

Co-rotating vortices with N fold symmetry for the inviscid surface quasi-geostrophic equation

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Abstract

We provide a variational construction of special solutions to the generalized surface quasi-geostrophic equations. These solutions take the form of N vortex patches with N -fold symmetry, which are steady in a uniformly rotating frame. Moreover, we investigate their asymptotic properties when the size of the corresponding patches vanishes. In this limit, we prove these solutions to be a desingularization of N Dirac masses with the same intensity, located on the N vertices of a regular polygon rotating at a constant angular velocity.

1 Introduction

In this paper, we consider the generalized surface quasi-geostrophic equation

$$\begin{cases} \partial_t \vartheta + v \cdot \nabla \vartheta = 0, \\ v = \nabla^\perp \Delta^{-s} \vartheta, \end{cases} \quad (\text{gSQG})$$

for $\vartheta : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$, where $0 < s < 1$ is a fixed parameter. We use here the notation Δ^{-s} as a shortcut for $-(-\Delta)^{-s}$.

The surface quasi-geostrophic equation corresponds to the case $s = \frac{1}{2}$, it is used as a model in the context of geophysical fluid mechanics with ϑ being the potential temperature in a rapidly rotating stratified fluid with uniform potential vorticity and subject to Brunt-Väisälä oscillations (see [13] and the references therein).

Formally at least, the family of nonlinear transport equations (gSQG) interpolates between the stationary equation $\partial_t \vartheta = 0$ when $s \rightarrow 0$, and the two-dimensional Euler equation for incompressible fluids in the limit $s \rightarrow 1$. In that last case, the vector field v is the flow velocity and the function ϑ is the flow vorticity, usually written as ω in that context.

From a mathematical point of view, the surface quasi-geostrophic equation has attracted interest in particular because, while being two dimensional and therefore simpler to analyze, it possesses some scaling analogy with the three-dimensional Euler equation (see [7] and the references therein). Local well-posedness of classical solutions was established in [7], but solutions with arbitrary Sobolev growth were constructed in [15] (in a periodic setting). To our knowledge, establishing global well-posedness of classical solutions, or alternatively describing their finite time singularities, remains an open problem. Note also that the global existence of weak solutions in $L^2(\mathbb{R}^2)$ was shown in [19], but their non-uniqueness below a certain regularity threshold was highlighted in [4].

As for the two-dimensional Euler equation, all the radially symmetric functions ϑ are stationary solutions to (gSQG). Exhibiting other global smooth solutions seems a challenging issue. A first example was recently provided in [6] by developing a bifurcation argument from a specific radially

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symmetric function. The variational construction of an alternative example in the form of a smooth traveling-wave solution was completed in [11, 9]. These latter results are the starting point of the following analysis.

It is well-known that the two-dimensional Euler equation exhibits point vortex solutions (see [17] and the references therein), which take the form

$$\vartheta(x, t) = \sum_{n=1}^N a_n \delta_{x=x_n(t)}.$$

In this formula, the notation ϑ stands for the vorticity of the ideal fluid under consideration, and a_n and $x_n(t)$ denote the intensity and the position at time t of the n -th vortex. When the number N of vortices is equal to 2, two special configurations are of interest. The first one corresponds to the case $a_1 + a_2 = 0$ for which the two vortices translate at a constant common velocity. When $a_1 - a_2 = 0$ instead, the two vortices rotate around each other at a constant angular velocity, while their midpoint remains fixed. This latter configuration generalizes to an arbitrary choice of the number $N \geq 3$, for which N vortices, with the same intensity and located on the N vertices of a regular polygon, co-rotate at a constant angular velocity.

Point vortices are singular weak solutions to the Euler equation. On finite time intervals, they can be approximated through regularization by classical solutions (see e.g. [17]). An interesting issue is to ask for the existence of regularized solutions which approximate the point vortices for all time. This desingularization issue was answered positively for numerous vortex configurations of the Euler equation (see e.g. [21] and the references therein). In particular, the construction of smooth traveling-wave solutions corresponding to a vortex pair in translation was achieved in [18] (see also [3, 5]). In parallel, less regular approximations called vortex patches were constructed both for the vortex pair in translation [22, 23] and for the ones in rotation [24].

The notion of point vortex solutions easily extends to the context of the generalized surface quasi-geostrophic equations. This was the starting point of the constructions in [11, 9], which exhibited a smooth traveling-wave solution to (gSQG) corresponding to a vortex pair in translation. In this direction, a natural question is to ask for a similar construction in the case of a vortex pair in rotation. For $1/2 < s < 1$, this question was answered positively in [14], where a pair of vortex patches was shown to exist by a perturbative argument. In [12], solutions with N -fold symmetry were constructed for $1/2 < s < 1$ (and also for the Euler equation). They are perturbations of one radially symmetric vortex patch and therefore are of different nature than desingularized point vortex solutions. While completing this work, smooth desingularized point vortex solutions of the form of translating vortex pairs or co-rotating vortices were also constructed using a perturbative method in [1]. An approximation result on finite time intervals for arbitrary solutions of the SQG point vortex system by weak solutions of SQG was also recently proved in [8]. In a different direction, beautiful rigidity results for uniformly-rotating solutions of the Euler and SQG equations are described in [10].

Our main goal in this paper, in the spirit of [24] for the Euler equation, is to provide a variational construction of families, for arbitrary $0 < s < 1$, of N co-rotating vortex patches with N -fold symmetry. Additionally, we prove these patches to be a desingularization of N point vortices in rotation.

Before stating our main result, we first explicit the equation satisfied by a solution to (gSQG), which remains steady in a uniformly rotating frame. We introduce an angular velocity α and we look for solutions under the form

$$\vartheta(x, t) = R_{\alpha t} \omega(x) := \omega(R_{-\alpha t} x), \tag{1}$$

where R_ϕ stands for the counterclockwise rotation of angle ϕ . Here, we make the choice to use the notation ω for the profile of the solution in the rotating frame in reference to the vorticity formulation of the two-dimensional Euler equation. In view of the equation for the velocity v in (gSQG), we next set

$$u = -\nabla^\perp (k_s \star \omega). \tag{2}$$

In this identity, the function k_s is the fundamental solution of the operator $(-\Delta)^s$ on \mathbb{R}^2 . This solution is explicitly given by the formula

$$k_s(x) = \frac{c_s}{|x|^{2(1-s)}}, \quad \text{with } c_s := \frac{\Gamma(1-s)}{2^{2s}\pi\Gamma(s)}, \quad (3)$$

it is often called the Riesz potential of order $2s$, and denoted also as I_{2s} .

Combining (1) and (2) yields

$$v(x, t) = R_{\alpha t} u(R_{-\alpha t} x),$$

so that the first equation in (gSQG) reduces to

$$\alpha x^\perp \cdot \nabla \omega(x) + \nabla^\perp(k_s \star \omega)(x) \cdot \nabla \omega(x) = 0.$$

Hence, a solution to (gSQG), which is steady in a uniformly rotating frame, satisfies the stationary equation

$$\nabla \omega(x) \cdot \nabla^\perp \left(k_s \star \omega(x) + \frac{\alpha}{2} |x|^2 \right) = 0. \quad (4)$$

A possible weak formulation for (4) is that ω satisfies

$$\int_{\mathbb{R}^2} \omega(x) \nabla \varphi(x) \cdot \nabla^\perp \left(k_s \star \omega(x) + \frac{\alpha}{2} |x|^2 \right) dx = 0, \quad (5)$$

for any function $\varphi \in C_c^\infty(\mathbb{R}^2)$. The vortex patch solutions which we will construct own a priori no more regularity than being in the Lebesgue spaces $L^p(\mathbb{R}^2)$ for any $1 \leq p \leq \infty$. Since they are also of compact support, due to standard elliptic theory for Riesz potentials the corresponding functions $k_s \star \omega$ belong to the Sobolev spaces $\dot{W}^{2s,p}(\mathbb{R}^2)$ for any $1 < p < \infty$. For $s < 1/2$, this regularity is not yet sufficient to provide a rigorous meaning to the weak formulation in (5). For that reason, we will rely on a different and restricted notion of weak solutions. More precisely, in view of (4) we shall say that if $\omega \in L^1(\mathbb{R}^2)$ has compact support and satisfies

$$\omega = f(k_s \star \omega + \frac{\alpha}{2} |x|^2),$$

for some Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, then it is a weak solution of (4). We will be particularly interested in the case where f is a step function, and in such cases we will show that these weak solutions are also weak solutions in the sense of (5) when $s \geq \frac{1}{2}$.

A vortex patch ω writes as

$$\omega = \lambda \mathbb{1}_\Omega.$$

In that identity, the notation $\mathbb{1}_\Omega$ refers to the characteristic function of the support Ω of the patch. The positive number λ is its intensity. Since we aim at desingularizing N point vortices with equal intensity, it is natural to impose that the N patches own the same intensity λ . Since the point vortices are located on the N vertices of a regular polygon, we also impose the symmetry condition

$$\omega = R_{\frac{2\pi}{N}} \omega. \quad (6)$$

In the sequel, this symmetry property is used in order to restrict the construction to only one vortex patch inside the angular sector

$$\mathfrak{S}_N := \left\{ (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 : -\frac{\pi}{N} < \theta < \frac{\pi}{N} \right\}. \quad (7)$$

Moreover, we also impose the vortex patch to be angular Steiner symmetric with respect to the angle $\theta = 0$. Recall that this condition is defined as

$$\omega(r, \theta) = \omega^\sharp(r, \theta), \quad (8)$$

where the notation (r, θ) , with $r > 0$ and $-\pi/N < \theta < \pi/N$, stands for the usual polar coordinates. In this definition, the angular Steiner symmetrization ω^\sharp of the function ω for the variable θ is the unique even function such that

$$\omega^\sharp(r, \theta) > \nu \quad \text{if and only if} \quad |\theta| < \frac{1}{2} \text{meas} \left\{ \theta' \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) : \omega(r, \theta') > \nu \right\},$$

for any positive numbers r and ν , and any $-\pi/N < \theta < \pi/N$. It turns out that the symmetry condition (8) provides a number of simplifications in our variational construction of N vortex patches (see Appendix A for some useful properties of the angular Steiner symmetrization). In the sequel, we denote by $L_{\text{sym}}^p(\mathbb{R}^2)$ the subset of functions in $L^p(\mathbb{R}^2)$, which satisfy both (6) and (8).

We are now in position to state our main result.

Theorem 1. *Let $0 < s < 1$ be given, and let $\lambda > 0$ be sufficiently large, depending only on s . There exist an angular velocity α_λ and a function $\omega_\lambda \in L_{\text{sym}}^\infty(\mathbb{R}^2)$, which satisfy the following properties.*

(i) *The vorticity ω_λ is a co-rotating vortex patch solution to (gSQG) with intensity λ , in the sense that there exists $\mu_\lambda \in \mathbb{R}$ such that*

$$\omega_\lambda = \lambda \mathbb{1}_{\{\psi_\lambda > 0\}}, \quad (9)$$

with

$$\psi_\lambda(x) := \frac{\Gamma(1-s)}{2^{2s}\pi\Gamma(s)} \int_{\mathbb{R}^2} \frac{\omega_\lambda(x')}{|x-x'|^{2(1-s)}} dx' + \frac{\alpha_\lambda}{2}|x|^2 - \mu_\lambda. \quad (10)$$

(ii) *If $1/2 \leq s < 1$, then ω_λ is also a weak solution to (4) in the sense of (5), where $\alpha \equiv \alpha_\lambda$.*

(iii) *Let $\Omega_\lambda := \{\psi_\lambda > 0\}$ be the support of ω_λ . There exists a positive number C , independent of λ , such that*

$$\Omega_\lambda \cap \mathfrak{S}_N \subseteq B\left((1,0), \frac{C}{\sqrt{\lambda}}\right).$$

In particular, the support Ω_λ has at least N connected components when λ is sufficiently large.

(iv) *In the limit $\lambda \rightarrow \infty$,*

$$\omega_\lambda \rightarrow \sum_{n=0}^{N-1} \delta_{R_{\frac{2n\pi}{N}}(1,0)}, \quad (11)$$

weakly in the sense of measures, and if $\frac{1}{2} \leq s < 1$ then the angular speeds satisfy

$$\alpha_\lambda \rightarrow \sum_{n=1}^{N-1} \frac{c_s(1-s)}{|(1,0) - R_{\frac{2n\pi}{N}}(1,0)|^{2(1-s)}}. \quad (12)$$

(v) *Concerning μ_λ , we have the asymptotic bounds*

$$0 < \liminf_{\lambda \rightarrow +\infty} \frac{\mu_\lambda}{\lambda^{1-s}} \leq \limsup_{\lambda \rightarrow +\infty} \frac{\mu_\lambda}{\lambda^{1-s}} < +\infty. \quad (13)$$

In common terms, ω_λ is therefore a desingularized solution of the (gSQG) equation with N patches rotating at a constant angular velocity α_λ .

Note that taking the limit $s \rightarrow 1^-$ of the expression on the right-hand-side of (12) in the particular case $N = 2$, yields the constant $\frac{1}{4\pi}$. That does correspond exactly to the speed of rotation of a pair of point vortices on the unit circle evolving according to the Euler equations.

We mention that in the case of the Euler equations the divergence speed of the parameter μ_λ with respect to λ is logarithmic (see [24]). In that case the full limit is known to exist for the quotient in (13).

We also do not know whether the support of each patch of ω_λ is convex or connected, even for λ large. For values of s smaller than one half this seems particularly challenging, since the weak formulation (5) a priori makes no sense.

In the next section we present the outline of the proof of Theorem 1. More precisely, we will describe the variational strategy which is followed, involving penalized regularized problems, and describe the various uniform and localization estimates which eventually allow us to pass to the limit and complete the proof of Theorem 1. The intermediate steps are all stated within that section, but the proofs are postponed to Section 3. Finally, Appendix A contains a number of properties relating the Steiner symmetrization and functionals which appear at some point in our analysis.

2 Outline of the proof of Theorem 1

In this section we describe in detail our strategy to prove Theorem 1, and we state all the intermediate results. The proofs of the later are postponed to Section 3.

2.1 Construction of the co-rotating vortex patches

The Hamiltonian nature of the Euler equations (see e.g. Arnold [2]) has long been a fruitful source for the construction of steady solutions. We follow a similar strategy in the context of the (gSQG) equation, and for that purpose we consider the energy of the fluid

$$E_s(\omega) := \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k_s(x - x') \omega(x) \omega(x') dx dx', \quad (14)$$

and its impulse

$$L(\omega) := \int_{\mathbb{R}^2} |x|^2 \omega(x) dx. \quad (15)$$

Both quantities are conserved for smooth solutions of equation (gSQG), and at least formally equation (4) arises as the Euler-Lagrange equation associated to these two functionals for some Lagrange multiplier α .

Similar to the Euler equation, equation (gSQG) is also a transport equation. In particular, any functional or constraint which is invariant by rearrangement is formally preserved by the flow. This applies to the total circulation of the fluid

$$M(\omega) := \int_{\mathbb{R}^2} \omega(x) dx, \quad (16)$$

and also to upper and lower bounds on the scalar function ω , constraints that we use in the sequel.

2.1.1 A constrained maximization problem with compact setting

There are a number of obstacles to build a meaningful and effective variational problem associated to the functionals E_s , L and M .

First, the energy functional E_s is unbounded under L^1 -type constraints such as those given by L or M . To overcome this, for $\lambda > 0$ we consider the additional L^∞ -type constraints set

$$L_\lambda^\infty(\mathbb{R}^2) := \left\{ \omega \in L^\infty(\mathbb{R}^2) : 0 \leq \omega \leq \lambda \text{ a.e.} \right\}.$$

Next, the physical domain is infinite and the functional spaces on which E_s , L and M are naturally defined are different. Following Turkington [24] in his study of vortex patches for the Euler equation, we consider the section of annulus

$$S := \left\{ (r \cos(\theta), r \sin(\theta)) : \frac{1}{2} \leq r \leq 2 \text{ and } -\frac{\pi}{2N} \leq \theta \leq \frac{\pi}{2N} \right\},$$

and further restrict our study to the function set

$$X_\lambda := \left\{ \omega \in L_\lambda^\infty(\mathbb{R}^2) \text{ s.t. } \omega \circ R_{\frac{2\pi}{N}} = \omega \text{ and } \omega = 0 \text{ a.e. on } \mathfrak{S}_N \setminus S \right\}. \quad (17)$$

The symmetry assumption is natural since it reflects the ones of the functionals, but the imposed vanishing of ω in $\mathfrak{S}_N \setminus S$ is evidently not a natural constraint, it makes the functional setting easier and provides some form of compactness to the problem, but eventually we will need to show that it is not activated. The introduction of S is motivated by the fact that, in the limit $\lambda \rightarrow \infty$ and in view of the symmetry assumption, the vortex patches which we will construct are expected to be supported into N small balls centered at the N vertices $x_n = (\cos(2n\pi/N), \sin(2n\pi/N))$ of a regular polygon.

Observe that for functions $\omega \in X_\lambda$ the energy E_s may be rewritten as

$$E_s(\omega) = \frac{N}{2} \int_S \int_S \kappa_s(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') r' dr' d\theta' r dr d\theta, \quad (18)$$

due to the symmetry and support properties of the function ω , and where the kernel κ_s is equal to

$$\kappa_s(r, r', \xi) := \sum_{n=0}^{N-1} \frac{c_s}{\left(r^2 + r'^2 - 2rr' \cos\left(\xi - \frac{2n\pi}{N}\right)\right)^{1-s}}. \quad (19)$$

For further use, let us also define the function

$$K_s \omega(r, \theta) := \int_S \kappa_s(r, r', \theta - \theta') \omega(r', \theta') r' dr' d\theta'.$$

We consider the maximization problem under constraint

$$\mathcal{E}_\lambda := \sup \left\{ E_s(\omega) : \omega \in X_\lambda \text{ s.t. } M(\omega) = L(\omega) = N \right\}. \quad (\mathcal{P}_\lambda)$$

It is straightforward to check that the constraint set is non empty when λ is chosen large enough. Let $\varepsilon \equiv \varepsilon(\lambda)$ be the length-scale defined by the identity

$$\lambda \pi \varepsilon^2 = 1. \quad (20)$$

There exists (a unique) $1 - \varepsilon \leq r_\varepsilon \leq 1 + \varepsilon$ such that the function ϖ_λ given by

$$\varpi_\lambda := \sum_{n=0}^{N-1} \lambda \mathbb{1}_{\mathcal{B}\left(R_{\frac{2n\pi}{N}}(r_\varepsilon, 0), \varepsilon\right)}, \quad (21)$$

satisfies the constraints $M(\varpi_\lambda) = L(\varpi_\lambda) = N$. It is clear also that this function belongs to X_λ for λ large enough, that is for ε small enough.

Invoking the boundedness of the set S and the Sobolev embedding theorem of $L^\infty(S)$ into $H^{-s}(S)$, it is also rather straightforward to conclude that the maximization problem (\mathcal{P}_λ) possesses a solution for all λ large enough. We will prove

Proposition 1. *Let $0 < s < 1$ be given, then for λ sufficiently large there exists an angular Steiner symmetric function $\omega_\lambda \in X_\lambda$ such that*

$$E_s(\omega_\lambda) = \mathcal{E}_\lambda.$$

Moreover, there exist two numbers α_λ and μ_λ such that

$$\omega_\lambda = \lambda \mathbb{1}_{\{\psi_\lambda > 0\}}, \quad (22)$$

where we have set

$$\psi_\lambda := K_s \omega_\lambda + \frac{\alpha_\lambda}{2} |\cdot|^2 - \mu_\lambda. \quad (23)$$

When $1/2 \leq s < 1$, the function ψ_λ belongs to $W^{1,q}(S)$ for any real number $q > 1$, and it satisfies

$$\int \omega_\lambda(x) \nabla^\perp \psi_\lambda(x) \cdot \nabla \varphi(x) dx = 0, \quad (24)$$

for any function $\varphi \in \mathcal{C}_c^\infty(S)$.

Note that at this level the weak formulation (24) is only known to be valid for test functions whose support is contained in S . This is related to the aforementioned vanishing constraint of ω in $\mathfrak{S}_N \setminus S$, and will be dealt with in Section 2.2 below.

2.1.2 The penalized problems

Our proof of Proposition 1 does not rely on a direct variational argument either. Because of the (eventually activated) L^∞ -constraint on ω , it seems indeed difficult to derive (22) and (24) in that way. Instead, we introduce the penalized functionals

$$E_{\lambda,p}(\omega) := E_s(\omega) - \frac{\lambda N}{p} \int_S \left(\frac{\omega}{\lambda} \right)^p, \quad (25)$$

for any number $p > 1/s$, as well as the corresponding maximization problems

$$\mathcal{E}_{\lambda,p} := \sup \left\{ E_{\lambda,p}(\omega) : \omega \in L^p_+(S) \text{ s.t. } M(\omega) = L(\omega) = 1 \right\}, \quad (26)$$

where the functionals M and L are defined here as in (16) and (15) but with integration domain restricted to S . Due to the boundedness of the set S , we observe that

$$\sup_{(r,\theta) \in S} \int_S \kappa_s(r, r', \theta - \theta')^q dr' d\theta' < \infty,$$

when $1 \leq q < 1/(1-s)$. Hence it follows from the Hölder inequality that both the terms in the right-hand side of (25) are well-defined when the vorticity ω lies in $L^p(S)$ with $p > 1/s$. In this case, the quantities $M(\omega)$ and $L(\omega)$ are also well-defined, so that the maximization problem (26) makes sense. Moreover, we can solve this problem as follows.

Lemma 1. *Let $0 < s < 1$, $\lambda > 0$ and $p > 1/s$. Denote by p' the Hölder conjugate of p defined by $1/p + 1/p' = 1$. There exists a function $\omega_{\lambda,p} \in L^p_+(S)$ such that*

$$E_{\lambda,p}(\omega_{\lambda,p}) = \max \left\{ E_{\lambda,p}(\omega) : \omega \in L^p_+(S) \text{ s.t. } M(\omega) = L(\omega) = 1 \right\}. \quad (27)$$

The function $\omega_{\lambda,p}$ is angular Steiner symmetric and bounded. Moreover, there exist two numbers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ such that the function $\omega_{\lambda,p}$ can be written as

$$\omega_{\lambda,p} = \lambda (\psi_{\lambda,p})_+^{p'-1}, \quad (28)$$

with

$$\psi_{\lambda,p} := K_s \omega_{\lambda,p} + \frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p}. \quad (29)$$

The proof of Lemma 1 is standard. Existence follows from bounding the L^p -norm of the minimizing sequences corresponding to (26) and from applying standard weak compactness results. The angular Steiner symmetry of the function $\omega_{\lambda,p}$ is a consequence of Lemmas 10 and 11. Equations (28) and (29) are no more than the Euler-Lagrange equations of the maximization problem (26). In particular, the numbers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are interpreted as the Lagrange multipliers of the problem. We refer to Subsection 3.1 for more details and the proofs.

2.1.3 Bounds on the maximizers $\omega_{\lambda,p}$ and the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$

Our goal is now to construct solutions of the maximization problems (\mathcal{P}_λ) as limits when $p \rightarrow \infty$ of the functions $\omega_{\lambda,p}$ of Lemma 1. Before passing to this limit, we establish that the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are bounded uniformly with respect to p .

Lemma 2. *Let $0 < s < 1$ and $\lambda > 0$. There exists a positive number $p_0 > 1/s$ such that there exists a positive number C_λ independent of p for which the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ in Lemma 1 satisfy*

$$|\alpha_{\lambda,p}| + |\mu_{\lambda,p}| \leq C_\lambda,$$

for any $p \geq p_0$.

This lemma follows from a uniform bound on the function $K_s \omega_{\lambda,p}$. As a consequence of the integrability properties of the kernel K_s , there exists a positive number C_λ independent of p such that

$$\|K_s \omega_{\lambda,p}\|_{L^\infty} \leq C_\lambda.$$

With this inequality at hand, we can control the support of the function $\psi_{\lambda,p}$ and then of the function $\omega_{\lambda,p}$. In case the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are not bounded independently of p , this control is enough to establish a contradiction with the constraints $M(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$.

We next apply standard regularity estimates to the Euler-Lagrange equations (28) and (29) in order to bound the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ uniformly with respect to p . When $1/2 \leq s < 1$, the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ own sufficient smoothness so as to satisfy the weak formulation of (4) in (24). As a matter of fact, this weak formulation follows from the collinearity of the gradients $\nabla \omega_{\lambda,p}$ and $\nabla \psi_{\lambda,p}$, which in turn is a consequence of (28) provided that $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ are smooth enough. More precisely, we show

Lemma 3. *Let $0 < s < 1$, $\lambda > 0$ and $p \geq p_0$, where p_0 is defined in Lemma 2. Consider the solution $\omega_{\lambda,p}$ to the maximization problem (27) constructed in Lemma 1, and set*

$$\psi_{\lambda,p} = K_s \omega_{\lambda,p} + \frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p},$$

where $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are the corresponding Lagrange multipliers. There exists a positive number C_λ independent of p such that

$$\|\omega_{\lambda,p}\|_{L^r} \leq C_\lambda, \quad (30)$$

for any $1 \leq r \leq \infty$, while there exist positive numbers $C_{\lambda,r}(R)$, not depending on p , such that

$$\|\psi_{\lambda,p}\|_{W^{2s,r}(B(0,R))} \leq C_{\lambda,r}(R), \quad (31)$$

for any $1 < r < \infty$ and any positive number R . In particular, when $1/2 \leq s < 1$, the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ satisfy the weak equation

$$\int_S \omega_{\lambda,p}(x) \nabla^\perp \psi_{\lambda,p}(x) \cdot \nabla \varphi(x) dx = 0,$$

for any function $\varphi \in C_c^\infty(S)$.

2.1.4 Convergence in the limit $p \rightarrow \infty$ towards vortex patch solutions

With Lemmas 2 and 3 at hand, we are in position to complete the proof of Proposition 1. Since the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$, as well as the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$, are uniformly bounded with respect to p , we can invoke a compactness argument in order to take the limit $p \rightarrow \infty$. This eventually provides a limit function $\omega_\lambda \in X_\lambda$, which maximizes the energy E_s under the constraints $M(\omega_\lambda) = L(\omega_\lambda) = 1$. We next derive from the following lemma that this function is a vortex patch.

Lemma 4 (Bathtub principle for Riesz integrals). *Let μ and ν be two positive numbers. Given a positive number η , set*

$$\mathcal{G}_\eta(\mathbb{R}) := \left\{ g \in L^\infty(\mathbb{R}, [0, 1]) : \int_{\mathbb{R}} g \leq \eta \right\},$$

and consider a function $f \in L^1(\mathbb{R})$, which is even and non-increasing on \mathbb{R}_+ . Then the maximization problem

$$\mathcal{I}_{\mu,\nu} := \sup_{(g,h) \in \mathcal{G}_\mu(\mathbb{R}) \times \mathcal{G}_\nu(\mathbb{R})} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(x) h(y) dx dy \quad (32)$$

is solved by the functions

$$g = \mathbb{1}_{\left[-\frac{\mu}{2}, \frac{\mu}{2}\right]} \quad \text{and} \quad h = \mathbb{1}_{\left[-\frac{\nu}{2}, \frac{\nu}{2}\right]}.$$

Moreover, when the function f is decreasing on \mathbb{R}_+ , this solution is unique (up to a translation).

The proof of this lemma consists in applying the bathtub principle (see e.g. [16, Theorem 1.14]) to the double integral in the right-hand side of (32). Double integrals taking this form are usually called Riesz integrals in reference to the Riesz rearrangement inequality [20]. For fixed numbers r and r' , the integrals depending on the variables θ and θ' in the expression (18) of the energy E_s are exactly of this form. In particular, it suffices to apply Lemma 4 to them in order to prove that the functions ω_λ are the characteristic functions of measurable sets Ω_λ .

The proof of Proposition 1 then reduces to establish that these sets are equal to the upper level sets $\{\psi_\lambda > 0\}$. In the limit $p \rightarrow \infty$, we deduce from (28) that

$$\{\psi_\lambda > 0\} \subset \Omega_\lambda \subset \{\psi_\lambda \geq 0\}.$$

It then results from the monotonicity properties of the kernel κ_s given by Lemma 12 that the functions $\theta \mapsto \psi_\lambda(r, \theta)$ are decreasing on $(0, \pi/2N)$ for any fixed number $\frac{1}{2} < r < 2$. As a consequence, the level sets $\{\psi_\lambda = 0\}$ have measure zero, which eventually gives (22).

Note also that the weak equation (24) follows as in the proof of Lemma 3 from the fact that ψ_λ belongs to the Sobolev spaces $W^{1,r}(S)$ for any $1 < r < \infty$ when $1/2 \leq s < 1$. We refer to Subsection 3.5 for more details and the proofs.

2.2 Description of the vortex patch support

The purpose of this section is to complete the description of the support of the vorticity ω_λ obtained in Proposition 1. We will eventually show in Proposition 2 that it is entirely contained in N small balls centered at the vertices of a regular polygon, at least when λ is sufficiently large. Combining Proposition 1 with Proposition 2, the proof of Theorem 1 will then easily follow in Section 2.3.

In a first step, we consider functions ω that are merely of the form

$$\omega = \lambda \mathbb{1}_\Omega, \quad \Omega \subset S \text{ measurable}, \quad M(\omega) = 1. \quad (33)$$

Clearly, $\omega_\lambda \mathbb{1}_S$ is such a function. For arbitrary measurable sets X, X' in \mathbb{R}^2 , we next define the localized energy

$$I_s(\omega, X, X') := \int_X \int_{X'} \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x') \omega(x) dx' dx$$

and the localized concentration of mass

$$M(\omega, X) := \int_X \omega(x) dx.$$

We also let

$$\mathcal{I}_s := \frac{1}{\pi^2} \int_{\mathcal{B}(0,1)} \int_{\mathcal{B}(0,1)} \frac{dx dx'}{|x - x'|^{2(1-s)}} = I_s(\lambda \mathbb{1}_{\mathcal{B}(0,\varepsilon)}, \mathbb{R}^2, \mathbb{R}^2). \quad (34)$$

The quantity \mathcal{I}_s is positive and elementary computations show that $\frac{1}{6} \leq s\mathcal{I}_s \leq 1$ for all $s \in (0, 1)$, in particular it is bounded and bounded away from zero independently of s .

We infer the following from the Riesz rearrangement inequality [20]. This provides lower bounds on the localized concentration of mass in terms of the localized energy.

Lemma 5. *Assume that ω satisfies (33) and $X, X' \subset \mathbb{R}^2$ are measurable. Then*

$$\begin{aligned} I_s(\omega, X, X) &\leq \mathcal{I}_s M(\omega, X)^{1+s}, \\ I_s(\omega, X, X') &\leq \frac{1}{s} M(\omega, X) M(\omega, X')^s, \\ I_s(\omega, X, X') &\leq \left(\frac{\varepsilon}{\text{dist}(X, X')} \right)^{2(1-s)} M(\omega, X) M(\omega, X'), \end{aligned}$$

where the last inequality is only meaningful provided that

$$\text{dist}(X, X') := \inf_{x \in X} \inf_{x' \in X'} |x - x'|$$

has positive value.

Being a solution of the minimisation problem (\mathcal{P}_λ) , we can derive some first tight local energy estimates for ω_λ using a comparison argument with the test function ϖ_λ introduced in (21).

Lemma 6. *Let ω_λ be a solution to (\mathcal{P}_λ) . There exists a positive number C , depending only on s , such that the bounds*

$$\mathcal{I}_s \leq I_s(\omega, S, S) \leq \mathcal{I}_s + C\varepsilon^{2(1-s)} \quad (35)$$

hold for $\omega := \omega_\lambda \mathbb{1}_S$.

For functions of the form (33) and for which (35) holds, one can obtain a first weak form of concentration by an elementary covering argument.

Lemma 7. *There exist positive numbers Λ_0 and η depending only on s such that if ω is of the form (33) and satisfies the bounds (35), then there exists $x \in S$ such that*

$$M(\omega, \mathcal{B}(x, \Lambda_0\varepsilon)) \geq \eta > 0. \quad (36)$$

Combining Lemmas 5, 6 and 7 with a more elaborate convexity and combinatory argument, we then show that splitting vorticity is not favourable and obtain a decay estimate for the vorticity density.

Lemma 8. *There exists a constant $C > 0$, depending only on s , such that if ω is of the form (33) and satisfies the bounds (35), then*

$$M(\omega, S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) \leq \frac{C}{\Lambda^{\gamma_s}}, \quad \forall \Lambda > 0,$$

where the decay rate γ_s is given by $\gamma_s := (1 + \frac{1}{2(1-s)})^{-1}$.

At this point we have only used arguments which rely on tight energy bounds for ω . It is hopeless to expect that complete concentration of vorticity such as the one stated in (iii) of Theorem 1 could hold under such assumptions only (it is indeed not complicated to build counter-examples). For the next step of the argument, we therefore restrict our focus to the solutions ω_λ of the maximization problem (\mathcal{P}_λ) , in order to exploit the equations (22)-(23).

Note that (22)-(23) only implies a very weak form of regularity, especially when s is small. For that reason, we rely on finite differences of ψ_λ at different pairs of well-chosen points inside and outside of $\text{supp}(\omega_\lambda)$ and combine the resulting identities with integral estimates based on the decay obtained in Lemma 8. The first localization result that we prove is the following.

Lemma 9. *There exists a positive constant Λ depending only on s such that if ω_λ is a solution of the maximization problem (\mathcal{P}_λ) then*

$$\text{supp}(\omega_\lambda) \cap \mathfrak{S}_N \subseteq \mathcal{B}((1, 0), \Lambda\varepsilon) \cup B, \quad (37)$$

where

$$B := \{(r, \theta) \in S : 2 - 1/8 \leq r \leq 2 \text{ or } 1/2 \leq r \leq 1/2 + 1/8\} \subseteq S.$$

As explained with more details in the proof, it is not possible to eliminate a possible residual accumulation of mass far from $\mathcal{B}((1, 0), \Lambda\varepsilon)$ with arguments using finite differences of ψ_λ . At this stage of the proof, the case $M(\omega_\lambda, B) > 0$ (even very small) cannot be excluded. Nevertheless, there is a gap of order 1 (with respect to ε) between the two sets $\mathcal{B}(x, \Lambda\varepsilon)$ and B and then, using again energetical arguments, we can prove that $M(\omega_\lambda, B) > 0$ contradicts the maximality of ω_λ . Here is the conclusive localization result.

Proposition 2. *There exists a positive constant Λ , depending only on s , such that if ω_λ is a solution of the maximization problem (\mathcal{P}_λ) then*

$$\text{supp}(\omega_\lambda) \cap \mathfrak{S}_N \subseteq \mathcal{B}((1, 0), \Lambda\varepsilon).$$

In particular, if λ is sufficiently large we have

$$\text{dist}(\text{supp}(\omega_\lambda), \mathfrak{S}_N \setminus S) > 0, \quad (38)$$

and therefore for $1/2 \leq s < 1$ the function ω_λ is a weak solution to (4) in the sense of (5).

2.3 Proof of Theorem 1 completed

Combining Propositions 1 and 2, we deduce that statements (i), (ii) and (iii) of Theorem 1 hold. It only remains to establish statements (iv) and (v), which concern the asymptotic as λ tends to $+\infty$, that is (11), (12) and (13).

The convergence in (11) is immediate. It follows from the constraints $M(\omega_\lambda) = L(\omega_\lambda) = N$, the rotational and Steiner symmetry assumptions, and Proposition 2.

Concerning the convergence in (12), it can be computed exactly using the equations (9) and (10) for ω_λ . This is the reason why we need to assume ¹ that $\frac{1}{2} \leq s < 1$.

Let $y = (y_1, y_2) \in \mathbb{R}^2$ be arbitrary, and consider a test function $\varphi \in C^\infty(S)$ such that

$$\varphi(x) = \frac{|x|^2}{2} + y \cdot x, \quad \forall x \in \text{supp}(\omega_\lambda).$$

From (5) we deduce therefore that

$$\alpha_\lambda T_{1,\lambda,y} = c_s(1-s)T_{2,\lambda,y},$$

where

$$\begin{aligned} T_{1,\lambda,y} &= \int_S \omega_\lambda(x) x^\perp \cdot y \, dx, \\ T_{2,\lambda,y} &= \int_S \int_S \omega_\lambda(x) \omega_\lambda(x') \sum_{n=1}^{N-1} \frac{(x - R_{\frac{2n\pi}{N}} x')^\perp}{|x - R_{\frac{2n\pi}{N}} x'|^{4-2s}} \cdot (y - R_{\frac{2n\pi}{N}} y) \, dx \, dx'. \end{aligned}$$

In view of (9) and statement (iii) of Theorem 1, we can pass to the limit $\lambda \rightarrow +\infty$ in the above two terms and obtain

$$T_{1,\lambda,y} \longrightarrow y_2, \quad \text{and} \quad T_{2,\lambda,y} \longrightarrow \sum_{n=1}^{N-1} \frac{((1,0) - R_{\frac{2n\pi}{N}}(1,0))^\perp}{|(1,0) - R_{\frac{2n\pi}{N}}(1,0)|^{4-2s}} \cdot (y - R_{\frac{2n\pi}{N}} y).$$

Taking $y = (0,1) = (1,0)^\perp$, the conclusion (12) follows.

Finally we consider (13), whose proof bares resemblance with the one of Proposition 2. We first take an arbitrary point $y \in \mathcal{B}((1,0), 2\varepsilon) \setminus \Omega_\lambda$ (such a point exists since by definition ε is the radius of a ball with the same area $1/\lambda$ as Ω_λ), so that $\psi_\lambda(y) \leq 0$, and therefore

$$K_s \omega_\lambda(y) + \frac{\alpha_\lambda}{2} |y|^2 \leq \mu_\lambda. \quad (39)$$

On the other hand, from (iii) we infer that

$$\begin{aligned} K_s \omega_\lambda(x) &\geq \int_S \frac{\omega_\lambda(x')}{|x - x'|^{2(1-s)}} \, dx' - C \\ &= \int_{\mathcal{B}((1,0), C\sqrt{\pi\varepsilon})} \frac{\omega_\lambda(x')}{|x - x'|^{2(1-s)}} \, dx' - C \geq \frac{1}{C\lambda^{1-s}} - C. \end{aligned}$$

Combining both inequalities and using the fact that α_λ is already known to be uniformly bounded by (12), we deduce the lower estimate in (13). Concerning the upper estimate, we consider instead a point $x \in \mathcal{B}((1,0), C/\sqrt{\lambda}) \cap \Omega_\lambda$. Then $\psi_\lambda(y) \geq 0$, and therefore

$$\mu_\lambda \leq K_s \omega_\lambda(x) + \frac{\alpha_\lambda}{2} |x|^2. \quad (40)$$

On the other hand by rearrangement

$$K_s \omega_\lambda(x) \leq \int_{\mathcal{B}((1,0), 2C/\sqrt{\lambda})} \frac{\omega_\lambda(x')}{|x - x'|^{2(1-s)}} \, dx' + C \leq \int_{\{|x'| \leq \varepsilon\}} \frac{\lambda}{|x'|^{2(1-s)}} \, dx' + C \leq \frac{C}{\lambda^{1-s}} + C,$$

and the upper estimate for μ_λ follows similarly. This ends the proof of Theorem 1. \square

¹We do not know if this is only technical.

3 Details of the proofs

We present in this section, in the order of appearance, the proofs of all the claims that were postponed from Section 2.

3.1 Proof of Lemma 1

We first establish the existence of the maximizing vorticity $\omega_{\lambda,p}$. Recall that the kernel κ_s is defined so as to satisfy

$$\kappa_s(r, r', \theta - \theta') = \sum_{n=0}^{N-1} \frac{c_s}{\left(r^2 + r'^2 - 2rr' \cos\left(\theta - \theta' - \frac{2n\pi}{N}\right)\right)^{1-s}},$$

for $x = (r \cos(\theta), r \sin(\theta))$ and $x' = (r' \cos(\theta'), r' \sin(\theta'))$. Hence, we derive from the boundedness of S that

$$\sup_{(r,\theta) \in S} \int_S \kappa_s(r, r', \theta - \theta')^{p'} r' dr' d\theta' < \infty, \quad (41)$$

under the condition $p > 1/s$. Given any function $\omega \in L^p(S)$, we infer from the Hölder inequality that

$$\|K_s \omega\|_{L^\infty} \leq C \|\omega\|_{L^p}, \quad (42)$$

where the number C only depends on s . The energy E_s is then controlled by

$$E_s(\omega) \leq \frac{CN}{2} \|\omega\|_{L^1} \|\omega\|_{L^p} = \frac{CN}{2} \|\omega\|_{L^p}, \quad (43)$$

under the conditions $\omega \geq 0$ and $M(\omega) = 1$. We next apply the Young inequality in order to obtain

$$E_s(\omega) \leq \left(\frac{C\lambda^{\frac{1}{p'}} N^{\frac{1}{p'}}}{2}\right) \left(\frac{N^{\frac{1}{p}} \|\omega\|_{L^p}}{\lambda^{\frac{1}{p'}}}\right) \leq K_{\lambda,p} + \frac{\lambda N}{p} \int_S \left(\frac{\omega}{\lambda}\right)^p.$$

Here, the constant $K_{\lambda,p} := C^{p'} \lambda N / (2^{p'} p')$ is uniformly bounded with respect to $p \rightarrow \infty$, and it controls the penalized energy as

$$E_{\lambda,p}(\omega) \leq K_{\lambda,p}. \quad (44)$$

In particular, the value of the supremum $\mathcal{E}_{\lambda,p}$ in (26) is finite.

In another direction, we also deduce from (43) that

$$\frac{\lambda^{1-p} N}{p} \|\omega\|_{L^p}^p \leq \frac{CN}{2} \|\omega\|_{L^p} - E_{\lambda,p}(\omega) \leq \frac{C^{p'} \lambda N}{2p'} + \frac{\lambda^{1-p} N}{2p} \|\omega\|_{L^p}^p - E_{\lambda,p}(\varpi_\lambda),$$

when $E_{\lambda,p}(\varpi_\lambda) \leq E_{\lambda,p}(\omega)$. Here, the notation ϖ_λ refers as before to the function in (21). We next obtain

$$\|\omega\|_{L^p}^p \leq (p-1) \lambda^p C^{p'} - \frac{2p\lambda^{p-1}}{N} E_{\lambda,p}(\varpi_\lambda), \quad (45)$$

so that there exists a maximizing sequence $(\omega_j)_{j \in \mathbb{N}}$ for (26), which is bounded in $L^p(S)$. Hence, there exists a function $\omega_{\lambda,p} \in L^p(S)$ such that, up to a subsequence, $(\omega_j)_{j \in \mathbb{N}}$ weakly tends to $\omega_{\lambda,p}$ in $L^p(S)$. This convergence is enough to guarantee that $\omega_{\lambda,p} \geq 0$ a.e., as well as the identities $M(\omega_{\lambda,p}) = 1$ and $L(\omega_{\lambda,p}) = 1$. Moreover,

$$\mathcal{E}_{\lambda,p} = \limsup_{j \rightarrow \infty} E_{\lambda,p}(\omega_j) \leq E_{\lambda,p}(\omega_{\lambda,p}). \quad (46)$$

Indeed, estimate (41) implies that $\kappa_s \in L^{p'}(S^2)$, and the functions $(x, y) \mapsto \omega_j(x) \omega_j(y)$ weakly tend to the function $(x, y) \mapsto \omega_{\lambda,p}(x) \omega_{\lambda,p}(y)$ in $L^p(S^2)$ as $j \rightarrow \infty$. Therefore, the quantities $E_s(\omega_j)$ converge to $E_s(\omega_{\lambda,p})$, and inequality (46) follows from the weak lower semi-continuity of the L^p -norm. Finally, this inequality guarantees that the limit vorticity $\omega_{\lambda,p}$ solves the maximization problem (26). Moreover, we can invoke Lemmas 10 and 11 in order to claim that the vorticity $\omega_{\lambda,p}$ is angular Steiner symmetric.

We now turn to the proof of (28) and (29). These two equations correspond to the Euler-Lagrange equations of the maximization problem (26). In order to write them, we observe that, given any measurable subset X of S with positive measure, the non-vanishing linear forms M and L are linearly independent on $L^\infty(X)$. Otherwise, there exists a number $a \neq 0$ such that

$$\int_X (a - |x|^2)\varphi(x) dx = 0,$$

for any $\varphi \in L^\infty(X)$. In particular, the function $x \mapsto a - |x|^2$ vanishes almost everywhere on X , which is only possible when a is positive and X is a subset of the circle of center $(0, 0)$ and radius \sqrt{a} . Since this circle is of measure 0, this contradicts the fact that X has positive measure. As a consequence of the linear independence of M and L on $L^\infty(X)$, we conclude that the linear mapping $\varphi \mapsto (M(\varphi), L(\varphi))$ is onto \mathbb{R}^2 .

Consider next a positive number δ such that $X_\delta = \{\omega_{\lambda,p} \geq \delta\}$ has positive measure. Note that all the numbers δ small enough satisfy this assumption, otherwise the vorticity $\omega_{\lambda,p}$ vanishes, which contradicts the condition $M(\omega_{\lambda,p}) = 1$. Taking into account the previous arguments, we derive the existence of two functions φ_1 and φ_2 in $L^\infty(X_\delta)$ with $M(\varphi_1) = L(\varphi_2) = 1$ and $M(\varphi_2) = L(\varphi_1) = 0$. In particular, given any function $\zeta \in L^\infty(S)$ such that $\zeta \geq 0$ on $S \setminus X_\delta$, we can introduce the function

$$\omega_h = \omega_{\lambda,p} + h(\zeta - M(\zeta)\varphi_1 - L(\zeta)\varphi_2),$$

for any positive number h . When this number is small enough, we check that this function belongs to $L^p_+(S)$, with $M(\omega_h) = L(\omega_h) = 1$. The maximizing nature of the vorticity $\omega_{\lambda,p}$ then gives

$$0 \geq \lim_{h \rightarrow 0^+} \frac{d}{dh} (E_{\lambda,p}(\omega_h)) = E'_{\lambda,p}(\omega_{\lambda,p}) \zeta - M(\zeta)E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_1 - L(\zeta)E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_2.$$

In this expression, the differential $E'_{\lambda,p}(\omega)$ is given by

$$E'_{\lambda,p}(\omega)\zeta = N \int_S \zeta \left(K_s \omega - \left(\frac{\omega}{\lambda} \right)^{\frac{1}{p'-1}} \right).$$

Setting $\mu_{\lambda,p} = E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_1 / N$, $\alpha_{\lambda,p} = -2E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_2 / N$ and $\psi_{\lambda,p}(x) := K_s \omega_{\lambda,p}(x) + \alpha_{\lambda,p} |x|^2 / 2 - \mu_{\lambda,p}$ as in (29), we obtain

$$\int_S \zeta \left(\psi_{\lambda,p} - \left(\frac{\omega_{\lambda,p}}{\lambda} \right)^{\frac{1}{p'-1}} \right) \leq 0.$$

Going back to the condition $\zeta \geq 0$ on $S \setminus X_\delta$, we infer that

$$\psi_{\lambda,p} = \left(\frac{\omega_{\lambda,p}}{\lambda} \right)^{\frac{1}{p'-1}} \text{ on } X_\delta, \quad \text{and} \quad \psi_{\lambda,p} \leq \left(\frac{\delta}{\lambda} \right)^{\frac{1}{p'-1}} \text{ on } S \setminus X_\delta.$$

Equation (28) follows by taking the limit $\delta \rightarrow 0$.

Finally, it follows from (42) that the function $\psi_{\lambda,p}$ is bounded on S . In view of (28), so is the vorticity $\omega_{\lambda,p}$. This completes the proof of Lemma 1. \square

3.2 Proof of Lemma 2

The proof relies on a uniform bound on the function $K_s \omega_{\lambda,p}$, which is derived from estimates (42) and (45). Indeed, we deduce from the definition of the function ϖ_λ that

$$E_{\lambda,p}(\varpi_\lambda) = E_s(\varpi_\lambda) - \frac{N}{p}.$$

Since $p > 1/s$, this gives

$$|E_{\lambda,p}(\varpi_\lambda)| \leq C_\lambda,$$

where C_λ denotes, here as in the sequel, a positive number not depending on p and possibly changing from line to line. Hence, we infer from (42) and (45) that

$$\|K_s \omega_{\lambda,p}\|_{L^\infty} \leq C_\lambda. \tag{47}$$

As a consequence of its definition, the function $\psi_{\lambda,p}$ then satisfies

$$\frac{\alpha_{\lambda,p}}{2}r^2 - \mu_{\lambda,p} - C_\lambda \leq \psi_{\lambda,p}(r, \theta) \leq \frac{\alpha_{\lambda,p}}{2}r^2 - \mu_{\lambda,p} + C_\lambda. \quad (48)$$

If the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ were not bounded independently of p , these estimates would provide a contradiction with the constraints $M(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$.

More precisely, let us now assume that

$$\mu_{\lambda,p} \rightarrow +\infty, \quad (49)$$

as $p \rightarrow \infty$. In this case, we can suppose that $\mu_{\lambda,p} > C_\lambda$ for p large enough. It then follows from inequalities (48) that $\alpha_{\lambda,p} > 0$. Otherwise, the function $\psi_{\lambda,p}$ is non-positive, which eventually contradicts the constraint $M(\omega_{\lambda,p}) = 1$. Consider next the number

$$r_0(\psi_{\lambda,p}) := \inf_{-\pi/4 \leq \theta \leq \pi/4} \left(\text{infess} \left(\text{supp}(r \mapsto (\psi_{\lambda,p})_+(r, \theta)) \right) \right).$$

Since the function $(\psi_{\lambda,p})_+$ is angular Steiner symmetric, this infimum is achieved for $\theta = 0$. Observe that the inequality $f \geq g$ implies that $r_0(f) \leq r_0(g)$. Therefore, we derive from (48) that

$$r_0(\omega_{\lambda,p}) = r_0(\psi_{\lambda,p}) \geq r_0 \left(\frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p} + C_\lambda \right) = \sqrt{\frac{2}{\alpha_{\lambda,p}} (\mu_{\lambda,p} - C_\lambda)} := r_0^*.$$

Similarly, we obtain

$$r_0(\omega_{\lambda,p}) = r_0(\psi_{\lambda,p}) \leq r_0 \left(\frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p} - C_\lambda \right) = \sqrt{\frac{2}{\alpha_{\lambda,p}} (\mu_{\lambda,p} + C_\lambda)} := r_1^*,$$

and we notice that

$$\frac{r_0^*}{r_1^*} = \sqrt{\frac{\mu_{\lambda,p} - C_\lambda}{\mu_{\lambda,p} + C_\lambda}} \rightarrow 1. \quad (50)$$

when $p \rightarrow \infty$. Given a positive number $\delta < 1$, we now face two possibilities.

Case 1. $\limsup_{p \rightarrow \infty} r_1^* \geq 1 + \delta$.

It then follows from (50) that

$$\limsup_{p \rightarrow \infty} r_0^* \geq 1 + \delta,$$

and we obtain the following contradiction

$$1 = \int_S |x|^2 \omega_{\lambda,p}(x) dx \geq r_0(\omega_{\lambda,p})^2 \int_S \omega_{\lambda,p} \geq (r_0^*)^2 \int_S \omega_{\lambda,p} = (r_0^*)^2 > 1, \quad (51)$$

with the constraints $M(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$ for a number p large enough.

Case 2. $\limsup_{p \rightarrow \infty} r_1^* \leq 1 + \delta$.

In this case, we have $r_1^* \leq 1 + \delta$ for any number p large enough. Combining the constraint $M(\omega_{\lambda,p}) = 1$ with (48), we get

$$\int_{r_1^*}^2 \left(\frac{\alpha_{\lambda,p}}{2} r^2 - \mu_{\lambda,p} - C_\lambda \right)^{p'-1} r dr \leq \frac{1}{\lambda\pi}. \quad (52)$$

The computation of the integral leads to

$$\frac{\alpha_{\lambda,p}}{2} 4 - \mu_{\lambda,p} - C_\lambda \leq \left(\frac{p' \alpha_{\lambda,p}}{\lambda\pi} \right)^{\frac{1}{p'}}.$$

Assuming that δ is small enough and applying the Young inequality to the right-hand side of this inequality give

$$\frac{\alpha_{\lambda,p}}{2} 4 - \mu_{\lambda,p} - C_\lambda \leq \frac{\delta \alpha_{\lambda,p}}{2} + \frac{1}{p} \left(\frac{2}{\delta \lambda \pi} \right)^{\frac{p}{p'}},$$

and then

$$(4 - \delta) \frac{\alpha_{\lambda,p}}{2} \leq \frac{1}{p} \left(\frac{2}{\delta \lambda \pi} \right)^{\frac{p}{p'}} + \mu_{\lambda,p} + C_\lambda.$$

In view of the definition of r_0^* , we are led to

$$(r_0^*)^2 \geq \frac{(4 - \delta)(\mu_{\lambda,p} - C_\lambda)}{\mu_{\lambda,p} + C_\lambda + \frac{1}{p} \left(\frac{2}{\delta \lambda \pi} \right)^{p-1}} \xrightarrow{\mu_{\lambda,p} \rightarrow \infty} 4^2 - \delta.$$

Since δ is arbitrary small, we finally get a contradiction by arguing as for (51).

We conclude that assumption (49) does not hold, and we can argue similarly in order to prove that $\mu_{\lambda,p}$ does not converge towards $-\infty$ when $p \rightarrow \infty$. This shows that the numbers $\mu_{\lambda,p}$ remain bounded in this limit, and so do the numbers $\alpha_{\lambda,p}$ due to the constraint $M(\omega_{\lambda,p}) = 1$. \square

3.3 Proof of Lemma 3

The proof relies on standard regularity theory. In view of the definition of the function $\psi_{\lambda,p}$, we can derive from (47) and Lemma 2 that

$$\|\psi_{\lambda,p}\|_{L^\infty} \leq C_\lambda, \quad (53)$$

where C_λ denotes, here as in the sequel, a positive number not depending on p and possibly changing from line to line. In view of (28), we obtain

$$\|\omega_{\lambda,p}\|_{L^\infty} \leq \lambda C_\lambda^{\frac{1}{p-1}}.$$

Since $p \geq p_0$, this is enough to get

$$\|\omega_{\lambda,p}\|_{L^\infty} \leq C_\lambda,$$

and to deduce (30) from the constraint $M(\omega_{\lambda,p}) = 1$ and the Hölder inequality. As a consequence of (53), we also observe that

$$\|\psi_{\lambda,p}\|_{L^r(B(0,R))} \leq C_\lambda R^{\frac{2}{r}}, \quad (54)$$

for any $1 \leq r \leq \infty$ and any positive number R .

For $1 < r < \infty$, we next recall the existence of a positive number $A_{s,r}$, depending only on s and r , such that

$$\|K_s f\|_{\dot{W}^{2s,r}(\mathbb{R}^2)} \leq A_{s,r} \|f\|_{L^r(\mathbb{R}^2)}, \quad (55)$$

for any function $f \in L^r(\mathbb{R}^2)$. Applying this inequality to the function $f = \omega_{\lambda,p} \mathbb{1}_{\text{supp}(\omega_{\lambda,p})}$, we obtain

$$\|K_s \omega_{\lambda,p}\|_{\dot{W}^{2s,r}(\mathbb{R}^2)} \leq A_{s,r} \|\omega_{\lambda,p}\|_{L^r(S)},$$

and (31) results from (54) and Lemma 2.

When $1/2 \leq s < 1$, this guarantees that the function $\psi_{\lambda,p}$ lies in $W^{1,r}(S)$ for any $1 < r < \infty$, so that its positive part $(\psi_{\lambda,p})_+$ is also in these spaces. Hence, we can compute

$$\nabla^\perp \left((\psi_{\lambda,p})_+^{p'} \right) = p' (\psi_{\lambda,p})_+^{p'-1} \nabla^\perp \psi_{\lambda,p} = \frac{p'}{\lambda} \omega_{\lambda,p} \nabla^\perp \psi_{\lambda,p},$$

and this identity holds in the sense of integrable functions. Given a function $\varphi \in \mathcal{C}_c^\infty(S)$, we conclude that

$$\int_S \omega_{\lambda,p}(x) \nabla^\perp \psi_{\lambda,p}(x) \cdot \nabla \varphi(x) dx = \int_S \nabla^\perp \left((\psi_{\lambda,p})_+^{p'} \right)(x) \cdot \nabla \varphi(x) dx = 0,$$

by integrating by parts. \square

3.4 Proof of Lemma 4

Invoking the Riesz rearrangement inequality, we reduce the analysis of the maximization problem (32) to the situation where both the functions g and h are even and non-increasing on \mathbb{R}_+ . In this case, since the function f is even and non-increasing on \mathbb{R}_+ , so is the function $f \star g$. In particular, the sets $\{f \star g > t\}$ have finite measure for any positive number t . Therefore, we can apply the bathtub principle in [16, Theorem 1.14] in order to show that the function $h_\mu = \mathbb{1}_{[-\frac{\mu}{2}, \frac{\mu}{2}]}$ is a solution to the problem

$$\mathcal{J}_\mu := \max_{h \in \mathcal{G}_\mu(\mathbb{R})} \int_{\mathbb{R}} f \star g(y) h(y) dy.$$

Moreover, when f is decreasing on \mathbb{R}_+ , so is the function $f \star g$, and the bathtub principle guarantees the uniqueness of the solution h_μ . We finally complete the proof of Lemma 4 by observing that the functions g and h play symmetric roles in the maximization problem (32). \square

3.5 Proof of Proposition 1

Let $p > 1/s$. Consider a solution $\omega_{\lambda,p}$ to the maximization problem (27) and the corresponding Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ given by Lemma 1. Up to an omitted subsequence, we first derive from Lemma 2 the existence of two numbers α_λ and μ_λ such that

$$\alpha_{\lambda,p} \rightarrow \alpha_\lambda \quad \text{and} \quad \mu_{\lambda,p} \rightarrow \mu_\lambda, \quad (56)$$

when $p \rightarrow \infty$. In view of Lemma 3, we can invoke the Sobolev embedding theorem in order to exhibit a continuous function $\psi_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that, up to a further subsequence,

$$\psi_{\lambda,p} \rightarrow \psi_\lambda \text{ in } L^\infty(B(0, R)), \quad (57)$$

as $p \rightarrow \infty$, for any positive number R . In particular, this convergence provides

$$\limsup_{p \rightarrow \infty} \|\omega_{\lambda,p}\|_{L^\infty(S)} \leq \lambda \lim_{p \rightarrow \infty} \|(\psi_{\lambda,p})_+\|_{L^\infty(S)}^{p'-1} \leq \lambda. \quad (58)$$

Going back to Lemma 3, we can apply the Banach-Alaoglu theorem to construct a non-negative function $\omega_\lambda \in L^\infty(S)$ such that, up to a further subsequence,

$$\omega_{\lambda,p} \overset{*}{\rightharpoonup} \omega_\lambda \text{ in } L^\infty(S), \quad (59)$$

when $p \rightarrow \infty$. In view of (58), the function ω_λ lies in $L^\infty_\lambda(S)$ and we can extend it as a function in X_λ by performing a N -fold symmetrization. Moreover, we infer from (56), (57) and (59) that the function ψ_λ is given by (23).

We now check that this function solves the maximization problem (\mathcal{P}_λ) . Since the maps $x \mapsto 1$ and $x \mapsto |x|^2$ are locally integrable, we derive from the constraints $M(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$ and (59) that $M(\omega_\lambda) = L(\omega_\lambda) = N$. Consider next a function $\omega \in X_\lambda$ such that $M(\omega) = L(\omega) = N$. It follows from Lemma 1 and (58) that

$$E_s(\omega) \leq E_s(\omega_{\lambda,p}) + \frac{\lambda N}{p} \int_S \left(\frac{\omega}{\lambda}\right)^p \leq E_s(\omega_{\lambda,p}) + \frac{\lambda N |S|}{p}.$$

Invoking the compact embedding of $L^\infty(S)$ into $\dot{H}^{-s}(S)$, we can extract a further subsequence for which

$$E_s(\omega_{\lambda,p}) \rightarrow E_s(\omega_\lambda),$$

when $p \rightarrow \infty$. Hence, we are led to

$$E_s(\omega) \leq E_s(\omega_\lambda),$$

by taking the limit $p \rightarrow \infty$ in the previous inequality. Therefore, the function ω_λ solves the maximization problem (\mathcal{P}_λ) .

At this point, we explain why the function ω_λ is a vortex patch. Given any function $\omega \in L^\infty(S)$, we define the function $\widehat{\omega}$ as being the only angular Steiner symmetric function of $L^\infty(S)$, with values into the pair $\{0, \lambda\}$, such that

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \omega(r, \theta) d\theta = \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \widehat{\omega}(r, \theta) d\theta,$$

for almost any $\frac{1}{2} < r < 2$. Using the decreasing nature of the map $\xi \mapsto \kappa_s(r, r', \xi)$ given by Lemma 12, we deduce from Lemma 4 that

$$\begin{aligned} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} k_s(r, r', \theta - \theta') \omega_1(r, \theta) \omega_2(r', \theta') d\theta d\theta' \\ \leq \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} k_s(r, r', \theta - \theta') \widehat{\omega}_1(r, \theta) \widehat{\omega}_2(r', \theta') d\theta d\theta', \end{aligned}$$

for any functions $(\omega_1, \omega_2) \in L^\infty(S)^2$ and almost any $(r, r') \in (\mathbb{R}_+)^2$. Moreover, this inequality is an equality if and only if $\omega_1(r, \cdot) = \widehat{\omega}_1(r, \cdot)$ and $\omega_2(r', \cdot) = \widehat{\omega}_2(r', \cdot)$ for almost any $-\pi/(2N) \leq \theta \leq \pi/(2N)$. Therefore, we deduce

$$E_s(\omega) \leq E_s(\widehat{\omega}),$$

with equality if and only if $\omega = \widehat{\omega}$. This guarantees that the function $\omega_\lambda = \widehat{\omega}_\lambda$ is angular Steiner symmetric, and up to the multiplicative factor λ , equal to the characteristic function of a measurable subset Ω_λ of S .

Since $\omega_{\lambda,p} = (\psi_{\lambda,p})_+^{p'-1}$, we derive from (57) that this set satisfies

$$\{\psi_\lambda > 0\} \subseteq \Omega_\lambda \subseteq \{\psi_\lambda \geq 0\}.$$

In order to conclude that this set matches (up to a set of measure 0) with the support of the function $(\psi_\lambda)_+$, we are reduced to show that the set $\{\psi_\lambda = 0\}$ has zero measure. Let $\frac{1}{2} < r < 2$ and $0 < \theta_1, \theta_2 < \pi/4$, with $\theta_1 < \theta_2$. In view of definition (23), we compute

$$\begin{aligned} \psi_\lambda(r, \theta_1) - \psi_\lambda(r, \theta_2) &= K_s \omega_\lambda(r, \theta_1) - K_s \omega_\lambda(r, \theta_2) \\ &= \int_{\frac{1}{2}}^2 \int_0^{\frac{\pi}{2N}} \left(k_s(r, r', \theta' - \theta_1) - k_s(r, r', \theta' - \theta_2) \right) \omega_\lambda(r', \theta') r' dr' d\theta, \end{aligned}$$

and this double integral is positive due to the fact that the map $\xi \mapsto k_s(r, r', \xi)$ is decreasing on $(0, \pi/2N)$ as a consequence of Lemma 12 below. By the Fubini theorem, we conclude that the measure of the set $\{\psi_\lambda = 0\}$ is equal to 0, which eventually provides (22).

We finally turn to the proof of (24). When $1/2 \leq s < 1$, Lemma 3 guarantees that the functions $\psi_{\lambda,p}$ are uniformly bounded in $W^{1,r}(B(0, R))$ with respect to $p \rightarrow \infty$ for any exponent $1 < r < \infty$ and any positive number R . As a consequence, we can assume that the function ψ_λ lies in $W_{loc}^{1,r}(\mathbb{R}^2)$. In particular, we are allowed to write

$$\nabla^\perp((\psi_{\lambda,p})_+) = \mathbb{1}_{\{\psi_\lambda > 0\}} \nabla^\perp \psi_{\lambda,p},$$

so that, by integrating by parts,

$$\int_S \omega_\lambda(x) \nabla^\perp \psi_\lambda(x) \cdot \nabla \varphi(x) dx = \lambda \int_S \nabla^\perp((\psi_\lambda)_+)(x) \cdot \nabla \varphi(x) dx = 0,$$

for any function $\varphi \in C_c^\infty(S)$. This completes the proof of Proposition 1. \square

3.6 Proof of Lemma 5

We infer from the Riesz rearrangement inequality that

$$\int_X \int_X \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx dx' \leq \lambda^2 \int_{\mathcal{B}(0, \varepsilon \sqrt{M(\omega, X)})} \int_{\mathcal{B}(0, \varepsilon \sqrt{M(\omega, X)})} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} dx dx',$$

and the first inequality in the statement of the lemma follows from the definition of \mathcal{I}_s . Regarding the second statement, since the map $x \mapsto |x|^{2(s-1)}$ is radially decreasing, we can rearrange the function ω so as to write

$$\begin{aligned} \int_X \omega(x') \int_{X'} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x) dx dx' &\leq \lambda \int_X \omega(x') \int_{\mathcal{B}(x, \varepsilon \sqrt{M(\omega, X')}}} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} dx dx' \\ &= \lambda M(\omega, X) \int_{\mathcal{B}(0, \varepsilon \sqrt{M(\omega, X')}}} \left(\frac{\varepsilon}{|y|} \right)^{2(1-s)} dy. \end{aligned}$$

and the conclusion follows from the identity $\lambda \pi \varepsilon^2 = 1$. Finally, the last statement is a direct consequence of the inequality

$$|x - x'| \geq d(X, X'),$$

which holds for any $x \in X$ and $x' \in X'$. \square

3.7 Proof of Lemma 6

On the one hand, since ω_λ is the maximizer of the energy E_s ,

$$E_s(\varpi_\lambda) \leq E_s(\omega_\lambda), \quad (60)$$

where ϖ_λ was defined in (21). Then,

$$E_s(\varpi_\lambda) \geq c_s \int_S \int_S \frac{\lambda \mathbb{1}_{\mathcal{B}((r_\varepsilon, 0), \varepsilon)}(x) \lambda \mathbb{1}_{\mathcal{B}((r_\varepsilon, 0), \varepsilon)}(x')}{|x-x'|^{2(1-s)}} dx dx' = \frac{c_s}{\varepsilon^{2(1-s)}} \mathcal{I}_s.$$

Together with (60), this gives the lower bound in (35). Concerning the upper bound,

$$E_s(\omega_\lambda) \leq c_s \int_S \int_S \frac{\omega_\lambda(x) \omega_\lambda(x')}{|x-x'|^{2(1-s)}} dx dx' + C.$$

Using the Riesz rearrangement inequality,

$$E_s(\omega_\lambda) \leq c_s \int_S \int_S \frac{\lambda \mathbb{1}_{\mathcal{B}(0, \varepsilon)}(x) \lambda \mathbb{1}_{\mathcal{B}(0, \varepsilon)}(x')}{|x-x'|^{2(1-s)}} dx dx' + C = \frac{c_s}{\varepsilon^{2(1-s)}} \mathcal{I}_s + C.$$

The lemma is proved. \square

3.8 Proof of Lemma 7

First, we expand the double integral in (35) as

$$\begin{aligned} I_s(\omega, S, S) &= \int_S \int_{\mathcal{B}(x, \Lambda_0 \varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x') dx' \omega(x) dx \\ &\quad + \int_S \int_{\mathcal{B}(x, \Lambda_0 \varepsilon)^c} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x') dx' \omega(x) dx. \end{aligned}$$

Let $\eta > 0$ and $C > 0$ to be precised later, and set

$$\Upsilon_\eta^{\Lambda_0} := \left\{ x \in \text{supp}(\omega) : \text{meas}(\mathcal{B}(x, \Lambda_0 \varepsilon) \cap \text{supp}(\omega)) \geq \eta \pi \varepsilon^2 \right\}. \quad (61)$$

Assume for the sake of a contradiction that $\Upsilon_\eta^{\Lambda_0}$ is empty. Then a rearrangement argument with respect to the variable x' yields

$$\int_S \int_{\mathcal{B}(x, \Lambda_0 \varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x') dx' \omega(x) dx < \lambda \int_S \int_{\mathcal{B}(x, \sqrt{\eta} \varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} dx' \omega(x) dx = \frac{\eta^s}{s}.$$

On the other hand, we check that

$$\int_S \int_{\mathcal{B}(x, \Lambda_0 \varepsilon)^c} \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x') dx' \omega(x) dx \leq \frac{1}{\Lambda_0^{2(1-s)}},$$

so that using (35) we obtain

$$s\mathcal{I}_s < \eta^s + \frac{s}{\Lambda_0^{2(1-s)}}.$$

In view of the inequality $s\mathcal{I}_s \geq \frac{1}{6}$, this provides the desired contradiction if we choose η sufficiently small and Λ_0 sufficiently large (both depending only on s). It suffices then to choose for x an arbitrary point in $\Upsilon_\eta^{\Lambda_0}$. \square

3.9 Proof of Lemma 8

Step 1. Let $A \subset B$ be arbitrary measurable subsets in \mathbb{R}^2 such that $K\varepsilon := d(A, B^c) > 0$. We decompose the double integral in Lemma 6 as

$$\begin{aligned} \mathcal{I}_s &\leq I_s(\omega, A, A) + I_s(\omega, B \setminus A, B \setminus A) + I_s(\omega, B^c, B^c) \\ &\quad + 2I_s(\omega, A \cup B^c, B \setminus A) + 2I_s(\omega, A, B^c). \end{aligned} \quad (62)$$

To further simplify the notations, we also set

$$\eta_0 := M(\omega, A), \quad \eta_1 := M(\omega, B \setminus A) \text{ and } \eta_2 := M(\omega, B^c),$$

and in particular since $M(\omega) = 1$ we have $\eta_0 + \eta_1 + \eta_2 = 1$.

We now apply the estimates in Lemma 5 to (62). More precisely, we apply the first one for the first three terms in (62), the second one for the fourth term, and the third one for the last term. This yields, after division by \mathcal{I}_s ,

$$1 \leq \eta_0^{1+s} + \eta_1^{1+s} + \eta_2^{1+s} + \frac{2}{\mathcal{I}_s} \left(\frac{\eta_1(\eta_0 + \eta_2)^s}{s} + \frac{\eta_0 \eta_2}{K^{2(1-s)}} \right). \quad (63)$$

Since η_0, η_1, η_2 and $\eta_0 + \eta_2$ are less than 1, we further deduce

$$1 \leq \eta_0^{s+1} + \eta_2^{s+1} + \left(1 + \frac{2}{s\mathcal{I}_s} \right) \eta_1 + \frac{2}{\mathcal{I}_s K^{2(1-s)}}. \quad (64)$$

In (64), we wish to view the last two terms as negligible, which requires in particular η_1 to be small, and then use a convexity inequality to deduce that the one on the left hand side cannot be too much spread across η_0 and η_2 . More precisely, we require K to satisfy $K \geq \Lambda_0$, where Λ_0 is the constant obtained from Lemma 7, and we define $A = \mathcal{B}(x, mK\varepsilon)$ and $B = \mathcal{B}(x, (m+1)K\varepsilon)$ where x is also given by Lemma 7 and $m \in \mathbb{N}_*$ is to be chosen hereafter. Note that since $\mathcal{B}(x, \Lambda_0\varepsilon) \subset A$, by Lemma 7 we have

$$\eta_0 \geq \eta > 0. \quad (65)$$

The integer $m \geq 1$ is chosen in such a way that $\eta_1 = M(\omega, B \setminus A)$ is small. By additivity of the integral, and since $M(\omega_\lambda, S) = 1$, we may find $1 \leq m \leq K^{2(1-s)}$ such that $\eta_1 \leq 1/K^{2(1-s)}$. From (64), we therefore obtain

$$1 - \eta_0^{s+1} - (1 - \eta_0)^{s+1} \leq \left(1 + \frac{2(1+s)}{s\mathcal{I}_s} \right) \frac{1}{K^{2(1-s)}}.$$

We now use the concavity of the function $t \mapsto 1 - t^{s+1} - (1-t)^{s+1}$ on the segment $[0, 1]$ in order to obtain

$$1 - t^{s+1} - (1-t)^{s+1} \geq (1 - 2^{-s}) \left(\frac{1}{2} - \left| t - \frac{1}{2} \right| \right),$$

so that we are reduced to the alternative

$$\eta_0 \leq \frac{C}{K^{2(1-s)}} \quad \text{or} \quad 1 - \eta_0 \leq \frac{C}{K^{2(1-s)}}, \quad (66)$$

for some constant C depending only on s . If K is chosen sufficiently large, only the first alternative can hold in view of (65). Note also that $mK \leq K^{1+2(1-s)}$. We finally fix the value of K in such a way that $K^{1+2(1-s)} = \Lambda$, provided Λ is sufficiently large so that the previous requirements on K hold with such a choice. If it is not the case, then the conclusion of Lemma 8 can simply be obtained by choosing C sufficiently large so that $C/\Lambda^{\gamma_s} \geq 1$, in which case it yields a trivial inequality. Therefore,

$$M(\omega, \mathcal{B}(x, \Lambda\varepsilon)^c) \leq 1 - \eta_0 \leq \frac{C}{K^{2(1-s)}} = \frac{C}{\Lambda^{\gamma_s}}. \quad (67)$$

Step 2. To conclude the proof this lemma, we show that in the above estimate, x can be chosen equal to $(1, 0)$. We start with the following lower and upper bound of $L(\omega)$ using $|x'| \leq 2$,

$$\begin{aligned} \left(\inf_{x' \in \mathcal{B}(x, \Lambda\varepsilon)} |x'|^2 \right) M(\omega, \mathcal{B}(x, \Lambda\varepsilon)) &\leq L(\omega) \\ &\leq \left(\sup_{x' \in \mathcal{B}(x, \Lambda\varepsilon)} |x'|^2 \right) M(\omega, \mathcal{B}(x, \Lambda\varepsilon)) + 4M(\omega, S \setminus \mathcal{B}(x, \Lambda\varepsilon)). \end{aligned} \quad (68)$$

We note that in the statement of Lemma 7, if in addition ω is assumed to be angular Steiner symmetric then the point x can be chosen of the form $x = (r, 0)$. Indeed, under Steiner symmetry the required inequality is improved by shifting x along the angular variable until it reaches the horizontal axis. With $x = (r, 0)$, the estimate (68) becomes

$$\left(r - \frac{\Lambda^2}{\pi\lambda} \right) M(\omega, \mathcal{B}(x, \Lambda\varepsilon)) \leq L(\omega) \leq \left(r + \frac{\Lambda^2}{\pi\lambda} \right) M(\omega, \mathcal{B}(x, \Lambda\varepsilon)) + 4M(\omega, S \setminus \mathcal{B}(x, \Lambda\varepsilon)).$$

In view of the constraints $M(\omega) = L(\omega) = 1$ and using (67),

$$\left(r - \frac{\Lambda^2}{\pi\lambda} \right) \left(1 - \frac{C}{\Lambda^{\gamma_s}} \right) \leq 1 \leq r + \frac{\Lambda^2}{\pi\lambda} + 4 \frac{C}{\Lambda^{\gamma_s}}.$$

Since Λ is any positive number, in the limit $\lambda \rightarrow \infty$, this gives $x = (r, 0) \rightarrow (1, 0)$. Using again the constraint $M(\omega) = L(\omega) = 1$ and (67) improves the convergence with $|x - (1, 0)| \leq C\varepsilon$. With such a convergence rate at hand, we can transform (67) into

$$M(\omega, \mathcal{B}((1, 0), \Lambda\varepsilon)^c) \leq \frac{C}{\Lambda^{\gamma_s}},$$

which is the announced estimate. \square

3.10 Proof of Lemma 9

In this proof, we focus on the support of the vorticity, that is the subset of S , which is composed of the points $x \in S$ such that $\psi_\lambda(x) \geq 0$. Under this condition, we can come back to the definition of the function ψ_λ in order to obtain the inequality

$$0 \geq \psi_\lambda(y) - \psi_\lambda(x) = K_s \omega_\lambda(y) - K_s \omega_\lambda(x) + \frac{\alpha}{2} (|y|^2 - |x|^2), \quad (69)$$

for any point $y \in S$ such that $\psi_\lambda(y) \leq 0$. In order to establish that the support of the vorticity has no intersection with the complementary set of $\mathcal{B}((1, 0), 2\Lambda\varepsilon) \cup B$, we rely on the previous inequality for two different choices of the point y . Technically, our argument first requires to prove the two following estimates on the function $K_s \omega_\lambda$.

Let Λ be a positive number. When $z \in \mathcal{B}((1, 0), \Lambda\varepsilon)$, we can use the fact that $\mathcal{B}((1, 0), \Lambda\varepsilon) \subseteq S$ in order to get the lower estimate

$$K_s \omega_\lambda(z) \geq c_s \int_S \frac{\omega_\lambda(y)}{|z - y|^{2(1-s)}} dy - C \geq \frac{c_s}{(2\Lambda\varepsilon)^{2(1-s)}} M(\omega_\lambda, \mathcal{B}((1, 0), \Lambda\varepsilon)) - C. \quad (70)$$

On the contrary, for $z \in \mathcal{B}((1, 0), 2\Lambda\varepsilon)^c$, we can split the integration domain S into the ball $\mathcal{B}((1, 0), \Lambda\varepsilon)$ and its complementary set, so as to obtain

$$K_s \omega_\lambda(z) \leq c_s \int_S \frac{\omega_\lambda(y)}{|z - y|^{2(1-s)}} dy + C \leq c_s \int_{S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)} \frac{\omega_\lambda(y)}{|z - y|^{2(1-s)}} dy + \frac{C_s}{(R\varepsilon)^{2(1-s)}} + C.$$

A rearrangement argument then provides the upper estimate

$$K_s \omega_\lambda(z) \leq \frac{c_s}{s\varepsilon^{2(1-s)}} M(\omega_\lambda, S \setminus \mathcal{B}((1,0), \Lambda\varepsilon))^s + \frac{c_s}{(\Lambda\varepsilon)^{2(1-s)}} + C. \quad (71)$$

With these estimates at hand, we now assume for the sake of a contradiction the existence of a point $x \in S \setminus (\mathcal{B}((1,0), 2\Lambda\varepsilon) \cup B)$ such that $\psi_\lambda(x) \geq 0$. Recall here that

$$B = \left\{ (r, \theta) \in S : \frac{1}{2} \leq r \leq \frac{1}{2} + \frac{1}{8} \text{ or } 2 - \frac{1}{8} \leq r \leq 2 \right\}. \quad (72)$$

When ε is small enough, we can find a point $y_1 \in S$ such that $\psi_\lambda(y_1) \leq 0$, and which satisfies the two conditions

$$\text{sign}(\alpha) = \text{sign}(|y_1|^2 - |x|^2) \quad \text{and} \quad ||y_1|^2 - |x|^2| \geq \frac{1}{10}. \quad (73)$$

Such a point necessarily exists due to the constraints $M(\omega_\lambda) = L(\omega_\lambda) = 1$, the vorticity concentration provided by Lemma 8 and the fact that $x \in S \setminus B$. Combining the inequalities (69) and (73) with the estimate (71) leads to the bound

$$\frac{|\alpha|}{20} \leq \frac{c_s}{s\varepsilon^{2(1-s)}} M(\omega_\lambda, S \setminus \mathcal{B}((1,0), \Lambda\varepsilon))^s + \frac{c_s}{(\Lambda\varepsilon)^{2(1-s)}} + C. \quad (74)$$

Consider now a further positive number Λ_1 , as well as a point $y_2 \in \mathcal{B}((1,0), \Lambda_1\varepsilon)$ such that $\psi_\lambda(y_2) \leq 0$. Provided that $\Lambda_1 > 1$, the existence of this point follows from the property that the support of the vorticity has measure $\pi\varepsilon^2$. In view of (70) and (71), we can estimate (69) so as to obtain

$$\frac{c_s}{(2\Lambda_1\varepsilon)^{2(1-s)}} M(\omega_\lambda, \mathcal{B}((1,0), \Lambda_1\varepsilon)) \leq \frac{c_s}{s\varepsilon^{2(1-s)}} M(\omega_\lambda, S \setminus \mathcal{B}((1,0), \Lambda\varepsilon))^s + \frac{c_s}{(\Lambda\varepsilon)^{2(1-s)}} + 4|\alpha| + C.$$

Hence, we deduce from (74) that

$$\frac{1}{(2\Lambda_1)^{2(1-s)}} M(\omega_\lambda, \mathcal{B}((1,0), \Lambda_1\varepsilon)) \leq C \left(M(\omega_\lambda, \mathcal{B}((1,0), S \setminus \Lambda\varepsilon))^s + \frac{1}{\Lambda^{2(1-s)}} + \varepsilon^{2(1-s)} \right).$$

We finally invoke the decay estimate provided by Lemma 8 in order to get

$$\frac{1}{(2\Lambda_1)^{2(1-s)}} \left(1 - \frac{1}{\Lambda_1^{\gamma_s}} \right) \leq C \left(\frac{1}{\Lambda^{s\gamma_s}} + \frac{1}{\Lambda^{2(1-s)}} + \varepsilon^{2(1-s)} \right).$$

For ε small enough, this inequality is false for a fixed number Λ_1 , when Λ is chosen large enough. This contradiction concludes the proof of Lemma 9. \square

3.11 Proof of Proposition 2

We argue by contradiction. We assume that the function ω_λ does not identically vanish on the set B and we derive a contradiction with the property that it is a solution of the maximization problem (\mathcal{P}_λ) . More precisely, our main argument lies in constructing a function $\omega_{\lambda,2} \in X_\lambda$, with

$$M(\omega_{\lambda,2}) = L(\omega_{\lambda,2}) = 1,$$

and such that

$$E_s(\omega_{\lambda,2}) > E_s(\omega_\lambda).$$

This construction is performed in two steps. The first step provides a function $\omega_{\lambda,1}$, which satisfies all the previous criteria except the constraint $L(\omega_{\lambda,2}) = 1$. This constraint is recovered in a second step without losing too much energy.

Step 1. We assume for the sake of a contradiction that $M(\omega_\lambda, B) > 0$. The construction of the function $\omega_{\lambda,1}$ then relies on Lemma 9, which guarantees that the function ω_λ identically vanishes on the subset $S \setminus (\mathcal{B}((1,0), \Lambda\varepsilon) \cup B)$. The idea underlying the construction is to rearrange the function ω_λ so that the positive mass $M(\omega_\lambda, B)$ localized in B is brought back into a small disk nearby $\mathcal{B}((1,0), \Lambda\varepsilon)$.

Observe first that the localized mass $M(\omega_\lambda, B)$ can be estimated as

$$M(\omega_\lambda, B) \leq C\varepsilon^{\gamma s}, \quad (75)$$

by Lemma 8. In particular, this quantity vanishes in the limit $\varepsilon \rightarrow 0$. Setting

$$\rho_1 = \sqrt{M(\omega_\lambda, B)} \quad \text{and} \quad x_1 = 1 + \Lambda\varepsilon + \rho_1\varepsilon, \quad (76)$$

we check that the ball $\mathcal{B}(x_1, \rho_1\varepsilon)$ lies in the subset $S \setminus (\mathcal{B}((1, 0), \Lambda\varepsilon) \cup B)$ for ε small enough, and that its intersection with the ball $\mathcal{B}((1, 0), \Lambda\varepsilon)$ reduces to the point $(1 + \Lambda\varepsilon, 0)$.

With these properties at hand, we can define the restriction of the function $\omega_{\lambda,1}$ to the angular sector \mathfrak{S}_N as

$$\omega_{\lambda,1} = \omega_\lambda \mathbb{1}_{\mathcal{B}((1,0),\Lambda\varepsilon)} + \lambda \mathbb{1}_{\mathcal{B}(x_1,\rho_1\varepsilon)}.$$

We next extend this definition to the whole space using the N -fold symmetry. In view of this definition, the function $\omega_{\lambda,1}$ belongs to the function set X_λ . Moreover, it is angular Steiner symmetric due to the angular Steiner symmetry of the function ω_λ , and it satisfies the condition $M(\omega_{\lambda,1}) = M(\omega_\lambda) = 1$.

We now estimate how the energy of the function ω_λ has been modified by this construction. Recall first that the function $\omega_{\lambda,1}$ is equal to the function ω_λ on the ball $\mathcal{B}((1, 0), \Lambda\varepsilon)$, so that

$$I_s(\omega_{\lambda,1}, \mathcal{B}((1, 0), \Lambda\varepsilon), \mathcal{B}((1, 0), \Lambda\varepsilon)) = I_s(\omega_\lambda, \mathcal{B}((1, 0), \Lambda\varepsilon), \mathcal{B}((1, 0), \Lambda\varepsilon)). \quad (77)$$

We next invoke the Riesz rearrangement inequality [20] in order to obtain

$$I_s(\omega_\lambda, S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) \leq I_s(\omega_{\lambda,1}, S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)). \quad (78)$$

We also check that

$$\text{dist}(\text{supp}(\omega_\lambda \mathbb{1}_S), \text{supp}(\omega_\lambda \mathbb{1}_{\mathbb{R}^2 \setminus S})) = \sin\left(\frac{\pi}{N}\right) \geq \frac{1}{2N}, \quad (79)$$

which gives

$$|I_s(\omega_\lambda, S, \mathbb{R}^2 \setminus S) - I_s(\omega_{\lambda,1}, S, \mathbb{R}^2 \setminus S)| \leq C\varepsilon^{2(1-s)} M(\omega_\lambda, B). \quad (80)$$

Finally, we turn to the last difference, namely $I_s(\omega_{\lambda,1}, \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) - I_s(\omega_\lambda, \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon))$. We observe that

$$\text{dist}(\mathcal{B}((1, 0), \Lambda\varepsilon), B) \geq \frac{1}{4},$$

when ε is small enough. Since the function ω_λ identically vanishes on $S \setminus (\mathcal{B}((1, 0), \Lambda\varepsilon) \cup B)$, it follows that

$$I_s(\omega_\lambda, \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) \leq C\varepsilon^{2(1-s)} M(\omega_\lambda, B). \quad (81)$$

On the other hand, we check that

$$\max_{x \in \mathcal{B}(x_1, \rho_1)} \max_{y \in \mathcal{B}((1,0), \Lambda\varepsilon)} |x - y| = 2\varepsilon(\Lambda + \rho_1),$$

and this inequality gives

$$\begin{aligned} & I_s(\omega_{\lambda,1}, \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) \\ & \geq c_s (2\Lambda + 2\rho_1)^{2(s-1)} M(\omega_{\lambda,1}, S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) M(\omega_{\lambda,1}, \mathcal{B}((1, 0), \Lambda\varepsilon)). \end{aligned}$$

In view of the decay estimate in Lemma 8, we know that

$$M(\omega_{\lambda,1}, \mathcal{B}((1, 0), \Lambda\varepsilon)) \geq \frac{1}{2},$$

for ε small enough, while $M(\omega_{\lambda,1}, S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) = M(\omega_\lambda, B)$. As a consequence, we are led to the lower bound

$$I_s(\omega_{\lambda,1}, \mathcal{B}((1, 0), \Lambda\varepsilon), S \setminus \mathcal{B}((1, 0), \Lambda\varepsilon)) \geq CM(\omega_\lambda, B), \quad (82)$$

and we can collect the energy estimates from (77) to (82) in order to write

$$E_s(\omega_{\lambda,1}) - E_s(\omega_\lambda) = \frac{N}{2\varepsilon^{2(1-s)}} (I_s(\omega_{\lambda,1}, S, \mathbb{R}^2) - I_s(\omega_\lambda, S, \mathbb{R}^2)) \geq C \left(\frac{1}{\varepsilon^{2(1-s)}} - 1 \right) M(\omega_\lambda, B). \quad (83)$$

However, the function $\omega_{\lambda,1}$ may not satisfy the constraint $L(\omega_{\lambda,1}) = 1$, so that we cannot consider it as a test function for the maximization problem (\mathcal{P}_λ) .

Step 2. We recover this property by a suitable translation of the restriction to the set S of the function $\omega_{\lambda,1}$. Consider the second order algebraic equation

$$r_0^2 + \left(2 \int_S x_1 \omega_{\lambda,1}(x) dx \right) r_0 + \int_S |x|^2 \omega_{\lambda,1}(x) dx - 1 = 0. \quad (84)$$

Going back to the definition of the function $\omega_{\lambda,1}$, we observe that the support of its restriction to S is included into the ball $\mathcal{B}((1,0), (\Lambda + 2\rho_1)\varepsilon)$. In view of (75) and (76), we deduce that

$$\int_S x_1 \omega_{\lambda,1}(x) dx = (1 + \mathcal{O}(\varepsilon)) \int_S \omega_{\lambda,1} = 1 + \mathcal{O}(\varepsilon),$$

in the limit $\varepsilon \rightarrow 0$. Similarly, we have

$$\int_S |x|^2 \omega_{\lambda,1}(x) dx = 1 + \mathcal{O}(\varepsilon),$$

as $\varepsilon \rightarrow 0$. As a consequence, the discriminant

$$\Delta = 4 \left(\int_S x_1 \omega_{\lambda,1}(x) dx \right)^2 - 4 \int_S |x|^2 \omega_{\lambda,1}(x) dx + 4$$

of (84) is positive for ε small enough, and we are authorized to set

$$r_0 = - \int_S x_1 \omega_{\lambda,1}(x) dx + \left(\left(\int_S x_1 \omega_{\lambda,1}(x) dx \right)^2 - \int_S |x|^2 \omega_{\lambda,1}(x) dx + 1 \right)^{\frac{1}{2}}. \quad (85)$$

Using the inequality $|\sqrt{v} - \sqrt{u}| \leq |v - u|$ for $u \geq 1/4$ and $v \geq 1/4$, we can estimate the number r_0 by

$$|r_0| \leq |L(\omega_{\lambda,1}) - 1|, \quad (86)$$

for ε small enough. Invoking once again the definition of the function $\omega_{\lambda,1}$, we can combine the property that $1/2 \leq |x| \leq 2$ for $x \in S$ with inequality (75) in order to estimate the quantity $L(\omega_{\lambda,1})$ as

$$|L(\omega_{\lambda,1}) - 1| = |L(\omega_{\lambda,1}) - L(\omega_\lambda)| \leq 4 M(\omega_\lambda, B) \leq C\varepsilon^{\gamma_s}. \quad (87)$$

Hence we obtain the estimate

$$|r_0| \leq C\varepsilon^{\gamma_s}. \quad (88)$$

At this stage, we set

$$\forall x \in \mathfrak{S}_N, \quad \omega_{\lambda,2}(x) = \omega_{\lambda,1}(x - (r_0, 0)). \quad (89)$$

Since the support of the restriction of $\omega_{\lambda,1}$ to S is included into the ball $\mathcal{B}((1,0), (\Lambda + 2\rho_1)\varepsilon)$, estimate (88) is enough to guarantee that the function $\omega_{\lambda,2}$ has a compact support in S when ε is small enough. In particular, we can extend this function to \mathbb{R}^2 by using the N -fold symmetry, and this extension amounts to translate the restriction of the function $\omega_{\lambda,1}$ to each angular sector $R_{(2n\pi)/N} \mathfrak{S}_N$ by the vector $y_n = R_{(2n\pi)/N}(r_0, 0)$. In particular, we can check that the function $\omega_{\lambda,2}$ belongs to the set X_λ . By construction, it is also angular Steiner symmetric and it satisfies the constraint $M(\omega_{\lambda,2}) = 1$. On the other hand, we compute

$$L(\omega_{\lambda,2}) = \int_S |x + (r_0, 0)|^2 \omega_{\lambda,1}(x) dx = r_0^2 + 2r_0 \int_S x_1 \omega_{\lambda,1}(x) dx + \int_S |x|^2 \omega_{\lambda,1}(x) dx,$$

using the property that $M(\omega_{\lambda,1}) = 1$. Since the number r_0 is a root of (84), we infer that $L(\omega_{\lambda,2}) = 1$.

We finally estimate the energy of the function $\omega_{\lambda,2}$. Since this function is constructed by translating the function $\omega_{\lambda,1}$ in S , we first have

$$I_s(\omega_{\lambda,2}, S, S) = I_s(\omega_{\lambda,1}, S, S). \quad (90)$$

Similarly, we also derive from the definition of the function $\omega_{\lambda,1}$ that

$$\begin{aligned} & I_s(\omega_{\lambda,2}, S, \mathbb{R}^2 \setminus S) - I_s(\omega_{\lambda,1}, S, \mathbb{R}^2 \setminus S) \\ &= \sum_{n=1}^{N-1} \int_S \int_{R_{\frac{2n\pi}{N}} S} \left(\left(\frac{\varepsilon}{|x - x' + y_0 - y_n|} \right)^{2(1-s)} - \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \right) \omega_{\lambda,1}(x') dx' \omega_{\lambda,1}(x) dx. \end{aligned}$$

Coming back to (79), we infer that

$$I_s(\omega_{\lambda,2}, S, \mathbb{R}^2 \setminus S) - I_s(\omega_{\lambda,1}, S, \mathbb{R}^2 \setminus S) \geq -C\varepsilon^{2(1-s)} \sum_{n=1}^{N-1} |y_0 - y_n|.$$

Since $|y_n| = r_0$ for any $0 \leq n \leq N-1$, we deduce from (86) and (87) that

$$I_s(\omega_{\lambda,2}, S, \mathbb{R}^2 \setminus S) \geq I_s(\omega_{\lambda,1}, S, \mathbb{R}^2 \setminus S) - C\varepsilon^{2(1-s)} M(\omega_\lambda, B).$$

Combining this estimate with (83) and (90) leads to the inequality

$$E_s(\omega_{\lambda,2}) - E_s(\omega_\lambda) = \frac{N}{2\varepsilon^{2(1-s)}} (I_s(\omega_{\lambda,2}, \mathbb{R}^2, \mathbb{R}^2) - I_s(\omega_\lambda, \mathbb{R}^2, \mathbb{R}^2)) \geq C \left(\frac{1}{\varepsilon^{2(1-s)}} - 1 \right) M(\omega_\lambda, B). \quad (91)$$

This inequality contradicts the maximality of the function ω_λ in the limit $\varepsilon \rightarrow 0$. This concludes the proof of Proposition 2. \square

A The angular Steiner symmetrization

Our previous construction of co-rotating vortices with N -fold symmetry relies several times on a reduction on each fold to angular Steiner symmetric functions. This reduction is made possible by the fact that the restriction to each fold of these co-rotating vortices are indeed angular Steiner symmetric. In this appendix, we collect the properties of this symmetrization, which we have used for our construction, and we give their proofs.

To have this appendix self contained, we recall here after the definitions we need. Consider a non-negative measurable function ω defined on the angular sector

$$\mathfrak{S}_N := \left\{ x = (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 : -\frac{\pi}{N} < \theta < \frac{\pi}{N} \right\}.$$

Its angular Steiner symmetrization ω^\sharp is defined as the unique even function for the variable θ such that

$$\omega^\sharp(r, \theta) > \nu \quad \text{if and only if} \quad |\theta| < \frac{1}{2} \text{meas} \left\{ \theta' \in \left(-\frac{\pi}{N}, \frac{\pi}{N} \right) : \omega(r, \theta') > \nu \right\},$$

for any positive numbers r and ν , and any angle $-\pi/N < \theta < \pi/N$. This function is well-defined and measurable on \mathfrak{S}_N . Moreover, the function ω is said to be angular Steiner symmetric if and only if $\omega^\sharp = \omega$ almost everywhere. In view of its definition, the angular Steiner symmetrization satisfies

$$\text{meas} \left\{ \theta \in \left(-\frac{\pi}{N}, \frac{\pi}{N} \right) : \omega^\sharp(r, \theta) > \nu \right\} = \text{meas} \left\{ \theta \in \left(-\frac{\pi}{N}, \frac{\pi}{N} \right) : \omega(r, \theta) > \nu \right\},$$

for any positive numbers r and ν , which means that it is a rearrangement with respect to the angular variable θ . As a consequence of the layer-cake representation of non-negative measurable functions, the angular Steiner symmetrization maps the set

$$L_+^p(\mathfrak{S}_N) := \left\{ \omega \in L^p(\mathfrak{S}_N) \text{ s.t. } \omega \geq 0 \text{ a.e.} \right\},$$

into itself. This rearrangement preserves several integral quantities as stated in the next lemma.

Lemma 10. Let $\omega \in L^1_{loc,+}(\mathfrak{S}_N)$ and let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous. We have

$$\int_{\mathfrak{S}_N} f(r) \omega(r, \theta) r dr d\theta = \int_{\mathfrak{S}_N} f(r) \omega^\sharp(r, \theta) r dr d\theta,$$

provided that these quantities are finite.

Proof. This is a direct consequence of the layer-cake formula

$$\omega(r, \theta) = \int_0^{+\infty} \mathbb{1}_{\{\omega(r, \cdot) > \nu\}}(\theta) d\nu,$$

the Fubini theorem and the definition of the angular Steiner symmetrization. \square

In contrast, the energy E is not conserved by the angular Steiner symmetrization, but it is increased by this transformation. The proof of this claim relies on the Riesz rearrangement inequality for the angular Steiner symmetrization.

Lemma 11. Let $\omega \in L^\infty_+(S)$. We have

$$E(\omega) \leq E(\omega^\sharp).$$

Proof. Recall first that the quantity $E(\omega)$ is defined by

$$E(\omega) = \int_{\frac{1}{2}}^2 r dr \int_{\frac{1}{2}}^2 r' dr' \left(\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') d\theta d\theta' \right). \quad (92)$$

In this expression, the kernel κ is equal to

$$\kappa(r, r', \xi) = \sum_{n=0}^{N-1} \frac{N c_s}{(r^2 + (r')^2 - 2rr' \cos(\xi - \frac{2\pi n}{N}))^{1-s}},$$

for $\frac{1}{2} < r \neq r' < 2$ and $-\pi/N < \xi < \pi/N$. This function is non-negative and even with respect to the variable ξ due to the identity

$$\cos\left(-\xi - \frac{2\pi n}{N}\right) = \cos\left(\xi - \frac{2\pi(N-n)}{N}\right),$$

which holds for $-\pi/N < \xi < \pi/N$ and $1 \leq n \leq N-1$.

Moreover, given two fixed numbers $\frac{1}{2} < r \neq r' < 2$, the map $\xi \mapsto \kappa(r, r', \xi)$ is smooth on $(-\pi/N, \pi/N)$, and its derivative is given by

$$\frac{\partial \kappa}{\partial \xi}(r, r', \xi) = -2c_s(1-s)rr' \sum_{n=0}^{N-1} \frac{\sin\left(\xi - \frac{2\pi n}{N}\right)}{(r^2 + (r')^2 - 2rr' \cos(\xi - \frac{2\pi n}{N}))^{2-s}}.$$

We claim that this quantity is negative on $(0, \pi/N)$, while it is positive on $(-\pi/N, 0)$. More precisely, we have

Lemma 12. Let $N \geq 2$, $0 \leq s \leq 1$ and $0 < \varrho < 1$. Consider the function $\phi : [-1, 1] \rightarrow \mathbb{R}$ defined by

$$\phi(x) := \frac{1}{(1 - \varrho x)^{2-s}}, \quad (93)$$

for any number $-1 \leq x \leq 1$, and set

$$\Phi(\xi) = \sum_{n=0}^{N-1} \sin\left(\xi - \frac{2\pi n}{N}\right) \phi\left(\cos\left(\xi - \frac{2\pi n}{N}\right)\right),$$

for any number $\xi \in \mathbb{R}$. There exists a continuous, positive and $2\pi/N$ -periodic function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\Phi(\xi) = \sin(N\xi) F(\xi), \quad (94)$$

for any number $\xi \in \mathbb{R}$.

Our previous claim directly follows from applying formula (94) with $\varrho = 2rr'/(r^2 + (r')^2) \in (0, 1)$. For sake of clarity, we postpone the proof of Lemma 12 and now complete the proof of Lemma 11.

Coming back to (92), we fix two positive numbers $r \neq r'$ and extend the functions $\theta \mapsto \omega(r, \theta)$ and $\theta \mapsto \omega(r', \theta)$ to \mathbb{R} by letting them equal to 0 outside the interval $(-\pi/2N, \pi/2N)$. Similarly, we extend the map $\theta \mapsto \kappa(r, r', \theta)$ to \mathbb{R} by letting it equal to 0 outside the interval $(-\pi/N, \pi/N)$. We then have

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') d\theta d\theta' = \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') d\theta d\theta',$$

and we can apply the Riesz rearrangement inequality (see e.g. [16, Lemma 3.6]) in order to obtain

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') d\theta d\theta' \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa^\sharp(r, r', \theta - \theta') \omega^\sharp(r, \theta) \omega^\sharp(r', \theta') d\theta d\theta'.$$

Since the map $\xi \mapsto \kappa(r, r', \xi)$ is even and non-increasing on \mathbb{R}_+ , we have

$$\kappa^\sharp(r, r', \theta - \theta') = \kappa(r, r', \theta - \theta'),$$

for any $(\theta, \theta') \in \mathbb{R}^2$. Moreover, since the functions $\theta \mapsto \omega(r, \theta)$ and $\theta \mapsto \omega(r', \theta)$ are supported in $(-\pi/2N, \pi/2N)$, so are the functions $\theta \mapsto \omega^\sharp(r, \theta)$ and $\theta \mapsto \omega^\sharp(r', \theta)$, and this property leads to

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') d\theta d\theta' \leq \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega^\sharp(r, \theta) \omega^\sharp(r', \theta') d\theta d\theta'.$$

Lemma 11 finally follows from introducing this inequality into (92). \square

We now provide the proof of Lemma 12.

Proof of Lemma 12. For sake of simplicity, we set

$$\xi_n = \xi - \frac{2\pi n}{N},$$

for any number $\xi \in \mathbb{R}$ and any integer $0 \leq n \leq N-1$. With this notation at hand, the function Φ rewrites as

$$\Phi(\xi) = \sum_{n=0}^{N-1} \sin(\xi_n) \phi(\cos(\xi_n)),$$

and we can express it as in (94) by developing the following inductive argument.

The first step, as the subsequent ones, relies on a Taylor expansion of the function ϕ . We define the maps

$$T_p[f](x) := \int_0^1 f^{(p)}(sx)(1-s)^{p-1} ds$$

for any integer $p \geq 1$, any number $-1 \leq x \leq 1$ and any function $f \in \mathcal{C}^\infty([-1, 1])$. These maps allow to write the remainder term in the Taylor formula as

$$f(x) = \sum_{k=0}^{p-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{x^p}{(p-1)!} T_p[f](x).$$

Since the function ϕ is smooth on \mathbb{R} , we infer from this formula that

$$\begin{aligned} \Phi(\xi) &= \sum_{k=0}^{N-2} \frac{\phi^{(k)}(0)}{k!} \sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^k \\ &\quad + \frac{1}{(N-2)!} \sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^{N-1} T_{N-1}[\phi](\cos(\xi_n)). \end{aligned} \tag{95}$$

We next claim that the trigonometric sums $\sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^k$ are equal to 0 when $0 \leq k \leq N-2$. For further use, we more generally set

$$\sigma_{k,\ell}(\xi) := \sum_{n=0}^{N-1} \sin(\ell\xi_n) \cos(\xi_n)^k, \quad (96)$$

for any integer $(k, \ell) \in \mathbb{N}^2$, and we check that

$$\sigma_{k,\ell}(\xi) = 0, \quad (97)$$

provided that $0 \leq k+\ell \leq N-1$. Indeed, the Euler formula for the cosine function and the binomial theorem give

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^k} \sum_{n=0}^{N-1} \sin(\ell\xi_n) \sum_{j=0}^k \binom{k}{j} e^{i(k-2j)\xi_n}.$$

Since the sum $\sigma_{k,\ell}(\xi)$ is real-valued, this expression reduces to

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \sum_{n=0}^{N-1} \sin(\ell\xi_n) \cos((k-2j)\xi_n),$$

so that

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^{k+1}} \sum_{j=0}^k \binom{k}{j} \sum_{n=0}^{N-1} \left(\sin((k+\ell-2j)\xi_n) + \sin((2j+\ell-k)\xi_n) \right). \quad (98)$$

We rewrite this formula as

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^{k+1}} \sum_{j=0}^k \binom{k}{j} \Im \left(e^{i(k+\ell-2j)\xi} \sum_{n=0}^{N-1} \left(e^{\frac{2\pi i(2j-k-\ell)}{N}} \right)^n + e^{i(2j+\ell-k)\xi} \sum_{n=0}^{N-1} \left(e^{\frac{2\pi i(k-\ell-2j)}{N}} \right)^n \right). \quad (99)$$

Recall that

$$\sum_{n=0}^{N-1} \left(e^{\frac{2\pi i p}{N}} \right)^n = \begin{cases} N & \text{if } p \text{ is a multiple of } N, \\ 0 & \text{elsewhere,} \end{cases}$$

and observe that the integers $2j-\ell-k$ and $k-\ell-2j$ lie in the interval $[-(N-1), N-1]$ due to the constraints $0 \leq k+\ell \leq N-1$ and $0 \leq j \leq k$. As a consequence, the sums with respect to n in (99) are equal to 0 except if $2j-\ell-k=0$ for the first sum, respectively $k-\ell-2j=0$ for the second one. In these two cases, the quantity in the imaginary part is real-valued, so that the quantity $\sigma_{k,\ell}(\xi)$ is indeed equal to 0.

Invoking (97) with $0 \leq k \leq N-2$ and $\ell=1$, we can simplify (95) as

$$\Phi(\xi) = \frac{1}{(N-2)!} \sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^{N-1} T_{N-1}[\phi](\cos(\xi_n)), \quad (100)$$

and put the focus on the trigonometric polynomial $\xi \mapsto \sin(\xi) \cos(\xi)^{N-1}$. As before, we more generally consider the functions $\xi \mapsto \sin(\ell\xi) \cos(\xi)^{N-\ell}$ for an integer $1 \leq \ell \leq N-1$. Arguing as for the proof of (98) and using the identity $\sin(N\xi) = \sin(N\xi)$ for $0 \leq n \leq N-1$, we observe that

$$\sin(\ell\xi_n) \cos(\xi_n)^{N-\ell} = \frac{1}{2^{N-\ell}} \left(\sin(N\xi) + \sum_{k=1}^{N-\ell} \binom{N-\ell}{k} \sin((N-2k)\xi_n) \right). \quad (101)$$

In this expression, the quantities $\sin((N-2k)\xi_n)$ can appear only once, with a positive multiplicative factor, or twice, with opposite multiplicative factors. For further use, we aim at eliminating this late behavior. For $2\ell \geq N$, this behavior is excluded, so that we now assume that $2\ell < N$. In

this case, we use the change of indices $j = N - k$ for $N/2 < k \leq N - \ell$ in order to decompose the sum over k in (101) as

$$\begin{aligned} \sum_{k=1}^{N-\ell} \binom{N-\ell}{k} \sin((N-2k)\xi_n) &= \sum_{1 \leq k < \ell} \binom{N-\ell}{k} \sin((N-2k)\xi_n) \\ &\quad + \sum_{\ell \leq k < \frac{N}{2}} \left(\binom{N-\ell}{k} - \binom{N-\ell}{N-k} \right) \sin((N-2k)\xi_n). \end{aligned} \quad (102)$$

Note here that

$$\binom{N-\ell}{k} - \binom{N-\ell}{N-k} = \frac{(N-\ell)!}{k!(N-k)!} \left(\prod_{j=0}^{\ell-1} (N-k-j) - \prod_{j=0}^{\ell-1} (k-j) \right) > 0, \quad (103)$$

since $N - k > k$ when $\ell \leq k < N/2$. Hence, rewriting (101) as in (102) provides an expression in which each quantity $\sin((N-2k)\xi_n)$ appears only once, with a positive multiplicative factor.

Going back to (100), we can combine (101) and (102) for $\ell = 1$ in order to obtain

$$\begin{aligned} \Phi(\xi) &= \frac{\sin(N\xi)}{2^{N-1}(N-2)!} \sum_{n=0}^{N-1} T_{N-1}[\phi](\cos(\xi_n)) \\ &\quad + \frac{1}{2^{N-1}(N-2)!} \sum_{1 \leq k < \frac{N}{2}} \left(\binom{N-1}{k} - \binom{N-1}{N-k} \right) \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) T_{N-1}[\phi](\cos(\xi_n)). \end{aligned} \quad (104)$$

At this stage, we set

$$F^1(\xi) := \frac{1}{2^{N-1}(N-2)!} \sum_{n=0}^{N-1} T_{N-1}[\phi](\cos(\xi_n)), \quad (105)$$

and

$$R_{N-2k}^1(x) = \frac{1}{2^{N-1}(N-2)!} \left(\binom{N-1}{k} - \binom{N-1}{N-k} \right) T_{N-1}[\phi](x), \quad (106)$$

for $1 \leq k < N/2$ and $-1 \leq x \leq 1$. With this notation at hand, we can write (104) as

$$\Phi(\xi) = \sin(N\xi) F^1(\xi) + \sum_{1 \leq k < \frac{N}{2}} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^1(\cos(\xi_n)), \quad (107)$$

and we observe that the sums

$$\sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^1(\cos(\xi_n))$$

have exactly the same form as the map Φ with the function ϕ being replaced by the functions R_{N-2k}^1 . As a consequence, we can expect that an inductive argument eventually provides an expression of the map Φ as in (94).

Before going into this inductive argument, we now check that the function F^1 is positive, while the remainder terms R_{N-2k}^1 are positive and absolutely monotone, which means that all their derivatives are positive. This claim originates into the two following properties.

First, in view of (93), the function ϕ is well-defined and smooth on $[-1, 1] \subset (-1/\varrho, 1/\varrho)$, and its successive derivatives are given by

$$\phi^{(p)}(x) = \frac{(2-s)(3-s)\dots(p+1-s)\varrho^p}{(1-\varrho x)^{p+2-s}} > 0,$$

for any $p \geq 1$ and any $x \in [-1, 1]$. Therefore, the function ϕ is absolutely monotone. Actually, it extends to an analytic function on the interval $(-1/\varrho, 1/\varrho)$ since it can be expanded as the power series. Let $R > 0$. We next assume for the sake of a contradiction the existence of a point $x \in \mathcal{B}((1, 0), 2R\varepsilon)^c$ such that $\psi_\lambda(x) \geq 0$. When ε is small enough, we can invoke Lemma 8 in order to find $y_1 \in S$ such that $\psi_\lambda(y_1) < 0$ and the two following conditions are satisfied

$$\phi(x) = \sum_{p=0}^{+\infty} \alpha_{p,s} \rho^p x^p, \quad (108)$$

for $x \in (-1/\varrho, 1/\varrho)$. In this formula, the coefficients $\alpha_{p,s}$ are given by

$$\alpha_{0,s} := 1 \quad \text{and} \quad \alpha_{p,s} := \frac{(2-s)(3-s)\dots(p+1-s)}{p!} \quad \text{for } p \geq 1, \quad (109)$$

and all of them are positive.

Second, we remark that, if a function $f \in C^\infty([-1, 1], \mathbb{R})$ is positive and absolutely monotone, so are the maps $T_p[f]$ for any $p \geq 1$. This is a direct consequence of the computation

$$T_p[f]^{(q)}(x) = \int_0^1 f^{(p+q)}(sx) s^q (1-s)^{p-1} ds > 0.$$

Hence, we can invoke (103), (105) and (106) to conclude that the function F^1 is positive, while the remainder terms R_{N-2k}^1 are positive and absolutely monotone.

We now go inside the inductive argument. Given an integer $p \geq 1$, we assume that we have constructed positive and smooth functions $(F^q)_{1 \leq q \leq p}$ on \mathbb{R} , and positive, smooth and absolutely monotone functions $(R_{N-2k}^q)_{1 \leq q \leq p, 1 \leq k < N/2}$ on $[-1, 1]$, such that

$$\Phi(\xi) = \sin(N\xi) \sum_{q=1}^p F^q(\xi) + \sum_{1 \leq k < \frac{N}{2}} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)), \quad (110)$$

for any $\xi \in \mathbb{R}$. In order to construct the function F^{p+1} and the remainders R_{N-2k}^{p+1} , we apply the same strategy as in the first inductive argument above. We fix an integer $1 \leq k < N/2$ and we invoke the Taylor formula so as to obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \sum_{j=0}^{2k-1} \frac{(R_{N-2k}^p)^{(j)}(0)}{j!} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) \cos(\xi_n)^j \\ &+ \frac{1}{(2k-1)!} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) \cos(\xi_n)^{2k} T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned}$$

In view of (97), the double sum in the right-hand side of this identity is equal to 0, so that

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \\ &= \frac{1}{(2k-1)!} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) \cos(\xi_n)^{2k} T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned}$$

When $k \leq N/4$, we next use (101) in order to get

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \frac{\sin(N\xi)}{4^k (2k-1)!} \sum_{n=0}^{N-1} T_{2k}[R_{N-2k}^p](\cos(\xi_n)) \\ &+ \frac{1}{4^k (2k-1)!} \sum_{j=1}^{2k} \binom{2k}{j} \sum_{n=0}^{N-1} \sin((N-2j)\xi_n) T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned} \quad (111)$$

For $k > N/4$, we similarly invoke (102) so as to obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \frac{\sin(N\xi)}{4^k(2k-1)!} \sum_{n=0}^{N-1} T_{2k}[R_{N-2k}^p](\cos(\xi_n)) \\ &+ \frac{1}{4^k(2k-1)!} \sum_{1 \leq j < N-2k} \binom{2k}{j} \sum_{n=0}^{N-1} \sin((N-2j)\xi_n) T_{2k}[R_{N-2k}^p](\cos(\xi_n)) \\ &+ \frac{1}{4^k(2k-1)!} \sum_{N-2k \leq j < \frac{N}{2}} \left(\binom{2k}{j} - \binom{2k}{N-j} \right) \sum_{n=0}^{N-1} \sin((N-2j)\xi_n) T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned}$$

Setting

$$F^{p+1}(\xi) := \sum_{1 \leq k < \frac{N}{2}} \frac{1}{4^k(2k-1)!} \sum_{n=0}^{N-1} T_{2k}[R_{N-2k}^p](\cos(\xi_n)), \quad (112)$$

as well as

$$\begin{aligned} R_{N-2k}^{p+1}(x) &= \sum_{\max\{1, \frac{k}{2}\} \leq j < \frac{N-k}{2}} \binom{2j}{k} \frac{T_{2j}[R_{N-2j}^p](x)}{4^j(2j-1)!} \\ &+ \sum_{\frac{N-k}{2} \leq j < \frac{N}{2}} \left(\binom{2j}{k} - \binom{2j}{N-k} \right) \frac{T_{2j}[R_{N-2j}^p](x)}{4^j(2j-1)!}, \end{aligned} \quad (113)$$

we conclude that

$$\Phi(\xi) = \sin(N\xi) \sum_{q=1}^{p+1} F^q(\xi) + \sum_{1 \leq k < \frac{N}{2}} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^{p+1}(\cos(\xi_n)).$$

Moreover, we can prove as for the functions F^1 and R_{N-2k}^1 that the map F^{p+1} is positive and smooth on \mathbb{R} , while the maps R_{N-2k}^{p+1} are positive, smooth and absolutely monotone on $[-1, 1]$. This completes the inductive argument and establishes the validity of (110) for any $p \geq 1$.

We finally conclude the proof of Lemma 12 by showing that

$$\sum_{1 \leq k < N/2} |R_{N-2k}^p(x)| \rightarrow 0, \quad (114)$$

as $p \rightarrow +\infty$, uniformly with respect to $x \in [-1, 1]$. In order to establish this convergence, we use the analytic expansion of the function ϕ in (108). More generally, we consider an analytic function f on the interval $(-1/\varrho, 1/\varrho)$, which we expand as the power series

$$f(x) = \sum_{k=0}^{+\infty} a_k x^k,$$

for $x \in (-1/\varrho, 1/\varrho)$. Given an integer $m \geq 1$ and a number $x \in [-1, 1]$, we compute

$$T_m[f](x) = \sum_{k=0}^{+\infty} a_{k+m} (k+m) \dots (k+1) x^k \int_0^1 s^k (1-s)^{m-1} ds.$$

Integrating by parts and arguing by induction provide

$$\int_0^1 s^k (1-s)^{m-1} ds = \frac{m-1}{k+1} \int_0^1 s^{k+1} (1-s)^{m-2} ds = \frac{(m-1)!}{(k+m) \dots (k+1)}.$$

Hence we obtain

$$T_m[f](x) = (m-1)! \tau_m f(x),$$

where the notation $\tau_m f$ refers to the analytic function f given by the power series

$$\tau_m f(x) = \sum_{k=0}^{+\infty} a_{k+m} x^k,$$

for any number $x \in (-1/\varrho, 1/\varrho)$.

With this expression at hand, we rewrite (113) as

$$R_{N-2k}^{p+1}(x) = \sum_{\frac{k}{2} \leq j < \frac{N}{2}} b_k^{2j} \tau_{2j} R_{N-2j}^p(x),$$

with

$$b_k^j = \begin{cases} \binom{j}{k} \frac{1}{2^j} & \text{if } k \leq j < N - k, \\ \left(\binom{j}{k} - \binom{j}{N-k} \right) \frac{1}{2^j} & \text{if } N - k \leq j < N. \end{cases}$$

Since

$$R_{N-2k}^1(x) = b_k^{N-1} \tau_{N-1} \phi(x),$$

by (106), a direct inductive argument provides

$$\begin{aligned} R_{N-2k}^{p+1}(x) = & \sum_{\max\{1, \frac{k}{2}\} \leq j_1 < \frac{N}{2}} b_k^{2j_1} \sum_{\max\{1, \frac{j_1}{2}\} \leq j_2 < \frac{N}{2}} b_{j_1}^{2j_2} \dots \\ & \dots \sum_{\max\{1, \frac{j_{p-1}}{2}\} \leq j_p < \frac{N}{2}} b_{j_{p-1}}^{2j_p} b_{j_p}^{N-1} \tau_{2j_1+2j_2+\dots+2j_p+N-1} \phi(x). \end{aligned} \quad (115)$$

In view of (108), we know that

$$\tau_m \phi(x) = \sum_{k=0}^{+\infty} \alpha_{k+m,s} \rho^{k+m} x^k.$$

Moreover, the radius of convergence of the power series $\sum_{k \geq 0} \alpha_{k,s} x^k$ is equal to 1. Therefore, given any number $0 < \sigma < 1$, there exists a positive number M_σ such that

$$\alpha_{k,s} \sigma^k \leq M_\sigma,$$

for any integer $k \in \mathbb{N}$. When $\tau < \sigma < 1$ and $-1 \leq x \leq 1$, we are led to the estimate

$$|\tau_m \phi(x)| \leq M_\sigma \frac{\tau^m}{\sigma^m} \sum_{k=0}^{+\infty} \frac{\rho^k}{\sigma^k} = \frac{M_\sigma \rho^m}{\sigma^{m-1}(\sigma - \rho)}.$$

Inserting this inequality into (115) gives

$$\begin{aligned} |R_{N-2k}^{p+1}(x)| \leq & \frac{M_\sigma \sigma}{\sigma - \rho} \sum_{\max\{1, \frac{k}{2}\} \leq j_1 < \frac{N}{2}} b_k^{2j_1} \sum_{\max\{1, \frac{j_1}{2}\} \leq j_2 < \frac{N}{2}} b_{j_1}^{2j_2} \dots \\ & \dots \sum_{\max\{1, \frac{j_{p-1}}{2}\} \leq j_p < \frac{N}{2}} b_{j_{p-1}}^{2j_p} b_{j_p}^{N-1} \left(\frac{\rho}{\sigma} \right)^{2j_1+2j_2+\dots+2j_p+N-1}. \end{aligned}$$

Hence, we deduce from the inequality $\rho < \sigma$ that

$$\begin{aligned} |R_{N-2k}^{p+1}(x)| \leq & \frac{M_\sigma \sigma}{\sigma - \rho} \left(\frac{\rho}{\sigma} \right)^{2p+N-1} \times \\ & \times \sum_{\max\{1, \frac{k}{2}\} \leq j_1 < \frac{N}{2}} b_k^{2j_1} \sum_{\max\{1, \frac{j_1}{2}\} \leq j_2 < \frac{N}{2}} b_{j_1}^{2j_2} \dots \sum_{\max\{1, \frac{j_{p-1}}{2}\} \leq j_p < \frac{N}{2}} b_{j_{p-1}}^{2j_p} b_{j_p}^{N-1}. \end{aligned} \quad (116)$$

At this stage, we deduce from the discrete Fubini theorem

$$\begin{aligned} \sum_{1 \leq k \leq \frac{N}{2}} \sum_{\max\{1, \frac{k}{2}\} \leq j < \frac{N}{2}} b_k^{2j} \alpha_j &\leq \sum_{1 \leq k \leq \frac{N}{2}} \sum_{\max\{1, \frac{k}{2}\} \leq j < \frac{N}{2}} \binom{2j}{k} \frac{\alpha_j}{2^{2j}} \\ &\leq \sum_{1 \leq j \leq \frac{N}{2}} \sum_{1 \leq k \leq 2j} \binom{2j}{k} \frac{\alpha_j}{2^{2j}} = \sum_{1 \leq j \leq \frac{N}{2}} \alpha_j, \end{aligned}$$

for any non-negative numbers $\alpha_1, \alpha_2, \dots$ and α_j . In view of (116), this yields

$$\sum_{1 \leq k < \frac{N}{2}} |R_{N-2k}^{p+1}(x)| \leq \frac{M_\sigma \sigma}{\sigma - \rho} \left(\frac{\rho}{\sigma}\right)^{2p+N-1} \sum_{1 \leq j < \frac{N}{2}} b_j^{N-1}.$$

Since

$$\sum_{1 \leq j < \frac{N}{2}} b_j^{N-1} \leq \frac{1}{2^{N-1}} \sum_{j=1}^{N-1} \binom{N-1}{j} \leq 1,$$

we conclude that

$$\sum_{1 \leq k < \frac{N}{2}} |R_{N-2k}^{p+1}(x)| \leq \frac{M_\sigma \sigma}{\sigma - \rho} \left(\frac{\rho}{\sigma}\right)^{2p+N-1},$$

which is enough to obtain the uniform convergence in (114).

Applying this convergence to (110) leads to the identity

$$\Phi(\xi) = \sin(N\xi) F(\xi),$$

for any number $\xi \in \mathbb{R}$. The map F in this formula is defined as the function series

$$F(\xi) = \sum_{q=1}^{+\infty} F^q(\xi).$$

Since the convergence of this series is uniform on \mathbb{R} , and all the functions F^q are continuous and positive, so is the map F . Its $2\pi/N$ -periodicity then follows from its continuity and the $2\pi/N$ -periodicity of the functions Φ and $\xi \mapsto \sin(N\xi)$. This completes the proof of Lemma 12. \square

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