_{N+2}(\mathbb{R}^N)$. The method just mentioned above now consists in writing for every $x \in \mathbb{R}^N$,

$$\begin{aligned} |x|^{N+2} \left| \int_{B(0,1)^c} \partial^2_{j,k} K_0(y) F(x-y) dy \right| \leq & A \Bigg(\int_{B(0,1)^c} |\partial^2_{j,k} K_0(y)| |y|^{N+2} |F(x-y)| dy \\ & + \int_{B(0,1)^c} |\partial^2_{j,k} K_0(y)| |x-y|^{N+2} |F(x-y)| dy \Bigg) \\ \leq & A(\|\partial^2_{j,k} K_0\|_{M^\infty_{N+2}(\mathbb{R}^N)} \|F\|_{L^1(\mathbb{R}^N)} \\ & + \|\partial^2_{j,k} K_0\|_{L^1(B(0,1)^c)} \|F\|^{\frac{N+2}{2N}}_{M^\infty_{N+2}(\mathbb{R}^N)} \|F\|^{\frac{N-2}{2N}}_{L^\infty(\mathbb{R}^N)}), \end{aligned}$$

and verifying that those norms are finite. Thus, this method connects the algebraic decay of the function $d^2\eta$ for instance, to the decay of the kernels d^2K_0 or d^2K_j. The main point is that in the case of superlinear nonlinearities (such as the almost quadratic nonlinearities F and G), the decay of the function is equal to the decay of the kernels. Applying this argument to each integral appearing in equations (21) and (31), we can obtain the optimal algebraic decay of the function $d^2\eta$, which is equal to the decay of the kernels d^2K_0 and d^2K_j. This yields Theorem 6 [3], from which we deduce the useful following corollary concerning the nonlinear functions F and G.

Corollary 2. *The functions F and G belong to $M^\infty_{2N}(\mathbb{R}^N)$, their gradients, to $M^\infty_{2N+1}(\mathbb{R}^N)$, and the second order derivatives of G, to $M^\infty_{2N+2}(\mathbb{R}^N)$.*

Finally, it completes the sketch of the proof of Theorem 1. Indeed, by Corollary 2, we now have sufficient decay rates for the nonlinear functions F and G to apply Lemmas 2 and 3 and prove the convergence at infinity of the functions $d^2\eta$, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$. Likewise, by Theorem 6, we also have sufficient decay rates for the functions $d^2\eta$, $d^2(\psi\theta)$ and $d^3(\psi\theta)$ to prove the uniformity of the convergences mentioned in Proposition 5.

[3] Theorem 6 is supposed to be optimal. Indeed, it is commonly conjectured that the functions $\partial^\alpha\eta$, $\partial^\alpha\nabla(\psi\theta)$ and $\partial^\alpha\nabla v$ are in $M^\infty_{N+|\alpha|}(\mathbb{R}^N)$, at least in the case where $|\alpha| \leq N$.

4 Sketch of the proofs of Theorems 2 and 3

Theorems 2 and 3 both rely on the same argument: the explicit integration of the system of equations (34) and (35). Indeed, this system presents the striking property to be integrable in dimension two and in the axisymmetric case. In both cases, it reduces to a system of linear ordinary differential equations of second order, which is entirely integrable in spherical coordinates, i.e.

$$\sigma = (\cos(\beta_1), \cos(\beta_2)\sin(\beta_1), \ldots, \sin(\beta_1)\ldots\sin(\beta_{N-1})).$$

In particular, the integration of this system yields formulae (11) and (13).

Proposition 7. *In the axisymmetric case, there is a constant α such that for every $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}$,*

$$\eta_\infty(\sigma) = \alpha c \left(\frac{1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - N \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}} \right), \tag{36}$$

$$\theta_\infty(\sigma) = \alpha \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}. \tag{37}$$

Likewise, in dimension two, there are constants α and β such that for every $\sigma = (\sigma_1, \sigma_2) \in \mathbb{S}^1$,

$$\eta_\infty(\sigma) = \alpha c \left(\frac{1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} - \frac{2\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^2} \right) - 2\beta c \frac{\sigma_1\sigma_2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^2}, \tag{38}$$

$$\theta_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}}. \tag{39}$$

Remark 9. The result above in dimension two holds for every subsonic travelling wave of finite energy, and not only for the axisymmetric ones.

The only remaining difficulty is now to compute the values of the coefficients α and β. We link them with the energy $E(v)$ and the momentum $\vec{P}(v)$ by some integral relations obtained by standard integrations by parts.

Lemma 6. *Let v, a travelling wave for the Gross-Pitaevskii equation in dimension $N \geq 2$ of finite energy and speed $0 < c < \sqrt{2}$. Then, we have*

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2c(1 - \frac{2}{c^2})p(v) = c(\frac{2N}{c^2} - 1) \int_{\mathbb{S}^{N-1}} \sigma_1 \theta_\infty(\sigma) d\sigma + \int_{\mathbb{S}^{N-1}} \sigma_1^2 \eta_\infty(\sigma) d\sigma, \tag{40}$$

$$\forall 2 \leq j \leq N, P_j(v) = \frac{c}{4} \int_{\mathbb{S}^{N-1}} \sigma_j \sigma_1 \eta_\infty(\sigma) d\sigma + \frac{N}{2} \int_{\mathbb{S}^{N-1}} \sigma_j \theta_\infty(\sigma) d\sigma. \tag{41}$$

Remark 10. Lemma 6 holds even if the travelling waves are not axisymmetric.

Theorems 2 and 3 then follow from equations (36), (37), (38), (39), (40) and (41), and from the standard Pohozaev identities, which were derived in [8].

Lemma 7 ([8]). *Let $0 < c < \sqrt{2}$. A finite energy solution v to equation* (2) *satisfies the two identities*

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2 \tag{42}$$

$$\forall 2 \leq j \leq N, E(v) = \int_{\mathbb{R}^N} |\partial_j v|^2 + cp(v). \tag{43}$$

Remark 11. Lemma 7 holds even if the travelling waves are not axisymmetric and if the speed c is not subsonic ($c = 0$ or $c > \sqrt{2}$).

5 Plan of the paper

The paper is divided in three parts. In the first part, we derive the improved decay estimates for the travelling waves in the Gross-Pitaevskii equation stated in Theorem 6. In a first section, we prove Lemmas 4 and 5 to obtain explicit integral expressions for some derivatives of the functions η and $\psi\theta$, on which the proof of Theorem 6 relies. In the second section, we compute the algebraic decay of those derivatives by the argument yet mentioned of J.L. Bona and Yi A. Li [4], A. de Bouard and J.C Saut [5] and M. Maris [15, 16]. Finally, we complete this section by inferring Corollary 2.

The proof of Theorem 1 forms the core of the second part. The first ingredient is the pointwise convergence at infinity of the kernels K_0, K_j and $L_{j,k}$: it follows from the proofs of Lemma 1 and Theorem 5 in the first section. The second and third sections are devoted to the proof of the pointwise convergence at infinity of the functions η, $\psi\theta$ and of some of their derivatives summed up in Proposition 4. It relies on Lemmas 2 and 3. In the fourth section, we deduce from Ascoli-Arzela's theorem and the improved decay estimates of the first part, the uniformity of the convergence yet described in Proposition 5. Finally, the last section is devoted to the proof of Proposition 6. Then, Theorem 1 follows from the remark that

$$v_\infty = \theta_\infty,$$

and the derivation of equation (10) from equations (34) and (35).

The third part is mainly concerned with the proofs of Theorems 2 and 3. In the first section, we integrate the system of equations (34) and (35) to deduce Proposition 7. In the second section, we infer Lemma 6 to compute the values of the coefficients α and β in function of the energy $E(v)$ and the momentum $\vec{P}(v)$. Finally, we end the paper by deducing Corollary 1 from Lemma 7.

1 Sharp decay of some derivatives of a travelling wave

We first improve the asymptotic decay estimates given in [9] by proving Theorem 6. We state integral representations of the functions $d^2\eta$, $d^2(\psi\theta)$ and $d^3(\psi\theta)$ and estimate their algebraic decay by the standard argument mentioned in the introduction.

1.1 Integral forms of the functions $d^2\eta$, $d^2\theta$ and $d^3\theta$

As mentioned above, the functions $d^2\eta$, $d^2(\psi\theta)$ and $d^3(\psi\theta)$ express as linear combinations of convolution integrals.

Proposition 8. *Let $1 \leq j, k, l \leq N$ and $x \in \mathbb{R}^N$. Then,*

$$\begin{aligned}\partial^2_{j,k}\eta(x) = &\int_{B(0,1)^c} \partial^2_{j,k}K_0(y)F(x-y)dy + \int_{B(0,1)} \partial^2_{j,k}K_0(y)(F(x-y)-F(x))dy \\ &+ \left(\int_{\mathbb{S}^{N-1}} \partial_j K_0(y)y_k dy\right) F(x) + 2c\sum_{i=1}^N \left(\int_{B(0,1)^c} \partial^2_{j,k}K_i(y)G_i(x-y)dy\right. \\ &\left.+ \int_{B(0,1)} \partial^2_{j,k}K_i(y)(G_i(x-y)-G_i(x))dy + \left(\int_{\mathbb{S}^{N-1}} \partial_j K_i(y)y_k dy\right) G_i(x)\right),\end{aligned} \tag{44}$$

$$\partial^2_{j,k}(\psi\theta)(x) = \frac{c}{2}\partial_k K_j * F(x) + c^2 \sum_{i=1}^N \partial_k L_{i,j} * G_i(x) + \sum_{i=1}^N \Bigg(\int_{B(0,1)^c} \partial_k R_{i,j}(y) G_i(x-y)dy$$
$$+ \int_{B(0,1)} \partial_k R_{i,j}(y)(G_i(x-y) - G_i(x) + y.\nabla G_i(x))dy + \int_{\mathbb{S}^{N-1}} R_{i,j}(y)y_k \tag{45}$$
$$(G_i(x) - y.\nabla G_i(x))dy \Bigg),$$

$$\partial^3_{j,k,l}(\psi\theta)(x) = \frac{c}{2} \Bigg(\int_{B(0,1)^c} \partial^2_{k,l} K_j(y) F(x-y)dy + \int_{B(0,1)} \partial^2_{k,l} K_j(y)(F(x-y) - F(x))dy$$
$$+ \left(\int_{\mathbb{S}^{N-1}} \partial_l K_j(y) y_k dy \right) F(x) \Bigg) + c^2 \sum_{i=1}^N \Bigg(\int_{B(0,1)^c} \partial^2_{k,l} L_{i,j}(y) G_i(x-y)dy$$
$$+ \int_{B(0,1)} \partial^2_{k,l} L_{i,j}(y)(G_i(x-y) - G_i(x))dy + \left(\int_{\mathbb{S}^{N-1}} \partial_l L_{i,j}(y) y_k dy \right) G_i(x) \Bigg)$$
$$+ \sum_{i=1}^N \Bigg(\int_{B(0,1)^c} \partial^2_{k,l} R_{i,j}(y) G_i(x-y)dy + \int_{B(0,1)} \partial^2_{k,l} R_{i,j}(y)(G_i(x-y) \tag{46}$$
$$- G_i(x) + y.\nabla G_i(x) - \frac{1}{2} d^2 G_i(x)(y,y))dy + \int_{\mathbb{S}^{N-1}} R_{i,j}(y) y_k (\partial_l G_i(x)$$
$$- y.\nabla \partial_l G_i(x))dy + \int_{\mathbb{S}^{N-1}} \partial_k R_{i,j}(y) y_l (G_i(x) - y.\nabla G_i(x)$$
$$+ \frac{1}{2} d^2 G_i(x)(y,y))dy \Bigg).$$

Proposition 8 is a straightforward consequence of Lemmas 4 and 5, so we postpone its proof after their proofs.

Proof of Lemma 4. Consider $t \in]-\frac{1}{2}, \frac{1}{2}[\setminus\{0\}$. On one hand, K is in $\mathcal{K}(\mathbb{R}^N)$, so, the function $\partial_k K$ belongs to $L^1(\mathbb{R}^N)$. On the other hand, f satisfies assumption (i), so, it is a continuous, bounded function on $\mathbb{R}^N$. Therefore, by standard convolution theory, the distribution $\partial_k g$ is actually a continuous function on $\mathbb{R}^N$, which writes

$$\forall x \in \mathbb{R}^N, \partial_k g(x) = \int_{\mathbb{R}^N} \partial_k K(y) f(x-y)dy.$$

Hence, we can compute

$$\frac{\partial_k g(x+te_j) - \partial_k g(x)}{t} = \int_{\mathbb{R}^N} \partial_k K(y) \frac{f(x+te_j-y) - f(x-y)}{t} dy$$
$$= \int_{\mathbb{R}^N} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} f(x-y)dy,$$

and therefore,

$$\frac{\partial_k g(x+te_j) - \partial_k g(x)}{t} = \int_{B(0,1)^c} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} f(x-y)dy$$
$$+ \left(\int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} dy \right) f(x) \tag{47}$$
$$+ \int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x))dy.$$

For the first term, we state

$$\forall y \in B(0,1)^c, \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} f(x-y) \underset{t\to 0}{\to} \partial^2_{j,k} K(y) f(x-y),$$

while, by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\begin{aligned}\forall y \in B(0,1)^c, \left| \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} f(x-y) \right| &\leq \frac{A}{t(1+|x-y|^{2N})} \int_0^t |\partial^2_{j,k} K(y+se_j)| ds \\ &\leq \frac{A}{(1+|x-y|^{2N})(|y|-\frac{1}{2})^{N+2}},\end{aligned}$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)^c} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} f(x-y) dy \underset{t\to 0}{\to} \int_{B(0,1)^c} \partial_{j,k} K(y) f(x-y) dy.$$

For the second term, we compute by integration by parts since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} dy = \int_{\mathbb{S}^{N-1}} \frac{K(y+te_j) - K(y)}{t} y_k dy.$$

K being in $\mathcal{K}(\mathbb{R}^N)$ once more, we get

$$\forall y \in \mathbb{S}^{N-1}, \left| \frac{K(y+te_j) - K(y)}{t} y_k \right| \leq \frac{A}{t} \int_0^t |\partial_j K(y+se_j)| ds \leq A,$$

hence, by the dominated convergence theorem,

$$\int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} dy \underset{t\to 0}{\to} \int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy.$$

For the last term, we find

$$\begin{aligned}&\int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) dy \\ &= \int_{|y|<2|t|} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) dy \\ &+ \int_{2|t|<|y|<1} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) dy.\end{aligned}$$

On one hand, by assumption (ii) and since $K \in \mathcal{K}(\mathbb{R}^N)$, we have

$$\begin{aligned}&\left| \int_{|y|<2|t|} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t} (f(x-y) - f(x)) dy \right| \\ &\leq \frac{A}{|t|} \int_{|y|<2|t|} \left(\frac{1}{|y+te_j|^{N-\frac{1}{2}}} + \frac{1}{|y|^{N-\frac{1}{2}}} \right) |y| dy \\ &\leq \frac{A}{|t|} \left(\int_{|y|<2|t|} \frac{dy}{|y|^{N-\frac{3}{2}}} + \int_{|y|<2|t|} \frac{dy}{|y+te_j|^{N-\frac{3}{2}}} + \int_{|y|<2|t|} \frac{|t| dy}{|y+te_j|^{N-\frac{1}{2}}} \right) \\ &\leq A\sqrt{|t|} \underset{t\to 0}{\to} 0.\end{aligned}$$

0n the other hand, we obtain likewise for $2|t| < |y| < 1$,

$$\begin{aligned}\left|\frac{\partial_k K(y+te_j) - \partial_k K(y)}{t}(f(x-y) - f(x))\right| &\leq \frac{A|y|}{t}\int_0^t |\partial^2_{j,k}K(y+se_j)|dy \\ &\leq \frac{A|y|}{(|y|-|t|)^{N+\frac{1}{2}}} \\ &\leq \frac{A}{|y|^{N-\frac{1}{2}}},\end{aligned}$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)} \frac{\partial_k K(y+te_j) - \partial_k K(y)}{t}(f(x-y) - f(x))dy \underset{t\to 0}{\to} \int_{B(0,1)} \partial^2_{j,k}K(y)(f(x-y) - f(x))dy.$$

Finally, the function $\partial_k g$ is differentiable in direction x_j and, by equation (47), its partial derivative $\partial^2_{j,k}g$ is given by formula (31). Moreover, the function $\partial_k g$ is actually of class C^1 on $\mathbb{R}^N$. Indeed, by formula (31), $\partial^2_{j,k}g$ is continuous on $\mathbb{R}^N$. It follows from the continuity of f, assumptions (i) and (ii), the fact that K belongs to $\mathcal{K}(\mathbb{R}^N)$ and a standard application of the dominated convergence theorem. □

We now turn to the proof of Lemma 5, which is similar.

Proof of Lemma 5. We begin by the proof of formula (32). Since f is a smooth function on $\mathbb{R}^N$ which satisfies assumptions (i) and (ii), we can state by standard Riesz operator theory,

$$\forall x \in \mathbb{R}^N, g(x) = \int_{B(0,1)^c} R_{j,k}(y)f(x-y)dy + \int_{B(0,1)} R_{j,k}(y)(f(x-y) - f(x))dy.$$

In particular, g is a continuous function on $\mathbb{R}^N$ (which can also be deduced from a standard application of the dominated convergence theorem thanks to the continuity of f and assumptions (i) and (ii)). Therefore, assuming $t \in]-\frac{1}{2}, \frac{1}{2}[\setminus\{0\}$, we compute

$$\begin{aligned}\frac{g(x+te_l) - g(x)}{t} = &\int_{B(0,1)^c} R_{j,k}(y)\frac{f(x+te_l-y) - f(x-y)}{t}dy + \int_{B(0,1)} R_{j,k}(y) \\ &\left(\frac{f(x+te_l-y) - f(x-y)}{t} - \frac{f(x+te_l) - f(x)}{t}\right)dy.\end{aligned} \tag{48}$$

On one hand, by assumption (ii),

$$\begin{aligned}\forall y \in B(0,1)^c, \left|R_{j,k}(y)\frac{f(x+te_l-y) - f(x-y)}{t}\right| &\leq \frac{A}{t|y|^N}\int_0^t |\partial_l f(x+se_l-y)|ds \\ &\leq \frac{A}{|y|^N(1+|x-y|^{2N})},\end{aligned}$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)^c} R_{j,k}(y)\frac{f(x+te_l-y) - f(x-y)}{t}dy \underset{t\to 0}{\to} \int_{B(0,1)^c} R_{j,k}(y)\partial_l f(x-y)dy.$$

On the other hand, by assumption (iii),

$$\forall y \in B(0,1), \left| R_{j,k}(y) \left(\frac{f(x+te_l-y)-f(x-y)}{t} - \frac{f(x+te_l)-f(x)}{t} \right) \right|$$
$$\leq \frac{A}{t|y|^N} \int_0^t |\partial_l f(x+se_l-y) - \partial_l f(x+se_l)|\, ds$$
$$\leq \frac{A}{|y|^{N-1}} \sup_{z\in\mathbb{R}^N} |d^2 f(z)|,$$

therefore, by the dominated convergence theorem,

$$\int_{B(0,1)} R_{j,k}(y) \left(\frac{f(x+te_l-y)-f(x-y)}{t} - \frac{f(x+te_l)-f(x)}{t} \right) dy$$
$$\underset{t\to 0}{\longrightarrow} \int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy.$$

Thus, the function g is differentiable in direction x_l and, by equation (48), its partial derivative $\partial_l g$ is given by

$$\forall x \in \mathbb{R}^N, \partial_l g(x) = \int_{B(0,1)^c} R_{j,k}(y)\partial_l f(x-y)dy + \int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy. \quad (49)$$

Now, we integrate by parts the first term of the right member:

$$\int_{B(0,1)^c} R_{j,k}(y)\partial_l f(x-y)dy = \int_{B(0,1)^c} \partial_l R_{j,k}(y) f(x-y)dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l f(x-y)dy. \quad (50)$$

It can be made rigorously by integrating by parts on $B(0,R)\setminus B(0,1)$ for some large R and taking the limit $R\to+\infty$, using assumptions (i) and (ii). Likewise, assumption (iii) yields for the second term

$$\int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy = \lim_{\epsilon\to 0} \int_{\epsilon<|y|<1} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy.$$

However, we find by integrating by parts,

$$\int_{\epsilon<|y|<1} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy$$
$$= \int_{\epsilon<|y|<1} R_{j,k}(y)\partial_l f(x-y)dy$$
$$= \int_{\epsilon<|y|<1} \partial_l R_{j,k}(y) f(x-y)dy + \int_{S(0,\epsilon)} R_{j,k}(y)\frac{y_l}{\epsilon} f(x-y)dy - \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l f(x-y)dy$$
$$= \int_{\epsilon<|y|<1} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy + \int_{S(0,\epsilon)} R_{j,k}(y)\frac{y_l}{\epsilon}(f(x-y) - f(x)$$
$$+ y.\nabla f(x))dy - \int_{\mathbb{S}^{N-1}} R_{j,k}(y) y_l (f(x-y) - f(x) + y.\nabla f(x))dy.$$

Now, we remark by assumption (iii)

$$\forall y \in B(0,1), |\partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))| \leq \frac{A}{|y|^{N-1}} \sup_{z\in\mathbb{R}^N} |d^2 f(z)|,$$

so,

$$\int_{\epsilon<|y|<1} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy$$
$$\underset{\epsilon\to 0}{\to} \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy.$$

We also notice by assumption (iii)

$$\forall y \in S(0,\epsilon), |R_{j,k}(y)y_l(f(x-y) - f(x) + y.\nabla f(x))| \leq \frac{A}{\epsilon^{N-3}} \sup_{z\in\mathbb{R}^N} |d^2 f(z)|,$$

therefore,

$$\frac{1}{\epsilon}\int_{S(0,\epsilon)} R_{j,k}(y)y_l(f(x-y) - f(x) + y.\nabla f(x))dy \underset{\epsilon\to 0}{\to} 0.$$

Finally, it leads to

$$\begin{aligned}\int_{B(0,1)} R_{j,k}(y)(\partial_l f(x-y) - \partial_l f(x))dy = &\int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x))dy \\ &- \int_{\mathbb{S}^{N-1}} R_{j,k}(y)y_l(f(x-y) - f(x) + y.\nabla f(x))dy.\end{aligned} \tag{51}$$

Finally, by combining equations (49), (50) and (51), the partial derivative $\partial_l g$ is given by formula (32). Thus, the function g is actually of class C^1 on $\mathbb{R}^N$. Indeed, by formula (32), $\partial_l g$ is continuous on $\mathbb{R}^N$. By a standard application of the dominated convergence theorem, it follows from the smoothness of f and assumptions (i), (ii) and (iii).

We now turn to formula (33) and we assume again that $t \in]-\frac{1}{2}, \frac{1}{2}[\setminus\{0\}$. Since f satisfies assumptions (i), (ii) and (iii), $\partial_l g$ is continuous on $\mathbb{R}^N$ and satisfies formula (32),

$$\begin{aligned}\forall x \in \mathbb{R}^N, \partial_l g(x) = &\int_{B(0,1)^c} \partial_l R_{j,k}(y)f(x-y)dy + \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y) - f(x) \\ &+ y.\nabla f(x))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y)y_l(f(x) - y.\nabla f(x))dy.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial_l g(x+te_m) - \partial_l g(x)}{t} = &\int_{B(0,1)^c} \partial_l R_{j,k}(y)\frac{f(x+te_m-y) - f(x-y)}{t}dy + \int_{B(0,1)} \partial_l R_{j,k}(y) \\ &\left(\frac{f(x+te_m-y) - f(x-y)}{t} - \frac{f(x+te_m) - f(x)}{t} + y.\right. \\ &\left.\frac{\nabla f(x+te_m) - \nabla f(x)}{t}\right) dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y)\left(\frac{f(x+te_m) - f(x)}{t}\right. \\ &\left.-y.\frac{\nabla f(x+te_m) - \nabla f(x)}{t}\right) y_l dy.\end{aligned} \tag{52}$$

On one hand, by assumption (ii),

$$\begin{aligned}\forall y \in B(0,1)^c, \left|\partial_l R_{j,k}(y)\frac{f(x+te_m-y) - f(x-y)}{t}\right| &\leq \frac{A}{t|y|^{N+1}}\int_0^t |\partial_m f(x+se_m-y)|ds \\ &\leq \frac{A}{|y|^{N+1}(1+|x-y|^{2N})},\end{aligned}$$

so, by the dominated convergence theorem,

$$\int_{B(0,1)^c} \partial_l R_{j,k}(y) \frac{f(x+te_m-y)-f(x-y)}{t} dy \underset{t\to 0}{\to} \int_{B(0,1)^c} \partial_l R_{j,k}(y)\partial_m f(x-y)dy.$$

On the other hand, assumption (iv) yields for every $y \in B(0,1)$,

$$\begin{aligned}&\left|\frac{\partial_l R_{j,k}(y)}{t}\big(f(x+te_m-y)-f(x-y)-f(x+te_m)+f(x)+y.(\nabla f(x+te_m)-\nabla f(x))\big)\right| \\ &\leq \frac{A}{t|y|^{N+1}}\int_0^t |\partial_m f(x+se_m-y)-\partial_m f(x+se_m)+y.\nabla\partial_m f(x+se_m)|ds \\ &\leq \frac{A}{|y|^{N-1}}\sup_{z\in\mathbb{R}^N}|d^3 f(z)|,\end{aligned}$$

hence, by the dominated convergence theorem,

$$\begin{aligned}&\int_{B(0,1)^c} \frac{\partial_l R_{j,k}(y)}{t}\Big(f(x+te_m-y)-f(x-y)-f(x+te_m)+f(x)+y.(\nabla f(x+te_m) \\ &-\nabla f(x))\Big)dy \underset{t\to 0}{\to} \int_{B(0,1)^c} \partial_l R_{j,k}(y)(\partial_m f(x-y)-\partial_m f(x)+y.\nabla\partial_m f(x))dy.\end{aligned}$$

Finally, f is in $C^\infty(\mathbb{R}^N)$, which gives

$$\begin{aligned}&\int_{\mathbb{S}^{N-1}} R_{j,k}(y)y_l\left(\frac{f(x+te_m)-f(x)}{t} - y.\frac{\nabla f(x+te_m)-\nabla f(x)}{t}\right)dy \\ &\underset{t\to 0}{\to} \int_{\mathbb{S}^{N-1}} y_l R_{j,k}(y)(\partial_m f(x) - y.\nabla\partial_m f(x))dy.\end{aligned}$$

Thus, the function $\partial_l g$ is differentiable in direction x_m and, by equation (52), its partial derivative $\partial^2_{l,m} g$ is given by

$$\begin{aligned}\forall x\in\mathbb{R}^N, \partial^2_{l,m}g(x) = &\int_{B(0,1)^c}\partial_l R_{j,k}(y)\partial_m f(x-y)dy + \int_{B(0,1)}\partial_l R_{j,k}(y)(\partial_m f(x-y) \\ &-\partial_m f(x)+y.\nabla\partial_m f(x))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y)y_l(\partial_m f(x)-y.\nabla\partial_m f(x))dy.\end{aligned} \tag{53}$$

Now, we integrate by parts the first term of the right member:

$$\begin{aligned}\int_{B(0,1)^c}\partial_l R_{j,k}(y)\partial_m f(x-y)dy = &\int_{B(0,1)^c}\partial^2_{l,m}R_{j,k}(y)f(x-y)dy + \int_{\mathbb{S}^{N-1}}\partial_l R_{j,k}(y)y_m \\ &f(x-y)dy.\end{aligned} \tag{54}$$

Similarly to equation (50), it can be made rigorously by integrating by parts on $B(0,R)\setminus B(0,1)$ for some large R and taking the limit $R\to+\infty$, using assumptions (i) and (ii). Likewise, assumption (iv) yields

$$\begin{aligned}&\int_{B(0,1)}\partial_l R_{j,k}(y)(\partial_m f(x-y)-\partial_m f(x)+y.\nabla\partial_m f(x))dy \\ &= \lim_{\epsilon\to 0}\int_{\epsilon<|y|<1}\partial_l R_{j,k}(y)(\partial_m f(x-y)-\partial_m f(x)+y.\nabla\partial_m f(x))dy.\end{aligned}$$

However, we compute by integrating by parts

$$\begin{aligned}
&\int_{\epsilon<|y|<1} \partial_l R_{j,k}(y)(\partial_m f(x-y) - \partial_m f(x) + y.\nabla\partial_m f(x))dy \\
&= \int_{\epsilon<|y|<1} \partial^2_{l,m} R_{j,k}(y) f(x-y)dy - \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m f(x-y)dy + \int_{S(0,\epsilon)} \partial_l R_{j,k}(y)\frac{y_m}{\epsilon} \\
&\quad f(x-y)dy - \int_{\epsilon<|y|<1} \partial_l R_{j,k}(y)dy\ \partial_m f(x) + \int_{\epsilon<|y|<1} \partial_l R_{j,k}(y) y.\nabla\partial_m f(x)dy \\
&= \int_{\epsilon<|y|<1} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy \\
&- \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy \\
&+ \int_{S(0,\epsilon)} \partial_l R_{j,k}(y)\frac{y_m}{\epsilon}(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy.
\end{aligned}$$

We then notice by assumption (iv) for every $y \in B(0,1)$,

$$|\partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))| \leq \frac{A}{|y|^{N-1}} \sup_{z\in\mathbb{R}^N} |d^3 f(z)|,$$

therefore,

$$\begin{aligned}
&\int_{\epsilon<|y|<1} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy \\
&\underset{\epsilon\to 0}{\to} \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy.
\end{aligned}$$

We also remark by assumption (iv) for every $y \in S(0,\epsilon)$,

$$|\partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))| \leq \frac{A}{\epsilon^{N-3}} \sup_{z\in\mathbb{R}^N} |d^3 f(z)|,$$

which gives

$$\frac{1}{\epsilon}\int_{S(0,\epsilon)} \partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy \underset{\epsilon\to 0}{\to} 0.$$

Thus, we find

$$\begin{aligned}
&\int_{B(0,1)} \partial_l R_{j,k}(y)(\partial_m f(x-y) - \partial_m f(x) + y.\nabla\partial_m f(x))dy \\
&= \int_{B(0,1)} \partial^2_{l,m} R_{j,k}(y)(f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy \\
&- \int_{\mathbb{S}^{N-1}} \partial_l R_{j,k}(y) y_m (f(x-y) - f(x) + y.\nabla f(x) - \frac{1}{2}d^2 f(x)(y,y))dy.
\end{aligned} \tag{55}$$

Finally, by equations (53), (54) and (55), the partial derivative $\partial^2_{l,m} g$ is given by formula (33). Thus, the function g is actually of class C^2 on $\mathbb{R}^N$. Indeed, by formula (33), $\partial^2_{l,m} g$ is continuous on $\mathbb{R}^N$: it follows from the smoothness of f, assumptions (i), (ii), (iii) and (iv), and a standard application of the dominated convergence theorem. □

We then complete the proof of Proposition 8.

Proof of Proposition 8. By formulae (19) and (20), and Proposition 1, the functions F and G are C^∞ on $\mathbb{R}^N$ and equal to

$$\begin{cases} F = \frac{|\nabla\eta|^2}{2(1-\eta)} + 2(1-\eta)|\nabla(\psi\theta)|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta), \\ G = \eta\nabla(\psi\theta), \end{cases}$$

on a neighbourhood of infinity, so, by Proposition 2, they satisfy all the assumptions of Lemmas 4 and 5.

Likewise, by Proposition 3, the kernels K_0, K_j and $L_{j,k}$ are in $\mathcal{K}(\mathbb{R}^N)$. Formula (44) is then a consequence of equation (21) and Lemma 4, while formulae (45) and (46) follow from invoking equation (24) and Lemmas 4 and 5. □

Remark 12. It seems possible to compute similar formulae for higher derivatives of the functions η and $\psi\theta$: since it is useless here, we are not going to investigate this point any further. However, it is probably a good way to prove the sharp decay of higher derivatives, i.e. to show that the functions $\partial^\alpha\eta$, $\partial^\alpha\nabla(\psi\theta)$ and $\partial^\alpha\nabla v$ are in $M^\infty_{N+|\alpha|}(\mathbb{R}^N)$, at least in the case where $|\alpha| \leq N$.

1.2 Sharp decay of the functions $d^2\eta$, $d^2\theta$ and $d^3\theta$

We now infer Theorem 6 from Proposition 8. We improve the asymptotic decay rate of the functions $d^2\eta$, $d^2\theta$, d^2v and $d^3\theta$ by the argument mentioned in the introduction. We first apply it in the following lemma.

Lemma 8. *Let $1 \leq j, k \leq N$ and $K \in \mathcal{K}(\mathbb{R}^N)$. Consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N)^N \cap M^\infty_{2N}(\mathbb{R}^N)^N$.

Then,

$$\partial^2_{j,k}(K * f) \in M^\infty_{N+2}(\mathbb{R}^N).$$

Proof. Let $g = K * f$. By assumptions (i) and (ii), Lemma 4 yields

$$\forall x \in \mathbb{R}^N, \partial^2_{j,k}g(x) = \int_{B(0,1)^c} \partial^2_{j,k}K(y)f(x-y)dy + \int_{B(0,1)} \partial^2_{j,k}K(y)(f(x-y) - f(x))dy + \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y) y_k dy\right) f(x).$$

By assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$, the first term satisfies

$$\begin{aligned} |x|^{N+2}\left|\int_{B(0,1)^c} \partial^2_{j,k}K(y)f(x-y)dy\right| \leq & A\Bigg(\int_{B(0,1)^c} |y|^{N+2}|\partial^2_{j,k}K(y)||f(x-y)|dy \\ & + \int_{B(0,1)^c} |\partial^2_{j,k}K(y)||x-y|^{N+2}|f(x-y)|dy\Bigg) \\ \leq & A(\|\partial^2_{j,k}K\|_{M^\infty_{N+2}(\mathbb{R}^N)}\|f\|_{L^1(\mathbb{R}^N)} \\ & + \|\partial^2_{j,k}K\|_{L^1(B(0,1)^c)}\|f\|^{\frac{N+2}{2N}}_{M^\infty_{N+2}(\mathbb{R}^N)}\|f\|^{\frac{N-2}{2N}}_{L^\infty(\mathbb{R}^N)}) \leq A. \end{aligned}$$

By assumption (ii) and since $K \in \mathcal{K}(\mathbb{R}^N)$, the second term verifies

$$|x|^{N+2}\left|\int_{B(0,1)} \partial^2_{j,k}K(y)(f(x-y)-f(x))dy\right| \leq A|x|^{N+2}\int_{B(0,1)}|y||\partial^2_{j,k}K(y)|dy \sup_{z\in B(x,1)}|\nabla f(z)| \leq A\frac{|x|^{N+2}}{1+|x|^{2N}} \leq A,$$

and likewise, by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$|x|^{N+2}\left|\int_{\mathbb{S}^{N-1}} \partial_j K(y)y_k dy\right||f(x)| \leq A\frac{|x|^{N+2}}{1+|x|^{2N}} \leq A.$$

Thus, the function g belongs to $M^\infty_{N+2}(\mathbb{R}^N)$. □

We next prove a similar lemma for the composed Riesz kernels $R_{j,k}$.

Lemma 9. *Let $1 \leq j,k,l,m \leq N$ and consider a function $f \in C^\infty(\mathbb{R}^N)$ such that*

(i) $f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(ii) $\nabla f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

(iii) $d^2 f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

Then,

$$\partial_l(R_{j,k} * f) \in M^\infty_{N+1}(\mathbb{R}^N).$$

Moreover, if f also satisfies

(iv) $d^3 f \in L^\infty(\mathbb{R}^N) \cap M^\infty_{2N}(\mathbb{R}^N)$,

then,

$$\partial^2_{l,m}(R_{j,k} * f) \in M^\infty_{N+2}(\mathbb{R}^N).$$

Proof. Let $g = R_{j,k} * f$. On one hand, by assumptions (i), (ii) and (iii), Lemma 5 leads to

$$\forall x \in \mathbb{R}^N, \partial_l g(x) = \int_{B(0,1)^c} \partial_l R_{j,k}(y)f(x-y)dy + \int_{B(0,1)} \partial_l R_{j,k}(y)(f(x-y)-f(x) + y.\nabla f(x))dy + \int_{\mathbb{S}^{N-1}} R_{j,k}(y)y_l(f(x)-y.\nabla f(x))dy.$$

By assumption (i), the first term verifies

$$|x|^{N+1}\left|\int_{B(0,1)^c} \partial_l R_{j,k}(y)f(x-y)dy\right| \leq A\int_{B(0,1)^c}(|y|^{N+1}|\partial_l R_{j,k}(y)||f(x-y)| + |\partial_l R_{j,k}(y)||x-y|^{N+1}|f(x-y)|)dy \leq A\left(\int_{\mathbb{R}^N}|f(t)|dt + \int_{B(0,1)^c}|\partial_l R_{j,k}(y)|dy\right) \leq A.$$

By assumption (iii), the second term satisfies

$$\begin{aligned}
&|x|^{N+1}\left|\int_{B(0,1)}\partial_l R_{j,k}(y)(f(x-y)-f(x)+y.\nabla f(x))dy\right| \\
&\leq A|x|^{N+1}\int_{B(0,1)}|y|^2|\partial_l R_{j,k}(y)|dy \sup_{z\in B(x,1)}|d^2f(z)| \\
&\leq A\frac{|x|^{N+1}}{1+|x|^{2N}}\leq A,
\end{aligned}$$

and likewise, by assumptions (i) and (ii),

$$|x|^{N+1}\left|\int_{\mathbb{S}^{N-1}}R_{j,k}(y)y_l(f(x)-y.\nabla f(x))dy\right|\leq A\frac{|x|^{N+1}}{1+|x|^{2N}}\leq A.$$

Hence, the derivative $\partial_l(R_{j,k}*f)$ is in $M^\infty_{N+1}(\mathbb{R}^N)$.

On the other hand, by assumptions (i), (ii), (iii) and (iv), Lemma 5 also gives

$$\begin{aligned}
\forall x\in\mathbb{R}^N, \partial^2_{l,m}g(x)=&\int_{B(0,1)^c}\partial^2_{l,m}R_{j,k}(y)f(x-y)dy+\int_{B(0,1)}\partial^2_{l,m}R_{j,k}(y)(f(x-y)-f(x) \\
&+y.\nabla f(x)-\frac{1}{2}d^2f(x)(y,y))dy+\int_{\mathbb{S}^{N-1}}R_{j,k}(y)y_l(\partial_m f(x) \\
&-y.\nabla\partial_m f(x))dy+\int_{\mathbb{S}^{N-1}}\partial_l R_{j,k}(y)y_m(f(x)-y.\nabla f(x) \\
&+\frac{1}{2}d^2f(x)(y,y))dy.
\end{aligned}$$

Likewise, by assumption (i), the first term satisfies

$$\begin{aligned}
|x|^{N+2}\left|\int_{B(0,1)^c}\partial^2_{l,m}R_{j,k}(y)f(x-y)dy\right|\leq& A\int_{B(0,1)^c}(|y|^{N+2}|\partial^2_{l,m}R_{j,k}(y)||f(x-y)| \\
&+|\partial^2_{l,m}R_{j,k}(y)||x-y|^{N+2}|f(x-y)|)dy \\
\leq& A\left(\int_{\mathbb{R}^N}|f(t)|dt+\int_{B(0,1)^c}|\partial^2_{l,m}R_{j,k}(y)|dy\right)\leq A.
\end{aligned}$$

For the second term, assumption (iv) yields

$$\begin{aligned}
&|x|^{N+2}\left|\int_{B(0,1)}\partial^2_{l,m}R_{j,k}(y)(f(x-y)-f(x)+y.\nabla f(x)-\frac{1}{2}d^2f(x)(y,y))dy\right| \\
&\leq A|x|^{N+2}\int_{B(0,1)}|y|^3|\partial^2_{l,m}R_{j,k}(y)|dy \sup_{z\in B(x,1)}|d^3f(z)| \\
&\leq A\frac{|x|^{N+2}}{1+|x|^{2N}}\leq A,
\end{aligned}$$

while for the third term, assumptions (ii) and (iii) give

$$|x|^{N+2}\left|\int_{\mathbb{S}^{N-1}}R_{j,k}(y)y_l(\partial_m f(x)-y.\nabla\partial_m f(x))dy\right|\leq A\frac{|x|^{N+2}}{1+|x|^{2N}}\leq A,$$

and likewise, for the last term, by assumptions (i), (ii) and (iii),

$$|x|^{N+2}\left|\int_{\mathbb{S}^{N-1}}\partial_l R_{j,k}(y)y_m(f(x)-y.\nabla f(x)+\frac{1}{2}d^2f(x)(y,y))dy\right|\leq A\frac{|x|^{N+2}}{1+|x|^{2N}}\leq A.$$

Thus, the function $\partial^2_{l,m}(R_{j,k}*f)$ belongs to $M^\infty_{N+2}(\mathbb{R}^N)$. □

Finally, Theorem 6 follows from Lemmas 8 and 9.

Proof of Theorem 6. Equation (21) writes

$$\eta = K_0 * F + 2c \sum_{j=1}^{N} K_j * G_j.$$

However, by Proposition 3, the kernels K_0 and K_j are in $\mathcal{K}(\mathbb{R}^N)$, whereas by formulae (19) and (20), and Propositions 1 and 2, the functions F and G satisfy all the assumptions of Lemma 8. Thus, the function $d^2\eta$ belongs to $M^\infty_{N+2}(\mathbb{R}^N)$ by Lemma 8.

Likewise, equation (24) states

$$\partial_j(\psi\theta) = \frac{c}{2} K_j * F + c^2 \sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k.$$

Then, Propositions 1, 2 and 3, and formulae (19) and (20) yield for every $l \in \{1, \ldots, N\}$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} |x|^{N+1}|\partial_l(K_j * F)(x)| =& |x|^{N+1}|(\partial_l K_j) * F(x)| \\ \leq& A \int_{\mathbb{R}^N} (|y|^{N+1}|\partial_l K_j(y)||F(x-y)| + |\partial_l K_j(y)| \\ & |x-y|^{N+1}|F(x-y)|)dy \\ \leq& A \left(\int_{\mathbb{R}^N} |F(t)|dt + \int_{\mathbb{R}^N} |\partial_l K_j(y)|dy \right) \leq A. \end{aligned}$$

Therefore, the function $\partial_l(K_j * F)$ is in $M^\infty_{N+1}(\mathbb{R}^N)$. Likewise, the functions $\partial_l(L_{j,k} * G_k)$ belong to $M^\infty_{N+1}(\mathbb{R}^N)$, so, since the functions G_k satisfy all the assumptions of Lemma 9, it follows from this lemma that the function $d^2(\psi\theta)$ also belongs to $M^\infty_{N+1}(\mathbb{R}^N)$.

The proof is identical for the function $d^3(\psi\theta)$ by Lemmas 8 and 9, and formula (24), so, we omit it.

Finally, by Proposition 1, the function $d^2 v$ is C^∞ on $\mathbb{R}^N$ and equal to

$$\partial^2_{j,k} v = \left(\sqrt{1-\eta}(i\partial^2_{j,k}\theta - \partial_j\theta\partial_k\theta) - \frac{\partial^2_{j,k}\eta + i(\partial_j\theta\partial_k\eta + \partial_k\theta\partial_j\eta)}{2\sqrt{1-\eta}} - \frac{\partial_j\eta\partial_k\eta}{4(1-\eta)^{\frac{3}{2}}} \right) e^{i\theta}$$

on a neighbourhood of infinity. Since the functions $\nabla\eta$, $\nabla(\psi\theta)$, $d^2\eta$ and $d^2(\psi\theta)$ are bounded and belong to respectively $M^\infty_{N+1}(\mathbb{R}^N)$, $M^\infty_N(\mathbb{R}^N)$, $M^\infty_{N+2}(\mathbb{R}^N)$ and $M^\infty_{N+1}(\mathbb{R}^N)$, and since η converges to 0 at infinity by Proposition 2, $d^2 v$ belongs to $M^\infty_{N+1}(\mathbb{R}^N)$. □

Before turning to the first order development at infinity of the function v, we establish Corollary 2.

Proof of Corollary 2. Corollary 2 is a consequence of the superlinear nature of F and G. By formulae (19) and (20), and Proposition 1, the functions F and G are C^∞ on $\mathbb{R}^N$ and equal to

$$\begin{cases} F = \frac{|\nabla\eta|^2}{2(1-\eta)} + 2(1-\eta)|\nabla\theta|^2 + 2\eta^2 - 2c\eta\partial_1\theta \\ G = \eta\nabla\theta \end{cases}$$

on $B(0, 3R_0)^c$. Thus, we compute for every $x \in B(0, 3R_0)^c$,

$$|x|^{2N}(|F(x)| + |G(x)|) \leq A|x|^{2N}(|\nabla\eta(x)|^2 + |\nabla\theta(x)|^2 + |\eta(x)|^2 + |\eta(x)||\nabla\theta(x)|),$$

$$\begin{aligned} |x|^{2N+1}(|\nabla F(x)| + |\nabla G(x)|) \leq & A|x|^{2N+1}(|d^2\eta(x)||\nabla\eta(x)| + |\nabla\eta(x)|^3 + |\eta(x)||\nabla\eta(x)| \\ & + |\nabla\eta(x)||\nabla\theta(x)|^2 + |\nabla\theta(x)||d^2\theta(x)| + |\nabla\eta(x)||\nabla\theta(x)| \\ & + |\eta(x)||d^2\theta(x)|), \end{aligned}$$

$$\begin{aligned} |x|^{2N+2}|d^2G(x)| \leq & A|x|^{2N+2}(|d^2\eta(x)||\nabla\theta(x)| + |\nabla\eta(x)||d^2\theta(x)| \\ & + |\eta(x)||d^3\theta(x)|). \end{aligned}$$

Corollary 2 then follows from Proposition 2 and Theorem 6. □

2 Asymptotic development at first order

Now, we consider the existence of a first order asymptotic expansion for the subsonic travelling waves of finite energy. By the method mentioned in the introduction, we first deduce the pointwise convergence of the Gross-Pitaevskii kernels, then, the pointwise convergence of all the convolution integrals which appear in formulae (21) and (24). We finish the proof of Theorem 1 by showing the convergences above are actually uniform on the sphere $\mathbb{S}^{N-1}$ and by computing a partial differential equation for the first order terms of this asymptotic expansion.

2.1 Pointwise convergence of Gross-Pitaevskii kernels

We first prove Theorem 5, i.e. the pointwise convergence of the Gross-Pitaevskii kernels K_0, K_j and $L_{j,k}$. As claimed previously in the introduction, it follows from the form of their Fourier transforms through Lemma 1, whose proof is mentioned below.

Proof of Lemma 1. Consider some integer $j \in \{1, \ldots, N\}$. The Fourier transform of K belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Therefore, the function f given by

$$\forall x \in \mathbb{R}^N, f(x) = (-ix_j)^{N-1}K(x),$$

is continuous on $\mathbb{R}^N$. Indeed, its Fourier transform

$$\widehat{f} = \partial_j^{N-1}\widehat{K}$$

belongs to $L^1(\mathbb{R}^N)$. Moreover, if $g \in S(\mathbb{R}^N)$, we compute

$$< x_j f, \widehat{g} >=< f, x_j\widehat{g} >= -i < f, \widehat{\partial_j g} >= -i < \widehat{f}, \partial_j g >,$$

so, $\widehat{f}$ being in $L^1(\mathbb{R}^N)$, we can write

$$< x_j f, \widehat{g} >= -i \int_{\mathbb{R}^N} \widehat{f}(\xi)\partial_j g(\xi)d\xi.$$

Then, we deduce from an integration by parts that for every $\lambda > 0$,

$$\begin{aligned} < x_j f, \widehat{g} > = & i \int_{B(0,\lambda)^c} \partial_j\widehat{f}(\xi)g(\xi)d\xi + i \int_{B(0,\lambda)} \partial_j\widehat{f}(\xi)(g(\xi) - g(0))d\xi \\ & + \frac{ig(0)}{\lambda} \int_{S(0,\lambda)} \xi_j\widehat{f}(\xi)d\xi. \end{aligned}$$

However, g is in $S(\mathbb{R}^N)$, therefore,

$$g(\xi) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{g}(x)e^{ix.\xi}dx,$$

which yields

$$< x_j f, \widehat{g} > = \frac{i}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{g}(x)\left(\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi}-1)d\xi \right.$$
$$\left. + \frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j \widehat{f}(\xi)d\xi\right)dx.$$

Therefore, by standard duality, the tempered distribution $x_j f$ is equal to the tempered distribution Ψ given for every $x \in \mathbb{R}^N$ by

$$\begin{aligned}\Psi(x) &= \frac{i}{(2\pi)^N}\left(\int_{B(0,\lambda)^c} \partial_j \widehat{f}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j \widehat{f}(\xi)(e^{ix.\xi}-1)d\xi + \frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j \widehat{f}(\xi)d\xi\right) \\ &= \frac{i}{(2\pi)^N}\left(\int_{B(0,\lambda)^c} \partial_j^N \widehat{K}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{K}(\xi)(e^{ix.\xi}-1)d\xi \right. \\ &\left. + \frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j \partial_j^{N-1}\widehat{K}(\xi)d\xi\right).\end{aligned}$$

Indeed, Ψ is a tempered distribution because, since $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, Ψ belongs to $L^1_{loc}(\mathbb{R}^N)$ and satisfies

$$\forall x \in \mathbb{R}^N, |\Psi(x)| \leq A(1+|x|).$$

Moreover, since $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$ once more, by a standard application of the dominated convergence theorem, Ψ is also continuous on $\mathbb{R}^N$. Thus, the function $x \mapsto x_j f(x) = x_j(-ix_j)^{N-1}K(x)$ is continuous on $\mathbb{R}^N$ and verifies for every $x \in \mathbb{R}^N$,

$$x_j(-ix_j)^{N-1}K(x) = \frac{i}{(2\pi)^N}\left(\int_{B(0,\lambda)^c} \partial_j^N \widehat{K}(\xi)e^{ix.\xi}d\xi + \int_{B(0,\lambda)} \partial_j^N \widehat{K}(\xi)(e^{ix.\xi}-1)d\xi \right.$$
$$\left. + \frac{1}{\lambda}\int_{S(0,\lambda)} \xi_j \partial_j^{N-1}\widehat{K}(\xi)d\xi\right).$$

Then, it only remains to choose $\lambda = \frac{1}{R}$ and $x = R\sigma - y$ to get formula (29). □

Theorem 5 is then a consequence of Lemma 1.

Proof of Theorem 5. Let $1 \leq j \leq N$ and let us first make the additional assumption

$$\alpha = 0.$$

We will remove it later. The function $\widehat{K}$ is a rational fraction only singular at the origin, so all its derivatives are also rational fractions only singular at the origin. Thus, we can state for every $i \in \{0,1,2\}$,

$$\partial_j^{N+i-1}\widehat{K} = \frac{\sum_{k=0}^{d_i} P_{k,i}}{\sum_{k=0}^{d'_i} Q_{k,i}} \tag{56}$$

where

- the functions $P_{k,i}$ and $Q_{k,i}$ are homogeneous polynomial functions either equal to 0 or of degree k.
- the polynomial functions $P_i = \sum_{k=0}^{d_i} P_{k,i}$ and $Q_i = \sum_{k=0}^{d'_i} Q_{k,i}$ are relatively prime.
- the polynomial function Q_i does not vanish on $\mathbb{R}^N \setminus \{0\}$.

Moreover, consider $\xi \in \mathbb{R}^N \setminus \{0\}$ and denote

- $l(\xi) = \begin{cases} \min\{k \in \{0, \ldots, d_i\}, P_{k,i}(\xi) \neq 0\}, \text{ if } \exists k \in \{0, \ldots, d_i\}, P_{k,i}(\xi) \neq 0, \\ +\infty, \text{ otherwise}, \end{cases}$
- $l'(\xi) = \min\{k \in \{0, \ldots, d'_i\}, Q_{k,i}(\xi) \neq 0\}$.

The functions l and l' are well-defined on $\mathbb{R}^N \setminus \{0\}$, and we can set

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, R_i(\xi) = \begin{cases} \delta_{l'(\xi), l(\xi)+N-1+i} \frac{P_{l(\xi),i}(\xi)}{Q_{l'(\xi),i}(\xi)}, \text{ if } l(\xi) \neq +\infty, \\ 0, \text{ otherwise.} \end{cases}$$

Now, we claim

Claim 1. *The function R_i belongs to $M^\infty_{N+i-1}(\mathbb{R}^N)$ and satisfies*

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \frac{1}{R^{N+i-1}} \partial_j^{N+i-1} \widehat{K}\left(\frac{\xi}{R}\right) \underset{R \to +\infty}{\to} R_i(\xi). \tag{57}$$

Proof of Claim 1. The case $l(\xi) = +\infty$ being straightforward since

$$\partial_j^{N+i-1} \widehat{K}\left(\frac{\xi}{R}\right) = \frac{\sum_{k=0}^{d_i} R^{-k} P_{k,i}(\xi)}{\sum_{k=0}^{d'_i} R^{N+i-1-k} Q_{k,i}(\xi)} = 0 = R_i(\xi),$$

consider $R > 0$ and $\xi \in \mathbb{R}^N \setminus \{0\}$ such that

$$l(\xi) \neq +\infty.$$

Formula (56) becomes

$$\frac{1}{R^{N+i-1}} \partial_j^{N+i-1} \widehat{K}\left(\frac{\xi}{R}\right) = \frac{\sum_{k=0}^{d_i} R^{-k} P_{k,i}(\xi)}{\sum_{k=0}^{d'_i} R^{N+i-1-k} Q_{k,i}(\xi)} \underset{R \to +\infty}{\sim} \frac{P_{l(\xi),i}(\xi)}{R^{N+i-1-l'(\xi)+l(\xi)} Q_{l'(\xi),i}(\xi)}.$$

However, the function $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, which means in particular that

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \left| \frac{1}{R^{N+i-1}} \partial_j^{N+i-1} \widehat{K}\left(\frac{\xi}{R}\right) \right| \leq \frac{A}{|\xi|^{N+i-1}}. \tag{58}$$

Thus, we first deduce

$$\frac{1}{R^{N+i-1}} \partial_j^{N+i-1} \widehat{K}\left(\frac{\xi}{R}\right) \underset{R \to +\infty}{\to} \delta_{N+i-1+l(\xi), l'(\xi)} \frac{P_{l(\xi),i}(\xi)}{Q_{l'(\xi),i}(\xi)} = R_i(\xi),$$

and secondly, by taking the limit $R \to +\infty$ in inequality (58),

$$|R_i(\xi)| \leq \frac{A}{|\xi|^{N+i-1}},$$

i.e. the function R_i belongs to $M^{\infty}_{N+i-1}(\mathbb{R}^N)$. $\square$

Now, we turn back to the proof of Theorem 5. Consider $(\sigma, y) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$ such that

$$\sigma_j \neq 0$$

and remark once again that the function $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$. By Lemma 1, we can state for every positive number R sufficiently large

$$\begin{aligned} R^N K(R\sigma - y) =& \frac{i^N}{(2\pi(\sigma_j - \frac{y_j}{R}))^N} \left(\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma - y)} d\xi + \int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi) \right. \\ & \left. (e^{i\xi.(R\sigma-y)} - 1) d\xi + R \int_{S(0,\frac{1}{R})} \xi_j \partial_j^{N-1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right). \end{aligned} \tag{59}$$

Our goal is to prove the convergence of each term of the right member towards a bounded measurable function independent of y.

Step 1. *The first term of the right member of equation* (59) *satisfies*

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{R\to+\infty}{\to} -\frac{1}{i\sigma_j} \left(\int_{B(0,1)^c} R_2(\xi) e^{i\xi.\sigma} d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi) e^{i\xi.\sigma} d\xi \right).$$

Indeed, for every $\lambda > \frac{1}{R}$,

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \lim_{\lambda\to+\infty} \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi.$$

Moreover, by integrating by parts,

$$\begin{aligned} \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi =& \frac{1}{i(R\sigma_j - y_j)} \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^N \widehat{K}(\xi) \partial_j (e^{i\xi.(R\sigma-y)}) d\xi \\ =& \frac{1}{i(R\sigma_j - y_j)} \left(- \int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right. \\ & + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi - R \int_{S(0,\frac{1}{R})} \xi_j \\ & \left. \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right). \end{aligned}$$

However, $\widehat{K}$ is in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, therefore,

$$\int_{\frac{1}{R}<|\xi|<\lambda} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{\lambda\to+\infty}{\to} \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi,$$

while

$$\left| \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right| \leq \frac{A\lambda^{N-1}}{\lambda^{N+2}} \underset{\lambda\to+\infty}{\to} 0.$$

Thus, we obtain

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \frac{1}{i(R\sigma_j - y_j)} \left(- \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right.$$
$$\left. - R \int_{S(0,\frac{1}{R})} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \right). \tag{60}$$

On one hand, the first term verifies

$$\frac{1}{R} \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \frac{1}{R^{N+1}} \int_{B(0,1)^c} \partial_j^{N+1} \widehat{K} \left(\frac{\xi}{R} \right) e^{i\xi.(\sigma - \frac{y}{R})} d\xi.$$

However, by assertion (57),

$$\frac{1}{R^{N+1}} \partial_j^{N+1} \widehat{K} \left(\frac{\xi}{R} \right) \underset{R\to+\infty}{\to} R_2(\xi),$$

and, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$\forall \xi \in B(0,1)^c, \left| \frac{1}{R^{N+1}} \partial_j^{N+1} \widehat{K} \left(\frac{\xi}{R} \right) e^{i\xi.(\sigma - \frac{y}{R})} \right| \leq \frac{A}{|\xi|^{N+1}},$$

hence, by the dominated convergence theorem,

$$\frac{1}{R} \int_{B(0,\frac{1}{R})^c} \partial_j^{N+1} \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{R\to+\infty}{\to} \int_{B(0,1)^c} R_2(\xi) e^{i\xi.\sigma} d\xi.$$

On the other hand, the second terms writes

$$\int_{S(0,\frac{1}{R})} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi = \frac{1}{R^N} \int_{\mathbb{S}^{N-1}} \xi_j \partial_j^N \widehat{K} \left(\frac{\xi}{R} \right) e^{i\xi.(\sigma - \frac{y}{R})} d\xi.$$

Likewise, by assertion (57),

$$\frac{1}{R^N} \partial_j^N \widehat{K} \left(\frac{\xi}{R} \right) \underset{R\to+\infty}{\to} R_1(\xi).$$

and, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$\forall \xi \in \mathbb{S}^{N-1}, \left| \frac{\xi_j}{R^N} \partial_j^N \widehat{K} \left(\frac{\xi}{R} \right) e^{i\xi.(\sigma - \frac{y}{R})} \right| \leq A,$$

which gives by the dominated convergence theorem,

$$\int_{S(0,\frac{1}{R})} \xi_j \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi) e^{i\xi.\sigma} d\xi.$$

In conclusion, equation (60) yields

$$\int_{B(0,\frac{1}{R})^c} \partial_j^N \widehat{K}(\xi) e^{i\xi.(R\sigma-y)} d\xi \underset{R\to+\infty}{\to} -\frac{1}{i\sigma_j} \left(\int_{B(0,1)^c} R_2(\xi) e^{i\xi.\sigma} d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi) e^{i\xi.\sigma} d\xi \right),$$

which ends the proof of Step 1.

Step 2. *The second term of the right member of equation* (59) *satisfies*

$$\int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi)(e^{i\xi.(R\sigma-y)}-1)d\xi \underset{R\to+\infty}{\to} \int_{B(0,1)} R_1(\xi)(e^{i\xi.\sigma}-1)d\xi.$$

Indeed, we have

$$\int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi)(e^{i\xi.(R\sigma-y)}-1)d\xi = \frac{1}{R^N}\int_{B(0,1)} \partial_j^N \widehat{K}\left(\frac{\xi}{R}\right)(e^{i\xi.(\sigma-\frac{y}{R})}-1)d\xi.$$

Likewise, by assertion (57),

$$\frac{1}{R^N}\partial_j^N \widehat{K}\left(\frac{\xi}{R}\right) \underset{R\to+\infty}{\to} R_1(\xi),$$

and, since $\widehat{K}\in\widehat{\mathcal{K}}(\mathbb{R}^N)$, we have for every $R>2|y|$,

$$\forall \xi\in B(0,1), \left|\frac{1}{R^N}\partial_j^N \widehat{K}\left(\frac{\xi}{R}\right)(e^{i\xi.(\sigma-\frac{y}{R})}-1)\right| \leq \frac{A}{|\xi|^N}\left|\xi.(\sigma-\frac{y}{R})\right| \leq \frac{A}{|\xi|^{N-1}}.$$

Hence, the dominated convergence theorem gives

$$\int_{B(0,\frac{1}{R})} \partial_j^N \widehat{K}(\xi)(e^{i\xi.(R\sigma-y)}-1)d\xi \underset{R\to+\infty}{\to} \int_{B(0,1)} R_1(\xi)(e^{i\xi.\sigma}-1)d\xi,$$

which is the desired result.

Step 3. *The last term of the right member of equation* (59) *verifies*

$$R\int_{S(0,\frac{1}{R})} \xi_j\partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \xi_j R_0(\xi)e^{i\xi.\sigma}d\xi.$$

Indeed, we compute

$$R\int_{S(0,\frac{1}{R})} \xi_j\partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi = \frac{1}{R^{N-1}}\int_{\mathbb{S}^{N-1}} \xi_j\partial_j^{N-1}\widehat{K}\left(\frac{\xi}{R}\right)e^{i\xi.(\sigma-\frac{y}{R})}d\xi.$$

However, by assertion (57),

$$\frac{1}{R^{N-1}}\partial_j^{N-1}\widehat{K}\left(\frac{\xi}{R}\right) \underset{R\to+\infty}{\to} R_0(\xi),$$

and, since $\widehat{K}\in\widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$\forall \xi\in\mathbb{S}^{N-1}, \left|\frac{1}{R^{N-1}}\xi_j\partial_j^{N-1}\widehat{K}\left(\frac{\xi}{R}\right)e^{i\xi.(\sigma-\frac{y}{R})}\right| \leq A,$$

which yields by the dominated convergence theorem,

$$R\int_{S(0,\frac{1}{R})} \xi_j\partial_j^{N-1}\widehat{K}(\xi)e^{i\xi.(R\sigma-y)}d\xi \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \xi_j R_0(\xi)e^{i\xi.\sigma}d\xi.$$

Finally, by equation (59), and Steps 1, 2 and 3, we conclude

$$R^N K(R\sigma-y) \underset{R\to+\infty}{\to} K_\infty(\sigma),$$

where K_∞ is given by

$$\begin{aligned}K_\infty(\sigma) =&\frac{i^N}{(2\pi\sigma_j)^N}\Bigg(\int_{B(0,1)} R_1(\xi)(e^{i\xi.\sigma}-1)d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_0(\xi)e^{i\xi.\sigma}d\xi \\ &-\frac{1}{i\sigma_j}\Big(\int_{B(0,1)^c} R_2(\xi)e^{i\xi.\sigma}d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi)e^{i\xi.\sigma}d\xi\Big)\Bigg).\end{aligned} \tag{61}$$

It then only remains to show that the function K_∞ is uniformly bounded on the sphere $\mathbb{S}^{N-1}$. Indeed, up to choose another integer $j \in \{1, \dots, N\}$, we can suppose that

$$\frac{1}{\sqrt{N}} \leq \sigma_j \leq 1.$$

We then deduce from Claim 1 and from this additional assumption that

$$|K_\infty(\sigma)| \leq A_N\left(\int_{B(0,1)} \frac{d\xi}{|\xi|^{N-1}} + \int_{\mathbb{S}^{N-1}} \frac{d\xi}{|\xi|^{N-2}} + \int_{B(0,1)^c} \frac{d\xi}{|\xi|^{N+1}} + \int_{\mathbb{S}^{N-1}} \frac{d\xi}{|\xi|^{N-1}}\right),$$

so, the function K_∞ is uniformly bounded on $\mathbb{S}^{N-1}$.

Now, we complete the proof of Theorem 5 by considering the case

$$\alpha \neq 0.$$

We first compute

$$\widehat{\partial^\alpha K}(\xi) = i^{|\alpha|}\xi^\alpha \widehat{K}(\xi).$$

We then consider $\beta \in \mathbb{N}^N$ such that $|\beta| = |\alpha|$ and denote L_β, the tempered distribution of Fourier transform

$$\widehat{L_\beta} = \partial^\beta \widehat{\partial^\alpha K}.$$

We claim that the function $\widehat{L_\beta}$ belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Indeed, by Leibnitz's formula,

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \widehat{L_\beta}(\xi) = \partial^\beta(i^{|\alpha|}\xi^\alpha \widehat{K}(\xi)) = \sum_{0\leq\gamma\leq\beta} A_{\gamma,\beta}\partial^\gamma(\xi^\alpha)\partial^{\beta-\gamma}\widehat{K}(\xi),$$

so, since $\widehat{K} \in \widehat{\mathcal{K}}(\mathbb{R}^N)$,

$$(1+|\xi|^2)|\widehat{L_\beta}(\xi)| \leq A(1+|\xi|^2)\sum_{0\leq\gamma\leq\beta} \frac{|\xi|^{|\alpha|-|\gamma|}}{(1+|\xi|^2)|\xi|^{|\beta|-|\gamma|}} \leq A.$$

Therefore, the function $\widehat{L_\beta}$ is in $L^\infty(\mathbb{R}^N) \cap M_2^\infty(\mathbb{R}^N)$. Likewise, a straightforward inductive argument for the derivatives of $\widehat{L_\beta}$ yields that $\widehat{L_\beta}$ is a rational fraction which is only singular at the origin and belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. By the proof ahead for the case $\alpha = 0$, there exists a bounded measurable function $L_{\beta,\infty}$ such that

$$R^N L_\beta(R\sigma - y) \underset{R\to+\infty}{\to} L_{\beta,\infty}(\sigma).$$

Moreover, we compute

$$L_\beta(x) = (-i)^{|\beta|}x^\beta \partial^\alpha K(x),$$

so,

$$R^N(-i)^{|\beta|}(R\sigma - y)^\beta \partial^\alpha K(R\sigma - y) \underset{R\to+\infty}{\to} L_{\beta,\infty}(\sigma),$$

and,

$$R^{N+|\alpha|}\sigma^\beta \partial^\alpha K(R\sigma - y) \underset{R\to+\infty}{\to} i^{|\alpha|} L_{\beta,\infty}(\sigma).$$

However, we can always choose β such that

$$|\sigma^\beta| \geq \frac{1}{N^{\frac{|\alpha|}{2}}},$$

so,

$$\left| \frac{i^{|\alpha|}}{\sigma^\beta} L_{\beta,\infty}(\sigma) \right| \leq N^{\frac{|\alpha|}{2}} \max_{|\beta|=|\alpha|} \|L_{\beta,\infty}\|_{L^\infty(\mathbb{S}^{N-1})}.$$

Thus, there is a bounded measurable function K_∞^α on the sphere $\mathbb{S}^{N-1}$ such that

$$R^{N+|\alpha|}\partial^\alpha K(R\sigma - y) \underset{R\to+\infty}{\to} K_\infty^\alpha(\sigma),$$

which completes the proof of Theorem 5. □

One application of Theorem 5 is given by the next corollary.

Corollary 3. *Let $1 \leq j,k \leq N$, $\alpha \in \mathbb{N}^N$ and $\sigma \in \mathbb{S}^{N-1}$. There exist bounded measurable functions $K_{0,\infty}^\alpha$, $K_{j,\infty}^\alpha$ and $L_{j,k,\infty}^\alpha$ on the sphere $\mathbb{S}^{N-1}$ such that*

$$\forall y \in \mathbb{R}^N, \begin{cases} R^{N+|\alpha|}\partial^\alpha K_0(R\sigma - y) \underset{R\to+\infty}{\to} K_{0,\infty}^\alpha(\sigma), \\ R^{N+|\alpha|}\partial^\alpha K_j(R\sigma - y) \underset{R\to+\infty}{\to} K_{j,\infty}^\alpha(\sigma), \\ R^{N+|\alpha|}\partial^\alpha L_{j,k}(R\sigma - y) \underset{R\to+\infty}{\to} L_{j,k,\infty}^\alpha(\sigma). \end{cases}$$

Proof. We infer from formulae (22), (23) and (25) that $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ are rational fractions which are only singular at the origin and belong to $\widehat{\mathcal{K}}(\mathbb{R}^N)$. Corollary 3 is then a consequence of Theorem 5. □

Remark 13. Formula (61) gives an expression of the limit K_∞ in function of the kernel K. It is quite involved to compute explicitly such an expression. However, we can conjecture the limit of the non-isotropic kernels K_0, K_j and $L_{j,k}$. Indeed, consider for instance the kernel K_0. By formula (22), its Fourier transform writes

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}.$$

Turning back to the proof of Theorem 5, we remark that the limit at infinity of K_0 is formally identical to the limit at infinity of the kernel R_0 whose Fourier transform is

$$\widehat{R_0}(\xi) = \frac{|\xi|^2}{2|\xi|^2 - c^2\xi_1^2}.$$

Indeed, the only terms which appear in the limit at infinity of the kernel K_0 are the homogeneous terms of lowest degree of the numerator and denominator of $\widehat{K_0}$. Moreover, up to a change of variables, the kernel R_0 is related to the composed Riesz kernels. Indeed, it is equal to

$$\widehat{R_0}(\xi) = \sum_{j=1}^N \frac{1}{2(1-\frac{c^2}{2})^{\delta_{j,1}}} \widehat{R_{j,j}}\Big(\sqrt{1-\frac{c^2}{2}}\xi_1, \ldots, \xi_N\Big).$$

Since we know the limit at infinity of the composed Riesz kernels by formula (30), we deduce that

$$K_{0,\infty}(\sigma) = R_{0,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-3}{2}} c^2}{8\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} \left(1 - \frac{N\sigma_1^2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\right). \tag{62}$$

Likewise, by formulae (23) and (25), we can compute formally the limit at infinity of the kernel K_j

$$K_{j,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-1}{2}}}{4\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} \left(\delta_{j,1}(1-\frac{c^2}{2})^{-\frac{\delta_{j,1}+1}{2}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}}\sigma_1\sigma_j}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\right), \tag{63}$$

and of the kernel $L_{j,k}$

$$\begin{aligned} L_{j,k,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})}{2c^2\pi^{\frac{N}{2}}} \Bigg(& (1-\frac{c^2}{2})^{\frac{N}{2}} \Bigg(\frac{\delta_{j,k}(1-\frac{c^2}{2})^{-\frac{\delta_{j,1}+\delta_{k,1}+1}{2}}}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}-\delta_{k,1}+\frac{1}{2}}\sigma_j\sigma_k}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}} \Bigg) \\ & - \delta_{j,k} + N\sigma_j\sigma_k \Bigg). \end{aligned} \tag{64}$$

Formulae (62), (63) and (64) lead to Conjecture 1 as we will see in Section 2.3.

2.2 Pointwise convergence of convolution integrals involving the Gross-Pitaevskii kernels

Now, we turn to the pointwise convergence of all the convolution integrals involving the Gross-Pitaevskii kernels K_0, K_j and $L_{j,k}$.

Proposition 9. *Let $\sigma \in \mathbb{S}^{N-1}$, $1 \leq j, k \leq N$ and $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$. Then, the following assertion holds*

$$R^{N+|\alpha|}\partial^\alpha(K * f)(R\sigma) \underset{R\to+\infty}{\to} K_\infty^\alpha(\sigma) \int_{\mathbb{R}^N} f(x)dx,$$

for K, either equal to K_0, K_j or $L_{j,k}$, and f either equal to F, G_j or G_k.

The proof of Proposition 9 is a straightforward consequence of Corollaries 2 and 3, and Lemma 2, so we postpone its proof after the proof of Lemma 2.

Proof of Lemma 2. We divide the proof in three steps which correspond to each desired assertion.

Step 1. *The next assertion holds*

$$R^N g(R\sigma) \underset{R\to+\infty}{\to} K_\infty(\sigma) \int_{\mathbb{R}^N} f(x)dx,$$

where K_∞ denotes the bounded measurable function given by Theorem 5.

Indeed, consider $R > 0$ and write the expression of the function g

$$\begin{aligned} R^N g(R\sigma) &= \int_{\mathbb{R}^N} R^N K(R\sigma - y)f(y)dy \\ &= \int_{|R\sigma-y|\leq\frac{R}{2}} R^N K(R\sigma - y)f(y)dy + \int_{|R\sigma-y|>\frac{R}{2}} R^N K(R\sigma - y)f(y)dy. \end{aligned} \tag{65}$$

On one hand, by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\begin{aligned}\left|\int_{|R\sigma-y|\le\frac{R}{2}} R^N K(R\sigma-y)f(y)dy\right| &= \left|\int_{|\sigma-z|\le\frac{1}{2}} R^{2N}K(R(\sigma-z))f(Rz)dz\right| \\ &\le A\int_{|\sigma-z|\le\frac{1}{2}} \frac{R^{2N}}{(1+R^{2N}|z|^{2N})(R^{N-1}|\sigma-z|^{N-1})}dz \\ &\le \frac{A}{R^{N-1}}\int_{|\sigma-z|\le\frac{1}{2}} \frac{dz}{|z|^{2N}|\sigma-z|^{N-1}} \underset{R\to+\infty}{\to} 0.\end{aligned}$$

On the other hand, by Theorem 5,

$$R^N 1_{|R\sigma-y|>\frac{R}{2}} K(R\sigma-y)f(y) \underset{R\to+\infty}{\to} K_\infty(\sigma)f(y),$$

while by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\forall y \in B\left(R\sigma, \frac{R}{2}\right)^c, |R^N K(R\sigma-y)f(y)| \le \frac{AR^N}{|R\sigma-y|^N(1+|y|^{2N})} \le \frac{A}{1+|y|^{2N}},$$

hence, by the dominated convergence theorem,

$$\int_{|R\sigma-y|>\frac{R}{2}} R^N K(R\sigma-y)f(y)dy \underset{R\to+\infty}{\to} K_\infty(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

which gives the desired result by equation (65).

Step 2. *The following assertion is valid*

$$R^{N+1}\partial_j g(R\sigma) \underset{R\to+\infty}{\to} K_\infty^j(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

where K_∞^j denotes the bounded measurable function given by Theorem 5.

The proof is quite similar to the proof of Step 1. Indeed, consider $R > 0$ and state likewise

$$R^{N+1}\partial_j g(R\sigma) = \int_{|R\sigma-y|\le\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y)f(y)dy + \int_{|R\sigma-y|>\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y) f(y)dy. \tag{66}$$

On one hand, by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\begin{aligned}\left|\int_{|R\sigma-y|\le\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y)f(y)dy\right| &\le A\int_{|\sigma-z|\le\frac{1}{2}} \frac{R^{2N+1}}{(1+R^{2N}|z|^{2N})(R^{N-\frac{1}{2}}|\sigma-z|^{N-\frac{1}{2}})}dz \\ &\le \frac{A}{R^{N-\frac{3}{2}}} \underset{R\to+\infty}{\to} 0.\end{aligned}$$

On the other hand, by Theorem 5,

$$R^{N+1}1_{|R\sigma-y|>\frac{R}{2}}\partial_j K(R\sigma-y)f(y) \underset{R\to+\infty}{\to} K_\infty^j(\sigma)f(y),$$

while by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\forall y \in B\left(R\sigma, \frac{R}{2}\right)^c, |R^{N+1}\partial_j K(R\sigma-y)f(y)| \le \frac{AR^{N+1}}{|R\sigma-y|^{N+1}(1+|y|^{2N})} \le \frac{A}{1+|y|^{2N}},$$

hence, by the dominated convergence theorem,

$$\int_{|R\sigma-y|>\frac{R}{2}} R^{N+1}\partial_j K(R\sigma-y)f(y)dy \underset{R\to+\infty}{\to} K_\infty^j(\sigma)\int_{\mathbb{R}^N} f(x)dx,$$

which ends the proof of Step 2 by equation (66).

Step 3. *The assertion*

$$R^{N+2}\partial_{j,k}^2 g(R\sigma) \underset{R\to+\infty}{\to} K_\infty^{j,k}(\sigma)\int_{\mathbb{R}^N} f(x)dx$$

holds if $K_\infty^{j,k}$ denotes the bounded measurable function defined in Theorem 5.

Indeed, Lemma 4 gives

$$\begin{aligned}\partial_{j,k}^2 g(R\sigma) = &\int_{B(0,1)^c} \partial_{j,k}^2 K(y)f(R\sigma-y)dy + \int_{B(0,1)} \partial_{j,k}^2 K(y)(f(R\sigma-y)-f(R\sigma))dy \\ &+ \left(\int_{\mathbb{S}^{N-1}} \partial_j K(y)y_k dy\right) f(R\sigma),\end{aligned}$$

which yields by an integration by parts and the change of variables $z = R\sigma - y$,

$$\begin{aligned}R^{N+2}\partial_{j,k}^2 g(R\sigma) =& R^{N+2}\int_{B(R\sigma,\frac{R}{2})^c} \partial_{j,k}^2 K(R\sigma-z)f(z)dz + R^{N+2}\int_{B(R\sigma,\frac{R}{2})} \partial_{j,k}^2 K(R\sigma-z) \\ &(f(z)-f(R\sigma))dz + 2R^{N+1}\left(\int_{S(0,\frac{R}{2})} \partial_j K(y)y_k dy\right) f(R\sigma).\end{aligned} \tag{67}$$

On one hand, we compute by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$R^{N+1}\left|\int_{S(0,\frac{R}{2})} \partial_j K(y)y_k dy\right| |f(R\sigma)| \leq \frac{AR^{N+1}}{1+R^{2N}}\int_{S(0,\frac{R}{2})} \frac{dy}{|y|^N} \leq \frac{A}{R^N} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by assumption (ii) and since $K \in \mathcal{K}(\mathbb{R}^N)$, we find

$$\begin{aligned}&R^{N+2}\left|\int_{B(R\sigma,\frac{R}{2})} \partial_{j,k}^2 K(R\sigma-z)(f(z)-f(R\sigma))dz\right| \\ \leq& AR^{N+2}\Bigg(\int_{B(R\sigma,1)} \frac{dz}{|R\sigma-z|^{N-\frac{1}{2}}} \sup_{y\in B(R\sigma,1)}|\nabla f(y)| + \int_{1\leq|R\sigma-z|\leq\frac{R}{2}} \frac{dz}{|R\sigma-z|^{N+1}} \\ &\sup_{y\in B(R\sigma,\frac{R}{2})}|\nabla f(y)|\Bigg) \\ \leq& \frac{A}{R^{N-1}} \underset{R\to+\infty}{\to} 0.\end{aligned}$$

Finally, Theorem 5 gives

$$R^{N+2}1_{|R\sigma-z|>\frac{R}{2}}\partial_{j,k}^2 K(R\sigma-z)f(z) \underset{R\to+\infty}{\to} K_\infty^{j,k}(\sigma)f(z),$$

while by assumption (i) and since $K \in \mathcal{K}(\mathbb{R}^N)$,

$$\forall z \in B\left(R\sigma, \frac{R}{2}\right)^c, |R^{N+2}\partial_{j,k}^2 K(R\sigma-z)f(z)| \leq \frac{AR^{N+2}}{|R\sigma-z|^{N+2}(1+|z|^{2N})} \leq \frac{A}{1+|z|^{2N}},$$

hence, by the dominated convergence theorem,

$$\int_{B(R\sigma,\frac{R}{2})^c} \partial^2_{j,k} K(R\sigma - z) f(z) dz \underset{R\to+\infty}{\to} K^{j,k}_\infty(\sigma) \int_{\mathbb{R}^N} f(x) dx,$$

which ends the proofs of Step 3 and Lemma 2 by equation (67). □

Before investigating the pointwise convergence of the convolution integrals involving the composed Riesz kernels, we complete the proof of Proposition 9.

Proof of Proposition 9. By Corollary 2, the functions F and G satisfy the assumptions (i) and (ii) of Lemma 2. Moreover, the functions K_0, K_j and $L_{j,k}$ belong to $\mathcal{K}(\mathbb{R}^N)$ by Proposition 3 and their Fourier transforms are rational fractions in $\widehat{\mathcal{K}}(\mathbb{R}^N)$, only singular at the origin by formulae (22), (23) and (25). Thus, Proposition 9 follows from Lemma 2 applied to the kernels K_0, K_j and $L_{j,k}$, and to the functions F and G. □

2.3 Pointwise convergence of convolution integrals involving the composed Riesz kernels

We now establish Proposition 4 by studying the pointwise convergence of the convolution integrals involving the composed Riesz kernels $R_{j,k}$.

Proposition 10. *Let $1 \leq j, k, l \leq N$ and $\sigma \in \mathbb{S}^{N-1}$. Then, we have*

$$\begin{cases} R^N R_{j,k} * G_k(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} (\delta_{j,k} - N\sigma_j\sigma_k) \int_{\mathbb{R}^N} G_k(x) dx, \\ R^{N+1}\partial_l R_{j,k} * G_k(R\sigma) \underset{R\to+\infty}{\to} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} ((N+2)\sigma_j\sigma_k\sigma_l - \delta_{j,k}\sigma_l - \delta_{j,l}\sigma_k - \delta_{k,l}\sigma_j) \int_{\mathbb{R}^N} G_k. \end{cases}$$

Proof. By Corollary 2, the functions G_k verify the assumptions (i), (ii) and (iii) of Lemma 3. Thus, Proposition 10 follows from Lemma 3 and it only remains to prove this lemma. □

Proof of Lemma 3. We split the proof in two steps which correspond to each desired assertion.

Step 1. *We have*

$$R^N g(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} (\delta_{j,k} - N\sigma_j\sigma_k) \int_{\mathbb{R}^N} f(x) dx.$$

Indeed, equation (30) yields for every $R > 0$,

$$R^N g(R\sigma) = \frac{\Gamma(\frac{N}{2}) R^N}{2\pi^{\frac{N}{2}}} \Bigg(\int_{|y|>\frac{R}{2}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} f(R\sigma - y) dy + \int_{|y|\leq\frac{R}{2}} \frac{\delta_{j,k}|y|^2 - Ny_jy_k}{|y|^{N+2}} (f(R\sigma - y) - f(R\sigma)) dy \Bigg),$$

so, by the change of variable $z = R\sigma - y$,

$$\begin{aligned} R^N g(R\sigma) = \frac{\Gamma(\frac{N}{2}) R^N}{2\pi^{\frac{N}{2}}} \Bigg(& \int_{|R\sigma-z|>\frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) dz \\ & + \int_{|R\sigma-z|\leq\frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} (f(z) - f(R\sigma)) dz \Bigg). \end{aligned} \tag{68}$$

However, on one hand, we compute

$$\begin{aligned}
&R^N \left| \int_{|R\sigma - z| \leq \frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} (f(z) - f(R\sigma)) dz \right| \\
\leq & AR^N \int_{|R\sigma - z| \leq \frac{R}{2}} \frac{dz}{|R\sigma - z|^{N-1}} \sup_{x \in B(R\sigma, \frac{R}{2})} |\nabla f(x)| \\
\leq & \frac{AR^{N+1}}{1 + R^{2N+1}} \underset{R \to +\infty}{\to} 0.
\end{aligned}$$

On the other hand, we find

$$R^N 1_{|R\sigma - z| > \frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) \underset{R \to +\infty}{\to} (\delta_{j,k} - N\sigma_j\sigma_k) f(z).$$

Moreover, assumption (i) yields

$$\forall z \in B\left(R\sigma, \frac{R}{2}\right)^c, R^N \left| \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) \right| \leq \frac{A}{1 + |z|^{2N}},$$

so, by the dominated convergence theorem,

$$R^N \int_{|R\sigma - z| > \frac{R}{2}} \frac{\delta_{j,k}|R\sigma - z|^2 - N(R\sigma_j - z_j)(R\sigma_k - z_k)}{|R\sigma - z|^{N+2}} f(z) dz \underset{R \to +\infty}{\to} (\delta_{j,k} - N\sigma_j\sigma_k) \int_{\mathbb{R}^N} f,$$

which leads to the desired result by equation (68).

Now, we show the second assertion, which relies on equation (32).

Step 2. *We have*

$$R^{N+1} \partial_l g(R\sigma) \underset{R \to +\infty}{\to} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} (-(\delta_{j,k}\sigma_l + \delta_{j,l}\sigma_k + \delta_{k,l}\sigma_j) + (N+2)\sigma_j\sigma_k\sigma_l) \int_{\mathbb{R}^N} f(x) dx.$$

The proof is rather similar to the previous one. Indeed, consider $R > 0$ and integrate equation (32) by parts:

$$\begin{aligned}
\partial_l g(R\sigma) = & \int_{B(0, \frac{R}{2})^c} \partial_l R_{j,k}(y) f(R\sigma - y) dy + \int_{B(0, \frac{R}{2})} \partial_l R_{j,k}(y) (f(R\sigma - y) - f(R\sigma) \\
& + y.\nabla f(R\sigma)) dy + \frac{2}{R} \int_{S(0, \frac{R}{2})} R_{j,k}(y) y_l (f(R\sigma) - y.\nabla f(R\sigma)) dy.
\end{aligned}$$

By the change of variable $z = R\sigma - y$, it becomes

$$\begin{aligned}
R^{N+1} \partial_l g(R\sigma) = & R^{N+1} \int_{B(R\sigma, \frac{R}{2})^c} \partial_l R_{j,k}(R\sigma - z) f(z) dz + R^{N+1} \int_{B(R\sigma, \frac{R}{2})} \partial_l R_{j,k}(R\sigma - z) \\
& (f(z) - f(R\sigma) + (R\sigma - z).\nabla f(R\sigma)) dz + 2R^N \int_{S(0, \frac{R}{2})} R_{j,k}(y) y_l (f(R\sigma) \\
& - y.\nabla f(R\sigma)) dy.
\end{aligned} \tag{69}$$

Now, by assumptions (i) and (ii),

$$R^N \left| \int_{S(0, \frac{R}{2})} R_{j,k}(y) y_l (f(R\sigma) + y.\nabla f(R\sigma)) dy \right| \leq AR^N \left(\frac{1}{1 + R^{2N}} + \frac{R}{1 + R^{2N+1}} \right) \underset{R \to +\infty}{\to} 0,$$

while by assumptions (iii),

$$\begin{aligned}
&R^{N+1}\left|\int_{B(R\sigma,\frac{R}{2})}\partial_l R_{j,k}(R\sigma-z)(f(z)-f(R\sigma)+(R\sigma-z).\nabla f(R\sigma))dz\right|\\
\leq & AR^{N+1}\int_{B(R\sigma,\frac{R}{2})}\frac{dz}{|R\sigma-z|^{N-1}}\sup_{x\in B(R\sigma,\frac{R}{2})}|d^2f(x)|\\
\leq & A\frac{R^{N+2}}{1+R^{2N+2}}\underset{R\to+\infty}{\to}0.
\end{aligned}$$

However, we compute

$$R^{N+2}1_{|R\sigma-z|>\frac{R}{2}}\partial_l R_{j,k}(R\sigma-z)f(z)\underset{R\to+\infty}{\to}((N+2)\sigma_j\sigma_k\sigma_l-(\delta_{j,k}\sigma_l+\delta_{j,l}\sigma_k+\delta_{k,l}\sigma_j))f(z),$$

and by assumption (i),

$$\forall z\in B\left(R\sigma,\frac{R}{2}\right)^c, R^{N+1}\left|\partial_l R_{j,k}(R\sigma-z)f(z)\right|\leq\frac{A}{1+|z|^{2N}},$$

so, by the dominated convergence theorem,

$$\begin{aligned}
&R^{N+1}\int_{|R\sigma-z|>\frac{R}{2}}\partial_l R_{j,k}(R\sigma-z)f(z)dz\\
\underset{R\to+\infty}{\to}&((N+2)\sigma_j\sigma_k\sigma_l-(\delta_{j,k}\sigma_l+\delta_{j,l}\sigma_k+\delta_{k,l}\sigma_j))\int_{\mathbb{R}^N}f(x)dx,
\end{aligned}$$

which completes the proofs of Step 2 and of Lemma 3 by equation (69). □

We are now in position to show Proposition 4.

Proof of Proposition 4. It follows from equations (21) and (24), and from Propositions 9 and 10 that there exist bounded measurable functions η_∞, η_∞^j, θ_∞^j, $\eta_\infty^{j,k}$ and $\theta_\infty^{j,k}$ such that for every $\sigma\in\mathbb{S}^{N-1}$,

$$\begin{cases}
R^N\eta(R\sigma)\underset{R\to+\infty}{\to}\eta_\infty(\sigma),\\
R^{N+1}\partial_j\eta(R\sigma)\underset{R\to+\infty}{\to}\eta_\infty^j(\sigma),\\
R^N\partial_j\theta(R\sigma)\underset{R\to+\infty}{\to}\theta_\infty^j(\sigma),\\
R^{N+2}\partial^2_{j,k}\eta(R\sigma)\underset{R\to+\infty}{\to}\eta_\infty^{j,k}(\sigma),\\
R^{N+1}\partial^2_{j,k}\theta(R\sigma)\underset{R\to+\infty}{\to}\theta_\infty^{j,k}(\sigma).
\end{cases}$$

In particular, we can compute for every $\sigma\in\mathbb{S}^{N-1}$,

$$\eta_\infty(\sigma)=K_{0,\infty}(\sigma)\int_{\mathbb{R}^N}F(x)dx+2c\sum_{j=1}^N K_{j,\infty}(\sigma)\int_{\mathbb{R}^N}G_j(x)dx \tag{70}$$

$$\theta_\infty^j(\sigma)=\frac{c}{2}K_{j,\infty}(\sigma)\int_{\mathbb{R}^N}F(x)dx+\sum_{k=1}^N(c^2L_{j,k,\infty}(\sigma)+\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k}-N\sigma_j\sigma_k))\int_{\mathbb{R}^N}G_k(x)dx. \tag{71}$$

Thus, it only remains to consider the existence of the function θ_∞. It follows from the next lemma.

Lemma 10. *Let $f \in C^1(\mathbb{R}^N, \mathbb{C})$ and $M > 1$. Assume that for every $j \in \{1, \dots, N\}$, there is a bounded function f_∞^j defined on the sphere $\mathbb{S}^{N-1}$ such that*

$$\forall \sigma \in \mathbb{S}^{N-1}, R^M \partial_j f(R\sigma) \underset{R \to +\infty}{\to} f_\infty^j(\sigma),$$

and that

$$f(x) \underset{|x| \to +\infty}{\to} \lambda_\infty \in \mathbb{C}.$$

Then,

$$\forall \sigma \in \mathbb{S}^{N-1}, R^{M-1}(f(R\sigma) - \lambda_\infty) \underset{R \to +\infty}{\to} f_\infty(\sigma) = -\frac{1}{M-1} \sum_{j=1}^N \sigma_j f_\infty^j(\sigma).$$

Proof of Lemma 10. Indeed, f belongs to $C^1(\mathbb{R}^N, \mathbb{C})$ and converges to λ_∞ at infinity, so, since $M > 1$, we can state

$$\forall R > 1, f(R\sigma) - \lambda_\infty = -\int_R^{+\infty} \sum_{j=1}^N \partial_j f(r\sigma)\sigma_j dr.$$

Moreover, we have

$$\sum_{j=1}^N \partial_j f(r\sigma)\sigma_j = \frac{1}{r^M} \sum_{j=1}^N f_\infty^j(\sigma)\sigma_j + \underset{r \to +\infty}{o}\left(\frac{1}{r^M}\right),$$

therefore,

$$\int_R^{+\infty} \sum_{j=1}^N \partial_j f(r\sigma)\sigma_j dr = \frac{1}{(M-1)R^{M-1}} \sum_{j=1}^N f_\infty^j(\sigma)\sigma_j + \underset{R \to +\infty}{o}\left(\frac{1}{R^{M-1}}\right),$$

which yields

$$R^{M-1}(f(R\sigma) - \lambda_\infty) \underset{R \to +\infty}{\to} -\frac{1}{M-1} \sum_{j=1}^N f_\infty^j(\sigma)\sigma_j = f_\infty(\sigma).$$

□

At this point, we notice that the function $\psi\theta$ satisfies all the assumptions of Lemma 10 with $M = N$ and $\lambda_\infty = 0$. Thus, there is a bounded measurable function θ_∞ such that

$$R^{N-1}\theta(R\sigma) \underset{R \to +\infty}{\to} \theta_\infty(\sigma) = -\frac{1}{N-1} \sum_{j=1}^N \sigma_j \theta_\infty^j(\sigma).$$

Moreover, by equation (71), we compute the next more explicit form of θ_∞

$$\begin{aligned}\theta_\infty(\sigma) = -\frac{1}{N-1}\Bigg(&\frac{c}{2}\Big(\sum_{j=1}^N \sigma_j K_{j,\infty}(\sigma)\Big)\int_{\mathbb{R}^N} F(x)dx + \sum_{k=1}^N \Big(c^2 \sum_{j=1}^N \sigma_j L_{j,k,\infty}(\sigma) \\ &- \frac{(N-1)\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\sigma_k\Big)\int_{\mathbb{R}^N} G_k(x)dx.\Bigg)\end{aligned} \tag{72}$$

□

Remark 14. Conjecture 1 follows from formulae (70) and (72). Indeed, in the first section of the second part, we computed formally the values of $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ (see formulae (62), (63) and (64)). By equations (70) and (72), it only remains to compute the values of $\int_{\mathbb{R}^N} F(x)dx$ and $\int_{\mathbb{R}^N} G_k(x)dx$ to get explicit expressions of the limits η_∞ and θ_∞. In the third part, we will compute such integrals and we will obtain that

$$\int_{\mathbb{R}^N} F(x)dx = 2((4-N)E(v) + c(N-3)p(v)),$$

and

$$\int_{\mathbb{R}^N} G_k(x)dx = 2P_k(v).$$

Finally, by equations (62), (63), (64), (70) and (72), it yields the value of the functions η_∞,

$$\eta_\infty(\sigma) = \frac{c\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Bigg(\Big(\frac{4-N}{2}cE(v) + (2+\frac{N-3}{2}c^2)p(v)\Big)\Big(\frac{1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N\sigma_1^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big) - 2\Big(1-\frac{c^2}{2}\Big)\sum_{j=2}^N P_j(v)\frac{N\sigma_1\sigma_j}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Bigg),$$

and θ_∞

$$\theta_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Bigg(\Big(\frac{4-N}{2}cE(v) + (2+\frac{N-3}{2}c^2)p(v)\Big)\frac{\sigma_1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + 2\Big(1-\frac{c^2}{2}\Big)\sum_{j=2}^N P_j(v)\frac{\sigma_j}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Bigg).$$

Since v_∞ is equal to θ_∞, it leads formally to Conjecture 1.

2.4 Uniformity of the convergence

Now, we show the uniformity of the previous pointwise convergence. Actually, Proposition 5 even yields a little more. Indeed, the functions $\sigma \mapsto R^N\eta(R\sigma)$ and $\sigma \mapsto R^{N-1}\theta(R\sigma)$ converge to η_∞, respectively θ_∞, in $C^1(\mathbb{S}^{N-1})$, respectively $C^2(\mathbb{S}^{N-1})$, when R tends to $+\infty$. As claimed in the introduction, it follows from the decay estimates of Theorem 6 and Ascoli-Arzela's theorem.

Proof of Proposition 5. Consider the functions $(\eta_R)_{R>0}$ and $(\theta_R)_{R>0}$ defined by

$$\forall \sigma \in \mathbb{S}^{N-1}, \begin{cases} \eta_R(\sigma) = R^N\eta(R\sigma) \\ \theta_R(\sigma) = R^{N-1}(\psi\theta)(R\sigma) \\ v_R(\sigma) = R^{N-1}(v(R\sigma)-1). \end{cases}$$

Step 1. *Computation of some derivatives of the functions η_R and θ_R and of their limits at infinity.*

We first compute some explicit expressions of some derivatives of η_R and θ_R and of their limits when $R \to +\infty$. It will be fruitful to prove the uniformity of the convergence and to deduce Proposition 6. By Proposition 4, we first get for every $\sigma \in \mathbb{S}^{N-1}$,

$$\begin{cases} \eta_R(\sigma) \underset{R\to+\infty}{\to} \eta_\infty(\sigma) \\ \theta_R(\sigma) \underset{R\to+\infty}{\to} \theta_\infty(\sigma). \end{cases}$$

Then, by definition, we have for every $j \in \{1, \dots, N\}$ and for every function $f \in C^1(\mathbb{S}^{N-1})$,

$$\partial_j^{\mathbb{S}^{N-1}} f(\sigma) = \lim_{t \to 0} \frac{f(\frac{\sigma + te_j}{|\sigma + te_j|}) - f(\sigma)}{t}.$$

Therefore, considering a function $f \in C^1(\mathbb{R}^N)$ and denoting for every $R > 0$ and $\sigma \in \mathbb{S}^{N-1}$,

$$f_R(\sigma) = f(R\sigma),$$

we compute

$$\partial_j^{\mathbb{S}^{N-1}} f_R(\sigma) = R(\partial_j f(R\sigma) - \sigma_j \sum_{i=1}^N \sigma_i \partial_i f(R\sigma)). \tag{73}$$

Likewise, we find for every $k \in \{1, \dots, N\}$ and $\sigma \in \mathbb{S}^{N-1}$,

$$\partial_j^{\mathbb{S}^{N-1}} \sigma_k = \delta_{j,k} - \sigma_j \sigma_k. \tag{74}$$

Thus, it follows from formula (73) that

$$\begin{cases} \partial_j^{\mathbb{S}^{N-1}} \eta_R(\sigma) = R^{N+1}(\partial_j \eta(R\sigma) - \sigma_j \sum\limits_{k=1}^N \sigma_k \partial_k \eta(R\sigma)), \\ \partial_j^{\mathbb{S}^{N-1}} \theta_R(\sigma) = R^N(\partial_j(\psi\theta)(R\sigma) - \sigma_j \sum\limits_{k=1}^N \sigma_k \partial_k(\psi\theta)(R\sigma)). \end{cases} \tag{75}$$

By Proposition 4, it gives

$$\begin{cases} \partial_j^{\mathbb{S}^{N-1}} \eta_R(\sigma) \underset{R \to +\infty}{\to} \eta_\infty^j(\sigma) - \sigma_j \sum\limits_{k=1}^N \sigma_k \eta_\infty^k(\sigma), \\ \partial_j^{\mathbb{S}^{N-1}} \theta_R(\sigma) \underset{R \to +\infty}{\to} \theta_\infty^j(\sigma) - \sigma_j \sum\limits_{k=1}^N \sigma_k \theta_\infty^k(\sigma). \end{cases}$$

Moreover, the functions η and $\psi\theta$ satisfy all the assumptions of Lemma 10 with $M = N + 1$, respectively $M = N$, and $\lambda_\infty = 0$. Therefore, Lemma 10 leads to

$$\begin{cases} \sum\limits_{k=1}^N \sigma_k \eta_\infty^k(\sigma) = -N\eta_\infty(\sigma), \\ \sum\limits_{k=1}^N \sigma_k \theta_\infty^k(\sigma) = -(N-1)\theta_\infty(\sigma), \end{cases} \tag{76}$$

and finally,

$$\begin{cases} \partial_j^{\mathbb{S}^{N-1}} \eta_R(\sigma) \underset{R \to +\infty}{\to} \eta_\infty^j(\sigma) + N\sigma_j \eta_\infty(\sigma), \\ \partial_j^{\mathbb{S}^{N-1}} \theta_R(\sigma) \underset{R \to +\infty}{\to} \theta_\infty^j(\sigma) + (N-1)\sigma_j \theta_\infty(\sigma). \end{cases} \tag{77}$$

Likewise, formulae (73) and (74) yield for every $(j, k) \in \{1, \dots, N\}^2$,

$$\begin{aligned} \partial_k^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}} \theta_R)(\sigma) =& R^{N+1} \Bigg(\partial_{j,k}^2 \theta(R\sigma) - \sum_{l=1}^N \sigma_l \Big(\sigma_k \partial_{j,l}^2 \theta(R\sigma) + \sigma_j \partial_{k,l}^2 \theta(R\sigma) - \sigma_k \sigma_j \\ & \sum_{m=1}^N \sigma_m \partial_{l,m}^2 \theta(R\sigma) \Big) \Bigg) - R^N \sum_{l=1}^N \Big((\delta_{j,k} - \sigma_j \sigma_k)\sigma_l + (\delta_{k,l} - \sigma_k \sigma_l)\sigma_j \Big) \\ & \partial_l \theta(R\sigma), \end{aligned} \tag{78}$$

so, by Proposition 4,

$$\partial_k^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}}\theta_R)(\sigma) \underset{R\to+\infty}{\to} \theta_\infty^{j,k}(\sigma) - \sum_{l=1}^N \sigma_l\Big(\sigma_k\theta_\infty^{j,l}(\sigma) + \sigma_j\theta_\infty^{k,l}(\sigma) - \sigma_k\sigma_j\sum_{m=1}^N \sigma_m \theta_\infty^{l,m}(\sigma)\Big) - \sum_{l=1}^N \Big((\delta_{j,k} - \sigma_j\sigma_k)\sigma_l + (\delta_{k,l} - \sigma_k\sigma_l)\sigma_j\Big)\theta_\infty^l(\sigma).$$

However, the function $\partial_j\theta$ also satisfies the assumptions of Lemma 10 with $M = N+1$ and $\lambda_\infty = 0$. Thus, we obtain likewise

$$\sum_{l=1}^N \sigma_l\theta_\infty^{j,l}(\sigma) = -N\theta_\infty^j(\sigma), \tag{79}$$

and

$$\partial_k^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}}\theta_R)(\sigma) \underset{R\to+\infty}{\to} \theta_\infty^{j,k}(\sigma) + N\sigma_k\theta_\infty^j(\sigma) + (N-1)\sigma_j\theta_\infty^k(\sigma) + (N-1)(\delta_{j,k} + (N-2)\sigma_j\sigma_k)\theta_\infty(\sigma). \tag{80}$$

Step 2. *Uniformity of the convergence.*

Now, assume by contradiction that $(\eta_R)_{R>0}$ does not converge to η_∞ in $C^1(\mathbb{S}^{N-1})$. Then, there is some real number $\epsilon > 0$, and a sequence of positive real number $(R_n)_{n\in\mathbb{N}}$ tending to $+\infty$, such that

$$\forall n \in \mathbb{N}, \|\eta_{R_n} - \eta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \|\nabla^{\mathbb{S}^{N-1}}\eta_{R_n} - \nabla^{\mathbb{S}^{N-1}}\eta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} > \epsilon.$$

However, on one hand, by Proposition 2 and equation (75), there is some real number A such that

$$\forall n \in \mathbb{N}, \begin{cases} \|\eta_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq A \\ \|\nabla^{\mathbb{S}^{N-1}}\eta_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq AR_n^{N+1}\|\nabla\eta(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})} \leq A. \end{cases}$$

On the other hand, formulae (73), (74) and (75), Proposition 2 and Theorem 6 yield that

$$\|d^{2,\mathbb{S}^{N-1}}\eta_{R_n}\|_{L^\infty(\mathbb{S}^{N-1})} \leq A(R_n^{N+1}\|\nabla\eta(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})} + R_n^{N+2}\|d^2\eta(R_n.)\|_{L^\infty(\mathbb{S}^{N-1})}) \leq A.$$

Therefore, by Ascoli-Arzela's theorem, up to a subsequence, $(\eta_{R_n})_{n\in\mathbb{N}}$ converges in the space $C^1(\mathbb{S}^{N-1})$. By Proposition 4, its limit is necessarily equal to η_∞, which yields a contradiction. Thus, $(\eta_R)_{R>0}$ converges to η_∞ in $C^1(\mathbb{S}^{N-1})$. In particular, η_∞ is of class C^1 on $\mathbb{S}^{N-1}$ and satisfies by equations (77) for every $j \in \{1, \ldots, N\}$,

$$\partial_j^{\mathbb{S}^{N-1}}\eta_\infty(\sigma) = \eta_\infty^j(\sigma) + N\sigma_j\eta_\infty(\sigma). \tag{81}$$

Likewise, by Proposition 2, Theorem 6 and equations (75) and (78), there is some real number A such that

$$\begin{cases} \|\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A, \\ \|\nabla^{\mathbb{S}^{N-1}}\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq AR^N\|\nabla(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} \leq A, \\ \|d^{2,\mathbb{S}^{N-1}}\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq AR^N(\|\nabla(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} + R\|d^2(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})}) \leq A. \end{cases}$$

Formulae (73), (74) and (78) then give

$$\|d^{3,\mathbb{S}^{N-1}}\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A(R^N\|\nabla(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} + R^{N+1}\|d^2(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})} + R^{N+2}\|d^3(\psi\theta)(R.)\|_{L^\infty(\mathbb{S}^{N-1})}),$$

so, by Proposition 2 and Theorem 6,

$$\|d^{3,\mathbb{S}^{N-1}}\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} \leq A.$$

Thus, up to the argument by contradiction above, the functions $(\theta_R)_{R>0}$ converge to θ_∞ in $C^2(\mathbb{S}^{N-1})$. In particular, θ_∞ is in $C^2(\mathbb{S}^{N-1})$ and satisfies by equations (77) and (80) for every $(j,k) \in \{1,\dots,N\}^2$,

$$\partial_j^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) = \theta_\infty^j(\sigma) + (N-1)\sigma_j\theta_\infty(\sigma), \tag{82}$$

and

$$\begin{aligned}\partial_k^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}}\theta_\infty)(\sigma) = \theta_\infty^{j,k}(\sigma) + N\sigma_k\theta_\infty^j(\sigma) + (N-1)\sigma_j\theta_\infty^k(\sigma)) + (N-1)(\delta_{j,k}\\ + (N-2)\sigma_k\sigma_j)\theta_\infty(\sigma).\end{aligned} \tag{83}$$

Finally, we consider the uniform convergence of the function v_R. By definition, we have for every $\sigma \in \mathbb{S}^{N-1}$ and $R > 3R_0$,

$$v_R(\sigma) = R^{N-1}(\sqrt{1-\eta(R\sigma)}e^{i\theta(R\sigma)} - 1),$$

so, by Proposition 2 and the proof of the uniform convergences of η_R and θ_R just above,

$$\begin{aligned}&\|v_R - i\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})}\\ \leq& R^{N-1}\|\sqrt{1-\eta(R.)} - 1\|_{L^\infty(\mathbb{S}^{N-1})} + \|R^{N-1}(e^{i\theta(R.)} - 1) - i\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})}\\ \leq& A\Big(\frac{1}{R}\|\eta_R\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R^{N-1}}\|\theta_R^2\|_{L^\infty(\mathbb{S}^{N-1})} + \|\theta_R - \theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})}\Big)\\ &\underset{R\to+\infty}{\to} 0.\end{aligned}$$

Likewise, we compute for every $j \in \{1,\dots,N\}$ by equation (73),

$$\begin{aligned}\partial_j^{\mathbb{S}^{N-1}}v_R(\sigma) =& R^N\Bigg(i\sqrt{1-\eta(R\sigma)}\partial_j\theta(R\sigma) - \frac{\partial_j\eta(R\sigma)}{2\sqrt{1-\eta(R\sigma)}} - \sigma_j\sum_{k=1}^N\sigma_k\Big(-\frac{\partial_k\eta(R\sigma)}{2\sqrt{1-\eta(R\sigma)}}\\ &+ i\sqrt{1-\eta(R\sigma)}\partial_k\theta(R\sigma)\Big)\Bigg)e^{i\theta(R\sigma)}\\ =& \Bigg(i\sqrt{1-\eta(R\sigma)}\partial_j^{\mathbb{S}^{N-1}}\theta_R(\sigma) - \frac{\partial_j^{\mathbb{S}^{N-1}}\eta_R(\sigma)}{2R\sqrt{1-\eta(R\sigma)}}\Bigg)e^{i\theta(R\sigma)}.\end{aligned}$$

Therefore, by Proposition 2 and the proof of the convergences in $C^1(\mathbb{S}^{N-1})$ of η_R and θ_R just above,

$$\begin{aligned}\|\partial_j^{\mathbb{S}^{N-1}}v_R - i\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leq& A\Big(\|\partial_j^{\mathbb{S}^{N-1}}\theta_R - i\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \|(\sqrt{1-\eta(R.)}\\ &e^{i\theta(R.)} - 1)\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R}\|\partial_j^{\mathbb{S}^{N-1}}\eta_R\|_{L^\infty(\mathbb{S}^{N-1})}\Big)\\ \leq& A\Big(\|\partial_j^{\mathbb{S}^{N-1}}\theta_R - i\partial_j^{\mathbb{S}^{N-1}}\theta_\infty\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R^N}\|\eta_R\|_{L^\infty(\mathbb{S}^{N-1})}\\ &+ \frac{1}{R^{N-1}}\|\theta_R\|_{L^\infty(\mathbb{S}^{N-1})} + \frac{1}{R}\|\partial_j^{\mathbb{S}^{N-1}}\eta_R\|_{L^\infty(\mathbb{S}^{N-1})}\Big)\\ &\underset{R\to+\infty}{\to} 0.\end{aligned}$$

Thus, denoting $v_\infty = \theta_\infty$, v_∞ is a smooth function on $\mathbb{S}^{N-1}$, which satisfies

$$\|v_R - v_\infty\|_{C^1(\mathbb{S}^{N-1})} \underset{R\to+\infty}{\to} 0.$$

This concludes the proof of Proposition 5. □

2.5 Partial differential equations satisfied by η_∞, θ_∞ and v_∞

Finally, we deduce from the proof of Proposition 5 just above the partial differential equations satisfied by η_∞ and θ_∞.

Proof of Proposition 6. Let $\sigma \in \mathbb{S}^{N-1}$. On one hand, we compute from equation (2) on a neighbourhood of infinity

$$\Delta\eta + 2|\nabla v|^2 + 2c\partial_1\theta - 2\eta - 2c\eta\partial_1\theta + 2\eta^2 = 0,$$

so, for every $R > 0$,

$$R^N(\Delta\eta(R\sigma) + 2|\nabla v(R\sigma)|^2 - 2c\partial_1\theta(R\sigma) - 2\eta(R\sigma) + 2c\eta(R\sigma)\partial_1\theta(R\sigma) + 2\eta(R\sigma)^2) = 0.$$

Taking the limit $R \to +\infty$, it gives by Propositions 2 and 4, and Theorem 6,

$$\eta_\infty(\sigma) = c\theta^1_\infty(\sigma),$$

which reduces to equation (34) by equation (82).

On the other hand, equation (18) yields on a neighbourhood of infinity

$$R^{N+1}(\Delta\theta(R\sigma) - \frac{c}{2}\partial_1\eta(R\sigma) - \nabla\eta(R\sigma).\nabla\theta(R\sigma) - \eta\Delta\theta(R\sigma)) = 0.$$

Therefore, Propositions 2 and 4, and Theorem 6 yield once again at the limit $R \to +\infty$

$$\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) = \frac{c}{2}\eta^1_\infty(\sigma),$$

which gives by equation (81),

$$\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) = \frac{c}{2}(\partial_1^{\mathbb{S}^{N-1}}\eta_\infty(\sigma) - N\sigma_1\eta_\infty(\sigma)).$$

However, by equations (82) and (83),

$$\begin{aligned}\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) &= \sum_{j=1}^N \partial_j^{\mathbb{S}^{N-1}}(\partial_j^{\mathbb{S}^{N-1}}\theta_\infty)(\sigma) - (2N-1)\sum_{j=1}^N \sigma_j\theta^j_\infty(\sigma)\\ &\quad - (N-1)\sum_{j=1}^N(1 + (N-2)\sigma_j^2)\theta_\infty(\sigma)\\ &= \Delta^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) - (2N-1)\sum_{j=1}^N \sigma_j\theta^j_\infty(\sigma) - (N-1)(2N-2)\theta_\infty(\sigma).\end{aligned}$$

Then, equation (76) states

$$\sum_{j=1}^N \sigma_j\theta^j_\infty(\sigma) = -(N-1)\theta_\infty(\sigma),$$

so,

$$\sum_{j=1}^N \theta^{j,j}_\infty(\sigma) = \Delta^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) + (N-1)\theta_\infty(\sigma).$$

Thus, we finally find equation (35)

$$\Delta^{\mathbb{S}^{N-1}}\theta_\infty(\sigma) + (N-1)\theta_\infty(\sigma) = \frac{c}{2}(\partial_1^{\mathbb{S}^{N-1}}\eta_\infty(\sigma) - N\sigma_1\eta_\infty(\sigma)).$$

Now, it only remains to prove that the functions θ_∞ and η_∞ are smooth on $\mathbb{S}^{N-1}$. Indeed, equations (34) and (35) give

$$\Delta^{\mathbb{S}^{N-1}}\theta_\infty - \frac{c^2}{2}\partial_1^{\mathbb{S}^{N-1}}(\partial_1^{\mathbb{S}^{N-1}}\theta_\infty) + c^2(N-1)\sigma_1\partial_1^{\mathbb{S}^{N-1}}\theta_\infty + (N-1)(1+\frac{c^2}{2} - (N+1)\frac{c^2}{2}\sigma_1^2)\theta_\infty = 0. \quad (84)$$

Thus, θ_∞ is solution on $\mathbb{S}^{N-1}$ of an elliptic partial differential system with smooth coefficients. By standard elliptic theory, it is of class C^∞ on $\mathbb{S}^{N-1}$. By equation (34), η_∞ is also smooth on $\mathbb{S}^{N-1}$. □

We conclude the second part by the proof of Theorem 1, which follows from Proposition 5 and equation (84).

Proof of Theorem 1. By Proposition 5, there exists a smooth function $v_\infty = \theta_\infty$ on $\mathbb{S}^{N-1}$ such that

$$|x|^{N-1}(v(x)-1) - iv_\infty\left(\frac{x}{|x|}\right) \underset{|x|\to+\infty}{\to} 0 \text{ uniformly}.$$

Moreover, by equation (84), v_∞ satisfies the linear partial differential equation (10). □

3 Asymptotics in dimension two and in the axisymmetric case

In the last part, we focus on the axisymmetric case and on the case of dimension two. In both cases, the system of equations (34) and (35) reduces to an entirely integrable system of linear ordinary differential equations of second order. In Proposition 7, we compute explicitly its solutions up to undetermined constants α and β. Lemma 6 in connection with the Pohozaev identities of Lemma 7 links the value of α and β with the energy $E(v)$ and the momentum $\vec{P}(v)$, which completes the proof of Theorems 2 and 3. Finally, we deduce Corollary 1 from Lemma 7.

3.1 Explicit expression for the first order term

This section is devoted to the integration of the system of equations (34) and (35) in dimension two and in the axisymmetric case. It relies on the use of spherical coordinates. That is the reason why we first recall some of their properties.

Indeed, let $\Phi_N : \Omega = \mathbb{R}_+ \times [0,\pi]^{N-2} \times [0,2\pi] \mapsto \mathbb{R}^N$, the function defined by

$$\Phi_N(r,\beta_1,\dots,\beta_{N-1}) = (r\cos(\beta_1), r\sin(\beta_1)\cos(\beta_2), \dots, r\prod_{i=1}^{N-1}\sin(\beta_i)).$$

The function Φ_N is smooth on Ω and its Jacobian matrix is

$$J(\Phi_N)(r,\beta_1,\dots,\beta_{N-1}) = (J_{i,j})_{1\leq i,j\leq N},$$

where

$$\begin{cases} J_{1,j} = \prod_{k=1}^{j-1}\sin(\beta_k)\cos(\beta_j), \\ J_{i,j} = 0, \text{ if } i\geq 2 \text{ and } j \leq i-2, \\ J_{i,i-1} = -r\prod_{k=1}^{i-1}\sin(\beta_k), \\ J_{i,j} = r\prod_{k=1}^{j-1}\sin(\beta_k)\cos(\beta_j)\cos(\beta_{i-1}), \text{ otherwise.} \end{cases}$$

Thus, $J(\Phi_N)$ is invertible if and only if $r \neq 0$ and $\beta_j \neq 0$ modulo π for every $j \in \{1, \ldots, N-2\}$. Moreover, its inverse is

$$J(\Phi_N)^{-1}(r, \beta_1, \ldots, \beta_{N-1}) = (J^{-1}_{i,j})_{1 \leq i,j \leq N},$$

where

$$\begin{cases} J^{-1}_{i,1} = \prod_{k=1}^{i-1} \sin(\beta_k) \cos(\beta_i), \\ J^{-1}_{i,j} = 0, \text{ if } j \geq 2 \text{ and } i \leq j-2, \\ J^{-1}_{j-1,j} = -\frac{\sin(\beta_{j-1})}{r \prod_{k=1}^{j-2} \sin(\beta_k)}, \\ J^{-1}_{i,j} = \frac{\prod_{k=j}^{i-1} \sin(\beta_k)}{r \prod_{k=1}^{j-2} \sin(\beta_k)} \cos(\beta_{j-1}) \cos(\beta_i), \text{ otherwise.} \end{cases}$$

Therefore, if we consider a smooth function $f \in C^\infty(\mathbb{R}^N)$ and denote

$$g = f \circ \Phi_N,$$

the chain rule theorem yields

$$\forall y \in \Omega, J(\Phi_N)(y) \begin{pmatrix} \partial_1 f(\Phi_N(y)) \\ \vdots \\ \partial_N f(\Phi_N(y)) \end{pmatrix} = \begin{pmatrix} \partial_r g(y) \\ \vdots \\ \partial_{\beta_{N-1}} g(y) \end{pmatrix}.$$

Moreover, assuming f is axisymmetric around axis x_1 or the dimension N is two, the function g is independent on the variables β_2, ... and β_N, which yields for every $j \in \{2, \ldots, N\}$,

$$\begin{aligned} \partial_1 f(\Phi_N(y)) &= \cos(\beta_1)\partial_r g(y) - \frac{\sin(\beta_1)}{r}\partial_{\beta_1} g(y), \\ \partial_j f(\Phi_N(y)) &= \prod_{k=1}^{j-1} \sin(\beta_k)\cos(\beta_j)\partial_r g(y) + \frac{\cos(\beta_1)\cos(\beta_j)}{r} \prod_{k=2}^{j-1} \sin(\beta_k)\partial_{\beta_1} g(y), \\ \partial^2_{1,1} f(\Phi_N(y)) &= \cos^2(\beta_1)\partial^2_{r,r} g(y) + \frac{2\sin(\beta_1)\cos(\beta_1)}{r^2}\partial_{\beta_1} g(y) - \frac{2\sin(\beta_1)\cos(\beta_1)}{r}\partial^2_{r,\beta_1} g(y) \\ &\quad + \frac{\sin^2(\beta_1)}{r}\partial_r g(y) + \frac{\sin^2(\beta_1)}{r^2}\partial^2_{\beta_1,\beta_1} g(y), \\ \partial^2_{j,j} f(\Phi_N(y)) &= \prod_{k=2}^{j-1} \sin(\beta_k)^2 \cos^2(\beta_j)\Big(\sin^2(\beta_1)\partial^2_{r,r} g(y) + \frac{2\sin(\beta_1)\cos(\beta_1)}{r}\partial^2_{r,\beta_1} g(y) \\ &\quad - \frac{2\sin(\beta_1)\cos(\beta_1)}{r^2}\partial_{\beta_1} g(y) + \frac{\cos^2(\beta_1)}{r}\partial_r g(y) + \frac{\cos^2(\beta_1)}{r^2}\partial^2_{\beta_1,\beta_1} g(y) \\ &\quad - \frac{1}{r}\partial_r g(y) - \frac{\cos(\beta_1)}{r^2\sin(\beta_1)}\partial_{\beta_1} g(y)\Big) + \frac{1}{r}\partial_r g(y) + \frac{\cos(\beta_1)}{r^2\sin(\beta_1)}\partial_{\beta_1} g(y), \\ \Delta f(\Phi_N(y)) &= \partial^2_{r,r} g(y) + \frac{N-1}{r}\partial_r g(y) + \frac{1}{r^2}(\partial^2_{\beta_1,\beta_1} g(y) + (N-2)\text{cotan}(\beta_1)\partial_{\beta_1} g(y)), \end{aligned}$$

provided that $r \neq 0$ and $\sin(\beta_1) \neq 0$. Finally, consider now a smooth function $f \in C^\infty(\mathbb{S}^{N-1})$ and denote

$$g(\beta_1, \ldots, \beta_{N-1}) = f(\Phi_N(1, \beta_1, \ldots, \beta_{N-1})).$$

Assuming f is axisymmetric around axis x_1 or the dimension N is two, we deduce that for every $y = (1, \beta_1, \ldots, \beta_{N-1})$ such that $\sin(\beta_1) \neq 0$,

$$\begin{aligned} \partial_1^{\mathbb{S}^{N-1}} f(\Phi_N(y)) &= -\sin(\beta_1)\partial_{\beta_1} g(y) \\ \partial_{1,1}^{2,\mathbb{S}^{N-1}} f(\Phi_N(y)) &= \sin^2(\beta_1)\partial^2_{\beta_1,\beta_1} g(y) + 2\sin(\beta_1)\cos(\beta_1)\partial_{\beta_1} g(y) \\ \Delta_{\mathbb{S}^{N-1}} f(\Phi_N(y)) &= \partial^2_{\beta_1,\beta_1} g(y) + (N-2)\text{cotan}(\beta_1)\partial_{\beta_1} g(y). \end{aligned} \tag{85}$$

Proposition 7 is then a consequence of formulae (85), and equations (34) and (35).

Proof of Proposition 7. In this proof, the dimension N is assumed to be two, or the travelling wave v is supposed to be axisymmetric around axis x_1. Thus, the functions η_∞ and θ_∞ only depend on the variable β_1 in spherical coordinates. Up to a misuse of notations, we will consider them as functions of β_1.

However, by Proposition 6, θ_∞ is smooth on $\mathbb{S}^{N-1}$ and satisfies equation (84). Therefore, in the new variables, it is smooth on $[0,\pi]$ in dimension $N \geq 3$, respectively $[0,2\pi]$ in dimension two. Moreover, by equation (84) and formulae (85), it verifies the second order ordinary differential equation

$$\begin{aligned}&(1-\frac{c^2}{2}\sin^2(\beta_1))\theta_\infty''(\beta_1)+((N-2)\mathrm{cotan}(\beta_1)-Nc^2\cos(\beta_1)\sin(\beta_1))\theta_\infty'(\beta_1)+(N-1)\\&(1+\frac{c^2}{2}-(N+1)\frac{c^2}{2}\cos^2(\beta_1))\theta_\infty(\beta_1)=0.\end{aligned}\tag{86}$$

The articles of C.A. Jones, S.J. Putterman and P.H. Roberts [13, 12] yield one particular solution of equation (86) in dimensions two and three. Generalising its form to every dimension, we find a first solution equal to

$$Sol_1(\beta_1)=\frac{\cos(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}}}.$$

However, the set of solutions on $]0,\pi[$ in dimension $N \geq 3$, respectively $]0,\pi[$ and $]\pi,2\pi[$ in dimension two, is a vectorial space of dimension two. In order to find another independent solution, we let

$$u(\beta_1)=\frac{\theta_\infty(\beta_1)}{Sol_1(\beta_1)},$$

for every $\beta_1 \in]0,\pi[\setminus\{\frac{\pi}{2}\}$ in dimension $N \geq 3$, respectively $\beta_1 \in]0,\pi[\setminus\{\frac{\pi}{2}\}\cup]\pi,2\pi[\setminus\{\frac{3\pi}{2}\}$ in dimension two. Then, we compute the next ordinary differential equation for the function u:

$$\sin(\beta_1)\cos(\beta_1)(1-\frac{c^2}{2}\sin^2(\beta_1))u''(\beta_1)+(N-2-N\sin^2(\beta_1)+c^2\sin^4(\beta_1))u'(\beta_1)=0.$$

After a first integration, we deduce that there is some real constant A such that

$$u'(\beta_1)=A\frac{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N-2}{2}}}{\cos^2(\beta_1)\sin^{N-2}(\beta_1)},$$

and, after another integration, we infer that there is another real constant B such that

$$u(\beta_1)=B+A\sum_{k=0}^{p-1}\frac{1}{2(k-p)+3}C_{p-1}^k\Big(1-\frac{c^2}{2}\Big)^{k+p-\frac{3}{2}}\tan^{2(k-p)+3}(\beta_1)$$

if $N=2p$, and if $N=2p+1$,

$$\begin{aligned}u(\beta_1)=&B+A\Big(1-\frac{c^2}{2}\Big)^{p-1}\Bigg(\sqrt{1+\Big(1-\frac{c^2}{2}\Big)\tan^2(\beta_1)}+\sum_{k=1}^{p}\frac{C_p^k}{a_k}\\&\Bigg(\ln\Bigg(\frac{\sqrt{1-\frac{c^2}{2}}\tan(\beta_1)}{1+\sqrt{1+(1-\frac{c^2}{2})\tan^2(\beta_1)}}\Bigg)-\sum_{q=1}^{k-1}\frac{a_{q+1}}{2q}\frac{\sqrt{1+(1-\frac{c^2}{2})\tan^2(\beta_1)}}{(1-\frac{c^2}{2})^q\tan^{2q}(\beta_1)}\Bigg)\Bigg),\end{aligned}$$

where

$$\forall k \in \mathbb{N}^*, a_k = \frac{(-4)^{k-1}((k-1)!)^2}{(2(k-1))!}.$$

Thus, we find another particular solution equal to

$$Sol_2(\beta_1) = \frac{\sin(\beta_1)}{1 - \frac{c^2}{2}\sin^2(\beta_1)}$$

if $N = 2$,

$$Sol_2(\beta_1) = \frac{\cos(\beta_1)}{(1 - \frac{c^2}{2}\sin^2(\beta_1))^p} \sum_{k=0}^{p-1} \frac{1}{2(k-p)+3} C_{p-1}^k \Big(1 - \frac{c^2}{2}\Big)^{k+p-\frac{3}{2}} \tan^{2(k-p)+3}(\beta_1)$$

if $N = 2p$ and $p > 1$, and if $N = 2p+1$,

$$\begin{aligned} Sol_2(\beta_1) =& \frac{\cos(\beta_1)}{(1 - \frac{c^2}{2}\sin^2(\beta_1))^{\frac{2p+1}{2}}} \Big(1 - \frac{c^2}{2}\Big)^{p-1} \Bigg(\sqrt{1 + \Big(1 - \frac{c^2}{2}\Big)\tan^2(\beta_1)} + \sum_{k=1}^{p} \frac{C_p^k}{a_k} \\ & \Bigg(\ln\Bigg(\frac{\sqrt{1 - \frac{c^2}{2}}\tan(\beta_1)}{1 + \sqrt{1 + (1 - \frac{c^2}{2})\tan^2(\beta_1)}}\Bigg) - \sum_{q=1}^{k-1} \frac{a_{q+1}}{2q} \frac{\sqrt{1 + (1 - \frac{c^2}{2})\tan^2(\beta_1)}}{(1 - \frac{c^2}{2})^q \tan^{2q}(\beta_1)}\Bigg)\Bigg). \end{aligned}$$

In particular, we remark that

$$Sol_2(\beta_1) \underset{\beta_1 \to 0}{\sim} \frac{(1 - \frac{c^2}{2})^{p-\frac{3}{2}}}{(3-2p)\beta_1^{2p-3}}, \tag{87}$$

if $N = 2p$ and $p > 1$,

$$Sol_2(\beta_1) \underset{\beta_1 \to 0}{\sim} \ln(\beta_1) \tag{88}$$

if $N = 3$, and if $N = 2p+1$ with $p > 1$,

$$Sol_2(\beta_1) \underset{\beta_1 \to 0}{\sim} \frac{1}{(2-2p)\beta_1^{2p-2}}. \tag{89}$$

Thus, every solution v of equation (86) writes as

$$v(\beta_1) = ASol_1(\beta_1) + BSol_2(\beta_1)$$

on $]0, \pi[\setminus\{\frac{\pi}{2}\}$ in dimension $N \geq 3$, respectively $]0, \pi[\setminus\{\frac{\pi}{2}\}$ and $]\pi, 2\pi[\setminus\{\frac{3\pi}{2}\}$ in dimension two.

Actually, θ_∞ is a smooth, bounded solution of equation (86). By assertions (87), (88) and (89), the functions Sol_2 are not bounded at the point $\beta_1 = 0$ in dimension $N \geq 3$, so, there is some real constant α such that

$$\theta_\infty(\beta_1) = \alpha Sol_1(\beta_1) = \frac{\alpha\cos(\beta_1)}{(1 - \frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}}},$$

which yields formula (37) in the axisymmetric case. On the other hand, in dimension two, both solutions Sol_1 and Sol_2 are smooth and bounded on $\mathbb{S}^1$. Therefore, there are some real constants α and β such that

$$\theta_\infty(\sigma) = \alpha\frac{\cos(\beta_1)}{1 - \frac{c^2}{2} + \frac{c^2\cos(\beta_1)^2}{2}} + \beta\frac{\sin(\beta_1)}{1 - \frac{c^2}{2} + \frac{c^2\cos(\beta_1)^2}{2}},$$

which is formula (39). Moreover, in dimension two, the axisymmetric travelling waves are even functions of β_1. Thus, if the travelling wave v is axisymmetric, the function θ_∞ is an even function of β_1, which means that the constant β vanishes and which leads to equation (37) in dimension two.

Now, equation (34) yields in spherical coordinates, up to a new misuse of notations,

$$\eta_\infty(\beta_1) = -c(\sin(\beta_1)\theta'_\infty(\beta_1) + (N-1)\cos(\beta_1)\theta_\infty(\beta_1)).$$

In dimension two, it gives equation (38)

$$\eta_\infty(\beta_1) = \alpha c\left(\frac{1}{1-\frac{c^2}{2}\sin^2(\beta_1)} - \frac{2\cos^2(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^2}\right) - 2\beta c\frac{\sin(\beta_1)\cos(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^2},$$

while in the axisymmetric case, it gives formula (36)

$$\eta_\infty(\beta_1) = \alpha c\left(\frac{1}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}}} - \frac{N\cos^2(\beta_1)}{(1-\frac{c^2}{2}\sin^2(\beta_1))^{\frac{N}{2}+1}}\right).$$

This ends the proof of Proposition 7. □

3.2 Value of the stretched dipole coefficient

Finally, we link the values of the coefficients α and β to the energy $E(v)$ and the momentum $\vec{P}(v)$. The proof essentially relies on integral equations which are summed up by Lemmas 6 and 7. In Lemma 7, we state Pohozaev's identities for equation (2). They follow from the multiplication of equation (2) by the standard Pohozaev multipliers $x_j\partial_j v(x)$ and several integrations by parts. They were already derived in [8], so, we omit their proof here. On the other hand, Lemma 6 provides integral equations (40) and (41). In particular, equation (40) is very similar to the new integral relation of [8]. The main difference is that the speed c is now supposed to be subsonic, whereas it was supersonic in [8].

Proof of Lemma 6.

Step 1. *Proof of equation* (40).

The proof relies on the multiplication of equation (2) by the standard multipliers v and iv. Indeed, consider the function defined by

$$\forall R > 0, \Phi(R) = \int_{B(0,R)} \eta(x)dx.$$

the multiplication of equation (2) by the function v gives after some integrations by parts

$$\int_{B(0,R)} (|\nabla v|^2 + \eta^2) = c\int_{B(0,R)} i\partial_1 v.v + \Phi(R) + \int_{S(0,R)} \partial_\nu v.v,$$

which also writes for R sufficiently large

$$\int_{B(0,R)} (|\nabla v|^2 + \eta^2) = c\int_{B(0,R)} (i\partial_1 v.v + \partial_1(\psi\theta)) + \Phi(R) - \frac{1}{2}\int_{S(0,R)} \partial_\nu \eta - c\int_{S(0,R)} \nu_1\theta. \quad (90)$$

By Proposition 1, we infer

$$\int_{B(0,R)} (|\nabla v|^2 + \eta^2) \underset{R\to+\infty}{\to} \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2),$$

while by definition,

$$\int_{B(0,R)} (i\partial_1 v.v + \partial_1(\psi\theta)) \underset{R\to+\infty}{\to} 2p(v). \tag{91}$$

However, Proposition 2 yields

$$\left|\int_{S(0,R)} \partial_\nu \eta\right| \leq \frac{AR^{N-1}}{R^{N+1}} \underset{R\to+\infty}{\to} 0,$$

while Proposition 5 gives

$$\int_{S(0,R)} \nu_1\theta = R^{N-1}\int_{\mathbb{S}^{N-1}} \sigma_1\theta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma. \tag{92}$$

Thus, equation (90) leads to

$$\Phi(R) \underset{R\to+\infty}{\to} \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2cp(v) + c\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma. \tag{93}$$

On the other hand, we can also multiply equation (2) by the function iv to find

$$\frac{c}{2}\partial_1\eta + \text{div}(i\nabla v.v) = 0. \tag{94}$$

Now, we multiply this equation by the function x_1 and integrate by parts to obtain

$$\frac{c}{2}\Phi(R) + \int_{B(0,R)} i\partial_1 v.v = \int_{S(0,R)} \Big(\frac{c}{2}R\nu_1^2\eta + R\nu_1 i\partial_\nu v.v\Big),$$

which also writes for R sufficiently large

$$\frac{c}{2}\Phi(R) = -\int_{B(0,R)} (\partial_1(\psi\theta) + i\partial_1 v.v) + \int_{S(0,R)} \Big(\frac{c}{2}R\nu_1^2\eta + R\nu_1 i\partial_\nu v.v + \nu_1\theta\Big). \tag{95}$$

By Proposition 5, we get

$$\int_{S(0,R)} R\nu_1^2\eta = R^N\int_{\mathbb{S}^{N-1}} \sigma_1^2\eta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1^2\eta_\infty(\sigma)d\sigma.$$

We then compute

$$\int_{S(0,R)} R\nu_1 i\partial_\nu v.v = -\int_{S(0,R)} R\nu_1\rho^2\partial_\nu\theta = \int_{S(0,R)} R\nu_1\eta\partial_\nu\theta - \int_{S(0,R)} R\nu_1\sum_{k=1}^N \nu_k\partial_k\theta.$$

However, on one hand, Proposition 2 gives

$$\left|\int_{S(0,R)} R\nu_1\eta\partial_\nu\theta\right| \leq \frac{AR^N}{R^{2N}} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by Propositions 2 and 4, equation (76) and the dominated convergence theorem, we compute

$$\begin{aligned}\int_{S(0,R)} R\nu_1\sum_{k=1}^N \nu_k\partial_k\theta = \int_{\mathbb{S}^{N-1}} R^N\sigma_1\sum_{k=1}^N \sigma_k\partial_k\theta(R\sigma)d\sigma &\underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1\sum_{k=1}^N \sigma_k\theta_\infty^k(\sigma)d\sigma \\ &= -(N-1)\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma.\end{aligned}$$

Thus, it follows from equations (91), (92) and (95) that

$$\Phi(R) \underset{R\to+\infty}{\to} -\frac{4}{c}p(v) + \int_{\mathbb{S}^{N-1}} \sigma_1^2\eta_\infty(\sigma)d\sigma + \frac{2N}{c}\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma.$$

By equation (93) and by uniqueness of the limit of the function Φ in $+\infty$, we finally find

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2cp(v) + c\int_{\mathbb{S}^{N-1}} \sigma_1\theta_\infty(\sigma)d\sigma = -\frac{4}{c}p(v) + \int_{\mathbb{S}^{N-1}} (\sigma_1^2\eta_\infty(\sigma) + \frac{2N}{c}\sigma_1\theta_\infty(\sigma))d\sigma,$$

which yields immediately equation (40).

Step 2. *Proof of equation* (41)

The proof relies once more on equation (94) just above. Here, we multiply it by the function x_j for any $j \in \{2, \ldots, N\}$ and integrate by parts on the ball $B(0,R)$ to obtain

$$\int_{B(0,R)} i\partial_j v.v = \int_{S(0,R)} \Big(\frac{c}{2}R\nu_1\nu_j\eta + R\nu_j i\partial_\nu v.v\Big),$$

which also writes for R sufficiently large

$$\int_{B(0,R)} (\partial_j(\psi\theta) + i\partial_j v.v) = \int_{S(0,R)} \Big(\frac{c}{2}R\nu_j\nu_1\eta + R\nu_j i\partial_\nu v.v + \nu_j\theta\Big). \tag{96}$$

By Proposition 5,

$$\int_{S(0,R)} R\nu_1\nu_j\eta = R^N\int_{\mathbb{S}^{N-1}} \sigma_1\sigma_j\eta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_1\sigma_j\eta_\infty(\sigma)d\sigma,$$

and,

$$\int_{S(0,R)} \nu_j\theta = R^{N-1}\int_{\mathbb{S}^{N-1}} \sigma_j\theta(R\sigma)d\sigma \underset{R\to+\infty}{\to} \int_{\mathbb{S}^{N-1}} \sigma_j\theta_\infty(\sigma)d\sigma.$$

Likewise, we compute

$$\int_{S(0,R)} R\nu_j i\partial_\nu v.v = -\int_{S(0,R)} R\nu_j\rho^2\partial_\nu\theta = \int_{S(0,R)} R\nu_j\eta\partial_\nu\theta - \int_{S(0,R)} R\nu_j\sum_{k=1}^N \nu_k\partial_k\theta.$$

However, on one hand, Proposition 2 gives

$$\left|\int_{S(0,R)} R\nu_j\eta\partial_\nu\theta\right| \leq \frac{AR^N}{R^{2N}} \underset{R\to+\infty}{\to} 0.$$

On the other hand, by Propositions 2 and 4, equation (76) and the dominated convergence theorem, we get

$$\begin{aligned}\int_{S(0,R)} R\nu_j\sum_{k=1}^N \nu_k\partial_k\theta = \int_{\mathbb{S}^{N-1}} R^N\sigma_j\sum_{k=1}^N \sigma_k\partial_k\theta(R\sigma)d\sigma \underset{R\to+\infty}{\to} & \int_{\mathbb{S}^{N-1}} \sigma_j\sum_{k=1}^N \sigma_k\theta_\infty^k(\sigma)d\sigma \\ & = -(N-1)\int_{\mathbb{S}^{N-1}} \sigma_j\theta_\infty(\sigma)d\sigma.\end{aligned}$$

Thus, it follows from the definition of the momentum and from equation (96) that

$$2\vec{P}_j(v) = \frac{c}{2}\int_{\mathbb{S}^{N-1}} \sigma_1\sigma_j\eta_\infty(\sigma)d\sigma + N\int_{\mathbb{S}^{N-1}} \sigma_j\theta_\infty(\sigma)d\sigma,$$

which is equation (41). □

Now, we state the proof of Theorem 2.

Proof of Theorem 2. By Proposition 7, we already know

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \theta_\infty(\sigma) = \frac{\alpha \sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}.$$

Thus, it only remains to deduce the value of the stretched dipole coefficient α from formula (40). Indeed, by Proposition 7, formula (40) writes

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2c\Big(1 - \frac{2}{c^2}\Big)p(v) = \alpha c\Bigg(\left(\frac{2N}{c^2} - 1\right)\int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} d\sigma + \int_{\mathbb{S}^{N-1}} \left(\frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N\sigma_1^4}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}}\right) d\sigma\Bigg).$$

Denoting

$$J_1 = \int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2c(1 - \frac{2}{c^2})p(v),$$

and

$$J_2 = \frac{2N}{c}\int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} d\sigma - Nc\int_{\mathbb{S}^{N-1}} \frac{\sigma_1^4}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}} d\sigma,$$

it also writes

$$J_1 = \alpha J_2. \tag{97}$$

Now, we express J_1 in function of the energy $E(v)$ and the momentum $p(v)$. Indeed, Lemma 7 yields

$$\int_{\mathbb{R}^N} |\partial_1 v|^2 = E(v),$$

and

$$\int_{\mathbb{R}^N} |\nabla_\perp v|^2 = (N-1)(E(v) - cp(v)),$$

where $\nabla_\perp v$ is defined by

$$\nabla_\perp v = (\partial_2 v, \ldots, \partial_N v).$$

However, by definition,

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\partial_1 v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \eta^2,$$

so,

$$\int_{\mathbb{R}^N} \eta^2 = 2(N-1)cp(v) - 2(N-2)E(v).$$

Thus, we conclude that

$$J_1 = (4-N)E(v) + ((N-3)c + \frac{4}{c})p(v). \tag{98}$$

On the other hand, we can explicitly compute the value of J_2 in function of c and N. Indeed, we have

$$J_2 = \frac{2N}{c}\left(1 - \frac{c^2}{2}\right)\int_{\mathbb{S}^{N-1}} \frac{\sigma_1^2}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}} d\sigma. \tag{99}$$

Therefore, we are reduced to estimate the integral defined by

$$I(N,c)=\int_{\mathbb{S}^{N-1}}\frac{\sigma_1^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}+1}}d\sigma. \tag{100}$$

In dimension two, we use the polar coordinates to compute such an integral:

$$\begin{aligned}I(N,c)=\int_0^{2\pi}\frac{\cos^2(\beta)}{(1-\frac{c^2}{2}\sin^2(\beta))^2}d\beta&=4\int_0^{+\infty}\frac{dt}{(1+(1-\frac{c^2}{2})t^2)^2}\\&=\frac{4}{\sqrt{1-\frac{c^2}{2}}}\int_0^{+\infty}\frac{du}{(1+u^2)^2}\\&=\frac{\pi}{\sqrt{1-\frac{c^2}{2}}},\end{aligned}$$

where we made the successive changes of variables $t=\tan(\beta)$ and $u=\sqrt{1-\frac{c^2}{2}}t$.

In dimension $N\geq 3$, we use the spherical coordinates:

$$I(N,c)=|\mathbb{S}^{N-2}|\int_0^{\pi}\frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1-\frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}}d\beta. \tag{101}$$

At this stage, the computations are different according to the parity of the dimension N. Assuming first that $N=2p+2$ is even, we find

$$|\mathbb{S}^{2p}|=\frac{2^{2p+1}\pi^p p!}{(2p)!},$$

and

$$\begin{aligned}\int_0^{\pi}\frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1-\frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}}d\beta&=2\int_0^{+\infty}\frac{t^{2p}}{(1+(1-\frac{c^2}{2})t^2)^{2+p}}dt\\&=\frac{2}{(1-\frac{c^2}{2})^{p+\frac{1}{2}}}\int_0^{+\infty}\frac{u^{2p}}{(1+u^2)^{2+p}}du\\&=\frac{2}{(1-\frac{c^2}{2})^{p+\frac{1}{2}}}\int_0^{+\infty}\frac{\mathrm{th}(s)^{2p}}{\mathrm{ch}(s)^3}ds,\end{aligned}$$

where we made the changes of variables $t=\tan(\beta)$, $u=\sqrt{1-\frac{c^2}{2}}t$ and $u=\mathrm{sh}(s)$. Then, consider

$$\forall p\in\mathbb{N}, I_p=\int_0^{+\infty}\frac{\mathrm{th}(s)^{2p}}{\mathrm{ch}(s)}ds.$$

An integration by parts gives

$$I_p-I_{p+1}=\int_0^{+\infty}\frac{\mathrm{th}(s)^{2p}}{\mathrm{ch}(s)^3}ds=\frac{I_{p+1}}{2p+1}.$$

Since $I_0=\frac{\pi}{2}$, the value of I_p is

$$I_p=\frac{(2p)!\pi}{2^{2p+1}(p!)^2},$$

and finally,

$$\int_0^{+\infty} \frac{\mathrm{th}(s)^{2p}}{\mathrm{ch}(s)^3} ds = \frac{(2(p+1))!\pi}{2^{2p+3}((p+1)!)^2(2p+1)}.$$

Thus, equation (101) writes

$$I(2p+2,c) = \frac{\pi^{p+1}}{(1-\frac{c^2}{2})^{p+\frac{1}{2}}(p+1)!}. \tag{102}$$

In particular, formula (102) remains valid when $p=0$.

On the other hand, assuming that $N = 2p+3$ is odd, we compute

$$|\mathbb{S}^{2p+1}| = \frac{2\pi^{p+1}}{p!},$$

and

$$\begin{aligned}\int_0^{\pi} \frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1-\frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}} d\beta &= 2\int_0^1 \frac{u^2(1-u^2)^p}{(1+\frac{c^2}{2}(u^2-1))^{p+\frac{5}{2}}} du \\ &= \frac{4\sqrt{2}}{c^{2p+3}(1-\frac{c^2}{2})^{p+1}} \int_0^{\frac{c}{\sqrt{2-c^2}}} \frac{v^2(c^2(1+v^2)-2v^2)^p}{(1+v^2)^{p+\frac{5}{2}}} dv \\ &= \frac{4\sqrt{2}}{c^{2p+3}(1-\frac{c^2}{2})^{p+1}} \int_0^{\frac{c}{\sqrt{2}}} (c^2-2w^2)^p w^2 dw \\ &= \frac{2}{(1-\frac{c^2}{2})^{p+1}} \int_0^{\frac{\pi}{2}} (\sin^{2p+1}(\theta) - \sin^{2p+3}(\theta)) d\theta,\end{aligned}$$

where we successively made the changes of variables $u = \cos(\beta)$, $v = \frac{cu}{\sqrt{2-c^2}}$, $w = \frac{v}{\sqrt{1+v^2}}$ and $w = \frac{c}{\sqrt{2}}\cos(\theta)$. Now, Wallis' formulae yield

$$\int_0^{\frac{\pi}{2}} (\sin^{2p+1}(\theta) - \sin^{2p+3}(\theta)) d\theta = \frac{4^p(p!)^2}{(2p+1)!(2p+3)},$$

which gives

$$\int_0^{\pi} \frac{\cos^2(\beta)\sin^{N-2}(\beta)}{(1-\frac{c^2}{2}\sin^2(\beta))^{\frac{N}{2}+1}} d\beta = \frac{2^{2p+1}(p!)^2}{(1-\frac{c^2}{2})^{p+1}(2p+1)!(2p+3)},$$

and finally, by equation (101),

$$I(2p+3,c) = \frac{(4\pi)^{p+1}p!}{(1-\frac{c^2}{2})^{p+1}(2p+1)!(2p+3)}. \tag{103}$$

In conclusion, if $N = 2p+2$, we have by equations (97), (98), (99), (100) and (102),

$$\alpha = \frac{(1-\frac{c^2}{2})^{p-\frac{1}{2}}p!}{2\pi^{p+1}} \Big((1-p)cE(v) + (2 + \frac{2p-1}{2}c^2)p(v) \Big),$$

and if $N = 2p+3$, by equations (97), (98), (99), (100) and (103),

$$\alpha = \frac{(1-\frac{c^2}{2})^p(2p+1)!}{(4\pi)^{p+1}p!} \Big(\frac{1-2p}{2}cE(v) + (2+pc^2)p(v) \Big).$$

It yields immediately equation (12) by using the definition of the function Γ, and completes the proof of Theorem 2. $\square$

By the same arguments as in the proof of Theorem 2, we complete the proof of Theorem 3.

Proof of Theorem 3. By Proposition 7, we already know that

$$\forall \sigma \in \mathbb{S}^1, v_\infty(\sigma) = \theta_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}}.$$

Thus, it only remains to deduce the values of the coefficients α and β from equations (40) and (41). Indeed, by Proposition 7, formula (40) writes in dimension two

$$\int_{\mathbb{R}^2} (|\nabla v|^2 + \eta^2) - 2c\Big(1 - \frac{2}{c^2}\Big)p(v) = \alpha \Big(\frac{4}{c} \int_{\mathbb{S}^1} \frac{\sigma_1^2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} d\sigma - 2c \int_{\mathbb{S}^1} \frac{\sigma_1^4}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} d\sigma \Big).$$

Actually, we remark that we recover formula (97) in dimension two. Therefore, the value of α is exactly the same as in the proof of Theorem 2, i.e.

$$\alpha = \frac{1}{2\pi\sqrt{1 - \frac{c^2}{2}}} \Big(cE(v) + \big(2 - \frac{c^2}{2}\big)p(v) \Big).$$

Likewise, by Proposition 7, formula (41) writes in dimension two

$$P_2(v) = \frac{\beta}{2} \Big(2 \int_{\mathbb{S}^1} \frac{\sigma_2^2}{1 - \frac{c^2\sigma_2^2}{2}} d\sigma - c^2 \int_{\mathbb{S}^1} \frac{\sigma_1^2\sigma_2^2}{(1 - \frac{c^2\sigma_2^2}{2})^2} d\sigma \Big). \tag{104}$$

Denoting

$$J_3 := 2 \int_{\mathbb{S}^1} \frac{\sigma_2^2}{1 - \frac{c^2\sigma_2^2}{2}} d\sigma - c^2 \int_{\mathbb{S}^1} \frac{\sigma_1^2\sigma_2^2}{(1 - \frac{c^2\sigma_2^2}{2})^2} d\sigma,$$

we compute

$$\begin{aligned} J_3 = (2 - c^2) \int_{\mathbb{S}^1} \frac{\sigma_2^2}{(1 - \frac{c^2\sigma_2^2}{2})^2} d\sigma &= 4(2 - c^2) \int_0^{\frac{\pi}{2}} \frac{\sin^2(t)}{(1 - \frac{c^2 \sin^2(t)}{2})^2} dt \\ &= \frac{8}{\sqrt{1 - \frac{c^2}{2}}} \int_0^{+\infty} \frac{u^2}{(1 + u^2)^2} du \\ &= \frac{8}{\sqrt{1 - \frac{c^2}{2}}} \int_0^{+\infty} \frac{\operatorname{sh}^2(v)}{\operatorname{ch}^3(v)} dv \\ &= \frac{2\pi}{\sqrt{1 - \frac{c^2}{2}}}, \end{aligned}$$

where we successively made the changes of variables $u = \sqrt{1 - \frac{c^2}{2}} \tan(t)$ and $u = \operatorname{sh}(v)$. Then, the computation of J_3 yields by equation (104)

$$\beta = \frac{\sqrt{1 - \frac{c^2}{2}}}{\pi} P_2(v),$$

which concludes the proof of Theorem 3. □

Finally, we conclude the paper by the proof of Corollary 1, which is an immediate consequence of Theorem 2 and Lemma 7.

Proof of Corollary 1. By equations (97), (99), (100), (102) and (103), there is some real number $A_{c,N} > 0$ such that

$$\alpha = A_{c,N} \left(\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) - 2cp(v) + \frac{4}{c^2} p(v) \right) = A_{c,N} J_1. \tag{105}$$

However, Lemma 7 gives on one hand

$$E(v) = \int_{\mathbb{R}^N} |\partial_1 v|^2.$$

On the other hand, by definition,

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\partial_1 v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{4} \int_{\mathbb{R}^N} \eta^2,$$

so,

$$E(v) = \int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \eta^2.$$

Thus, we compute

$$J_1 = 2(E(v) - cp(v)) + \frac{4}{c^2} p(v) + \frac{1}{2} \int_{\mathbb{R}^N} \eta^2. \tag{106}$$

Moreover, Lemma 7 once more yields

$$E(v) - cp(v) = \frac{1}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 \geq 0,$$

and likewise,

$$cp(v) = E(v) - \frac{1}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 = \frac{N-2}{N-1} \int_{\mathbb{R}^N} |\nabla_\perp v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} \eta^2 \geq 0.$$

Therefore, J_1 is the sum of three non negative terms.

Now assume that α is equal to 0. $A_{c,N}$ being strictly positive, J_1 is equal to 0. By formula (106), it follows that

$$E(v) - cp(v) = p(v) = \int_{\mathbb{R}^N} \eta^2 = 0,$$

so the energy $E(v)$ vanishes, and the travelling wave v is a complex constant of modulus one.

Reciprocally, if v is constant, the energy $E(v)$ and the momentum $p(v)$ vanish and α is equal to 0 by equation (105), which ends the proof of Corollary 1. □

Remark 15. By the proof of Corollary 1, the stretched dipole coefficient α is always non negative.

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